LOGARITHMIC SOBOLEV INEQUALITIES FOR UNBOUNDED SPIN SYSTEMS REVISITED

M. Ledoux

University of Toulouse, France

Abstract. — *We analyze recent proofs of decay of correlations and logarithmic Sobolev inequalities for unbounded spin systems in the perturbative regime developed by B. Zegarlinski, N. Yoshida, B. Helffer, Th. Bodineau. We investigate to this task a simple analytic model. Proofs are short and self-contained.*

Let μ be a probability measure on $\mathbb{R}$ satisfying, for some constant $C > 0$ and for every smooth enough function f on $\mathbb{R}$, either the Poincaré (or spectral gap) inequality

$$\mathrm{Var}_\mu(f) \leq C \int f'^2 d\mu$$

where $\mathrm{Var}_\mu(f)$ is the variance of f with respect to μ (see below), or the logarithmic Sobolev inequality

$$\mathrm{Ent}_\mu(f^2) \leq C \int f'^2 d\mu$$

where $\mathrm{Ent}_\mu(f^2)$ is the entropy of f^2 with respect to μ (see below). It is well-known that the product measure μ^n of μ on $\mathbb{R}^n$ then satisfies the preceding inequalities (with the Euclidean length of the gradient of the function f on $\mathbb{R}^n$) with the same constant C, in particular independent of the dimension n.

Let now H be a smooth function on $\mathbb{R}^n$ such that $\int \mathrm{e}^{-H} d\mu^n < \infty$. Define Q the probability measure on $\mathbb{R}^n$ with density

$$\frac{1}{Z}\,\mathrm{e}^{-H}$$

with respect to μ^n, where Z is the normalization factor. It is a natural question to ask under which conditions on H, the probability measure Q will satisfy a Poincaré or logarithmic Sobolev inequality, and to control the dependence of the constants on H. For example, one may consider potentials H of the form

$$H(x) = \langle Ax, x\rangle + \langle B, x\rangle, \quad x \in \mathbb{R}^n,$$

where A is an $n \times n$ matrix and $B \in \mathbb{R}^n$. In particular, it might be of interest to describe classes of matrices A and vectors B for which the spectral gap and logarithmic Sobolev constants are independent on the dimension n. The simple example of

$$H(x) = \sum_{i=1}^{n} x_i x_{i+1}, \quad x \in \mathbb{R}^n,$$

where $x_{n+1} = x_1$, discussed at the end of Section 1 already raises a number of non-trivial questions.

This setting includes classical examples of spin systems in statistical mechanics. Logarithmic Sobolev inequalities for compact spin systems have been studied extensively during the past years, in particular in the papers [S-Z1], [S-Z2] by D. Stroock and B. Zegarlinski, [L-Y] by S. L. Lu and H. T. Yau and [M-O1], [M-O2] by F. Martinelli and E. Olivieri. Recently, B. Zegarlinski [Ze1], N. Yoshida [Yo1], [Yo3], B. Helffer [He2] and Th. Bodineau [B-H1], [B-H2] investigated the more general and delicate unbounded case. For example, for a finite subset Λ in $\mathbb{Z}^d$, $d \geq 1$, and boundary condition $\omega \in \mathbb{R}^{\mathbb{Z}^d}$, consider the measure $Q = Q^{\Lambda,\omega}$ with density, with respect to the product measure μ^Λ of μ on Λ, $\frac{1}{Z}\,\mathrm{e}^{-JH}$ where $J \in \mathbb{R}$ and

$$H(x) = H^{\Lambda,\omega}(x) = \sum_{p,q \in \Lambda, p \sim q} x_p x_q + \sum_{p \in \Lambda, q \notin \Lambda, p \sim q} x_p \omega_q, \quad x = (x_p)_{p \in \Lambda} \in \mathbb{R}^\Lambda,$$

where the summations are taken on nearest neighbors $p \sim q$ in $\mathbb{Z}^d$. Results in the preceding references assert that for one-dimensional phase measures $d\mu = \frac{1}{Z}\,\mathrm{e}^{-u} dx$ where u is strictly convex at infinity, both Poincaré and logarithmic Sobolev inequalities hold uniformly in cubes Λ and boundary conditions ω provided the interaction coefficient J is small enough (perturbative regime). A typical example of phase u is given by the double-well function $u(x) = x^4 - \beta x^2$, $x \in \mathbb{R}$, $\beta > 0$. Spectral gap and logarithmic Sobolev inequalities represent smoothing properties of the associated stochastic dynamic of fundamental importance in the control of convergence to equilibrium for various spin systems (cf. [S-Z4], [Ze1], [Yo2]), thus providing strong motivation for their investigation.

Our aim in this work is to analyze these results on the preceding abstract model, and to describe at each step the conditions required on H. Spectral gaps and decays of correlations are presented following the Witten Laplacian approach by Helffer-Sjöstrand [He-S], that we however describe in an elementary way by classical semi-group methods. This global analysis does not seem to work for logarithmic Sobolev inequalities for which the usual induction procedure on the dimension has to be performed as developed in [Ze1] and [Yo1]. Together with appropriate correlation bounds, the proof may actually be described in a rather simple minded way.

The purpose of this work is a mere clarification and simplication of the arguments of the papers [Ze1], [He2], [Yo1] and [B-H1] (see also [Yo3], [B-H2]). We adopt the convexity assumptions on the phase μ of [B-H1]. We only consider the perturbative regime where the coupling constants are small. For the matter of clarity, all the constants are explicit. We do not study here the non-perturbative case, for which spectral gaps and logarithmic Sobolev inequalities usually do not hold uniformly,

but for which a formal equivalence between spectral gap, decay of correlation, and logarithmic Sobolev inequality has been shown in [S-Z3] for the compact spins and in [Yo3] for the unbounded case.

Section 1 collects a number of classical results on spectral gaps and logarithmic Sobolev inequalities, tensorization, convexity and Γ_2 conditions etc, essentially by means of simple semigroup arguments taken from [Ba1], [Le2]. The next section emphasizes some correlations inequalities from [He2], [Yo1], [Yo3], [B-H1]. In Section 3, we present Helffer's method for spectral gaps that we describe in the generality of our analytic model. This method unfortunately breaks down for logarithmic Sobolev inequalities so that we have to develop the usual inductive proof. To this end, we describe, in Section 4, marginal distributions when the phase is strictly convex at infinity following [B-H1]. We then proceed to the Markov tensorization of the logarithmic Sobolev inequality (martingale method). In Section 6 we present the main result about uniform logarithmic Sobolev inequalities for the more specific spin systems mentioned above. We conclude with some remarks and extensions. While the paper might look long for what it announces, note that the first part (Section 1) is a self-contained review on known facts and results on spectral gaps and logarithmic Sobolev inequalities that can be skipped by readers familiar with the theory (and aware for example of references [Ba], [Ro], [He3], [An], [G-Z]...).

1. General results and known facts

Throughout this work, if m is a probability measure on a measurable space $(E, \mathcal{E})$, we denote by

$$\mathrm{Var}_m(f) = \int f^2 dm - \left(\int f dm \right)^2 = \int \left(f - \int f dm\right)^2 dm$$

the variance of a square integrable real-valued function f on $(E, \mathcal{E})$ and by

$$\mathrm{Ent}_m(f) = \int f \log f dm - \int f dm \log \left(\int f dm \right)$$

the entropy of a non-negative function f on $(E, \mathcal{E})$ such that $\int f \log(1+f) dm < \infty$.

Let m be a probability measure on $\mathbb{R}^n$ equipped with its Borel σ-field. We say that m satisfies a Poincaré inequality if there exists $\lambda > 0$ such that for all smooth enough functions f on $\mathbb{R}^n$,

$$\lambda \, \mathrm{Var}_m(f) \leq \int |\nabla f|^2 dm \tag{1.1}$$

where $|\nabla f|$ is the Euclidean norm of the gradient of f. We denote by $\mathrm{SG}(m)$ the largest $\lambda > 0$ such that (1.1) holds for all smooth functions f. (By smooth, we understand here and throughout this work, enough regularity in order the various expressions we are dealing with are well defined and finite.) Similarly, we say that

m satisfies a logarithmic Sobolev inequality if there exists $\rho > 0$ such that for all smooth enough functions f on $\mathbb{R}^n$,

$$\rho \operatorname{Ent}_m(f^2) \leq 2 \int |\nabla f|^2 dm. \tag{1.2}$$

We denote by $\mathrm{LS}(m)$ the largest $\rho > 0$ such that (1.2) holds. The normalization in (1.2) in chosen in such a way that the classical inequality

$$\mathrm{LS}(m) \leq \mathrm{SG}(m) \tag{1.3}$$

holds. In particular, logarithmic Sobolev inequalities are stronger than Poincaré inequalities. The proof of (1.3) follows by applying (1.2) to $1 + \varepsilon f$ and by letting ε tend to 0. Logarithmic Sobolev inequalities go back to the foundation paper [Gr] by L. Gross where they are shown to describe equivalently smoothing properties in the form of hypercontractivity. The prime example of measures satisfying (1.1) and (1.2) is the Gaussian measure with density $(2\pi)^{-n/2}\mathrm{e}^{-|x|^2/2}$ with respect to Lebesgue measure on $\mathbb{R}^n$.

In this section, we review basic facts on spectral gaps and logarithmic Sobolev inequalities as well as known criteria in order for these inequalities to hold. To describe measures satisfying either Poincaré or logarithmic Sobolev inequalities is a challenging question. Equivalent conditions in dimension one in terms of the distribution function of m are presented in [B-G]. These conditions are however difficult to tract and in any case do not extend to higher dimensions. Poincaré and logarithmic Sobolev inequalities are however well suited to product measures.

Lemma 1.1. *If m is a probability measure on $\mathbb{R}$, and if m^n denotes the product measure of m on $\mathbb{R}^n$, then, for each n,*

$$\mathrm{SG}(m^n) = \mathrm{SG}(m) \quad \text{and} \quad \mathrm{LS}(m^n) = \mathrm{LS}(m).$$

Although classical, let us briefly present the argument leading to Lemma 1.1 since it plays a crucial role in the investigation of logarithmic Sobolev inequalities in dependent cases (cf. Section 5). Let f be a smooth function on $\mathbb{R}^n$, and let f_k on $\mathbb{R}^k$, $k = 1, \ldots, n$, be the conditional expectation of f given $x_1, \ldots, x_k$. In other words, in this independent case,

$$f_k(x_1, \ldots, x_k) = \int f(x_1, \ldots, x_n) dm(x_{k+1}) \cdots dm(x_n), \quad (x_1, \ldots, x_k) \in \mathbb{R}^k. \tag{1.4}$$

Now,

$$\mathrm{Var}_{m^n}(f) = \sum_{k=1}^{n} \left[\int f_k^2 dm^n - \int f_{k-1}^2 dm^n \right]$$

where we agree that $f_0 = \int f dm^n$. Since f_{k-1} is also the conditional expectation of f_k given $x_1, \ldots, x_{k-1}$, and since m^n is a product measure,

$$\int f_k^2 dm^n - \int f_{k-1}^2 dm^n = \int \mathrm{Var}_{m_k}(f_k) dm^n$$

where we denote by m_k the measure m acting on the k-th coordinate x_k. Therefore,

$$\mathrm{SG}(m)\,\mathrm{Var}_{m^n}(f) \leq \sum_{k=1}^{n} \int \left|\partial_k f_k\right|^2 dm^n$$

where ∂_k denotes partial derivative with respect to the k-th coordinate. Now,

$$\partial_k f_k = \int \partial_k f_{k+1} dm(x_{k+1}) = \cdots = \int \partial_k f dm(x_{k+1}) \cdots dm(x_n)$$

so that, by Jensen's inequality,

$$\mathrm{SG}(m)\,\mathrm{Var}_{m^n}(f) \leq \sum_{k=1}^{n} \int \left|\partial_k f\right|^2 dm^n$$

from which the claim concerning $\mathrm{SG}(m^n)$ follows.

To reach a similar conclusion for $\mathrm{LS}(m^n)$, we have to modify (1.4) into

$$f_k(x_1, \ldots, x_k) = \left(\int f^2(x_1, \ldots, x_n) dm(x_{k+1}) \cdots dm(x_n) \right)^{1/2}$$

that does not induce any fundamental changes in the argument. However, since now

$$2 f_k \partial_k f_k = \int 2 f_{k+1} \partial f_{k+1} dm(x_{k+1}) = \cdots = \int 2 f \partial_k f dm(x_{k+1}) \cdots dm(x_n), \quad (1.5)$$

it is necessary to make use of the Cauchy-Schwarz inequality to get

$$|\partial_k f_k|^2 \leq \int |\partial_k f|^2 dm(x_{k+1}) \cdots dm(x_n)$$

where we used that $f_k^2 = \int f^2 dm(x_{k+1}) \cdots dm(x_n)$. In the dependent cases we study in this paper, the derivatives $\partial_k f_k$ involve correlation terms (cf. (5.4)) that have to be handled separately by the arguments developed in Sections 2 and 3. The use of f_k^2 instead of f_k induces furthermore a number of difficulties in the dependent case that motivate Proposition 2.2 below (cf. Sections 4 and 5).

If m is the product measure of $m_1, \ldots, m_n$, we have similarly that

$$\mathrm{SG}(m^n) = \min_{1 \leq i \leq n} \mathrm{SG}(m_i) \quad \text{and} \quad \mathrm{LS}(m^n) = \min_{1 \leq i \leq n} \mathrm{LS}(m_i).$$

Spectral gap and logarithmic Sobolev constants are stable by simple perturbations. Let U be a smooth potential on $\mathbb{R}^n$ such that $Z = \int \mathrm{e}^{-U} dx < \infty$ and let m be the probability measure on the Borel sets of $\mathbb{R}^n$ defined by $dm = \frac{1}{Z}\,\mathrm{e}^{-U} dx$. Assume m satisfies a spectral gap or logarithmic Sobolev inequality with respective constants $\mathrm{SG}(m)$ and $\mathrm{LS}(m)$. We then have

Lemma 1.2. *Let m' be the probability measure defined by $dm' = \frac{1}{Z'}\,\mathrm{e}^{-U'}dx$ where $\|U - U'\|_\infty \leq C$. Then m' satisfies a Poincaré inequality and a logarithmic Sobolev inequality with constants*

$$\mathrm{SG}(m') \geq \mathrm{e}^{-4C}\mathrm{SG}(m) \quad \text{and} \quad \mathrm{LS}(m') \geq \mathrm{e}^{-4C}\mathrm{LS}(m).$$

Proof. First note that $\mathrm{e}^{-C}Z' \leq Z \leq \mathrm{e}^{C}Z'$. Now, for a given smooth function f,

$$\mathrm{Var}_m(f) = \inf_{a\in\mathbb{R}} \int \big|f(x) - a\big|^2 dm$$

and similarly for $\mathrm{Var}_{m'}(f)$. Therefore, for every $\lambda < \mathrm{SG}(m)$,

$$\begin{aligned}
\lambda\,\mathrm{Var}_{m'}(f) &= \lambda \inf_{a\in\mathbb{R}} \int \big|f(x) - a\big|^2 \mathrm{e}^{U-U'} Z Z'^{-1} dm \\
&\leq \mathrm{e}^{2C}\,\lambda\,\mathrm{Var}_m(f) \\
&\leq \mathrm{e}^{2C} \int |\nabla f|^2 dm \\
&\leq \mathrm{e}^{2C} \int |\nabla f|^2 \mathrm{e}^{U'-U} Z' Z^{-1} dm' \\
&\leq \mathrm{e}^{4C} \int |\nabla f|^2 dm'.
\end{aligned}$$

Similarly, as put forward in [H-S], for every $a, b > 0$, $b\log b - b\log a - b + a \geq 0$ and

$$\mathrm{Ent}_m(f^2) = \inf_{a>0} \int \left[f^2 \log f^2 - f^2 \log a - f^2 + a\right] dm.$$

Therefore, for every $\rho < \mathrm{LS}(m)$,

$$\begin{aligned}
\rho\,\mathrm{Ent}_{m'}(f^2) &= \rho \inf_{a>0} \int \left[f^2 \log f^2 - f^2 \log a - f^2 + a\right] \mathrm{e}^{U-U'} Z Z'^{-1} dm \\
&\leq \mathrm{e}^{2C}\,\rho\,\mathrm{Ent}_m(f^2) \\
&\leq 2\mathrm{e}^{2C} \int |\nabla f|^2 dm \\
&\leq 2\mathrm{e}^{4C} \int |\nabla f|^2 dm'.
\end{aligned}$$

Lemma 1.2 is established. □

Known examples where Poincaré and logarithmic Sobolev inequalities hold have been described by the so-called Bakry-Emery Γ_2 criterion [Ba-E], [Ba1] that involves log-concavity assumptions on the measure (rather its density). Assume as before that m is a probability measure on $\mathbb{R}^n$ with smooth strictly positive density with respect to Lebesgue measure $dm(x) = \frac{1}{Z}\,\mathrm{e}^{-U(x)}dx$ where U is a smooth potential on $\mathbb{R}^n$ such that $\int \mathrm{e}^{-U}dx = Z < \infty$. Let the second order differential operator $\mathrm{L} = \Delta - \langle \nabla U, \nabla\rangle$ that satisfies the integration by parts formula

$$\int f(-\mathrm{L}g)dm = \int \langle \nabla f, \nabla g\rangle dm \tag{1.6}$$

for smooth functions f and g on $\mathbb{R}^n$. Under mild growth conditions on U, we may consider the invariant and time reversible semigroup $(P_t)_{t\geq 0}$ with infinitesimal generator L (cf. [Ba1], [Ro] for details in this respect). Strict convexity (or only strict convexity at infinity) of U assumed throughout this work easily enters this framework. Now, since for a smooth function f on $\mathbb{R}^n$, $P_0 f = f$ and $P_\infty f = \int f dm$, we may write that

$$\begin{aligned}\mathrm{Var}_m(f) &= \int f\Big(-\int_0^\infty \mathrm{L}P_t f dt dm\Big)dt \\ &= \int_0^\infty \Big(\int P_{t/2} f(-\mathrm{L}P_{t/2} f)dm\Big)dt \\ &= \int_0^\infty \Big(\int |\nabla P_{t/2} f|^2 dm\Big)dt.\end{aligned} \tag{1.7}$$

Set $F(t) = \int |\nabla P_t f|^2 dm$, $t \geq 0$. By (1.6) again,

$$F'(t) = 2\int \langle \nabla P_t f, \nabla \mathrm{L}P_t f\rangle dm = -2\int (\mathrm{L}P_t f)^2 dm.$$

Assume now that for some $\kappa > 0$ and every f,

$$\kappa \int |\nabla f|^2 dm \leq \int (\mathrm{L}f)^2 dm.$$

Then $-F'(t) \geq 2\kappa F(t)$ for every $t \geq 0$ so that $F(t) \leq \mathrm{e}^{-2\kappa t} F(0)$ and

$$\mathrm{Var}_m(f) \leq \int_0^\infty \mathrm{e}^{-\kappa t} F(0) dt = \frac{1}{\kappa}\int |\nabla f|^2 dm.$$

Hence, $\mathrm{SG}(m) \geq \kappa$. On the other hand, by invariance and the Cauchy-Schwarz inequality,

$$\begin{aligned}\int |\nabla f|^2 dm &= \int f(-\mathrm{L}f)dm \\ &= \int \big(f - \textstyle\int f dm\big)(-\mathrm{L}f)dm \leq \mathrm{Var}_m(f)^{1/2}\Big(\int (\mathrm{L}f)^2 dm\Big)^{1/2}\end{aligned}$$

so that $\mathrm{SG}(m) \leq \kappa$. Therefore, the largest $\kappa > 0$ is exactly $\mathrm{SG}(m)$. This is one simple instance of the Witten Laplacian approach of J. Sjöstrand and B. Helffer [He-S], [He1] summarized in the next statement.

Proposition 1.3. *The spectral gap* $\mathrm{SG}(m)$ *of* m *is equal to the largest* $\kappa > 0$ *such that*

$$\kappa \int |\nabla f|^2 dm \leq \int (\mathrm{L}f)^2 dm$$

for every smooth function f *on* $\mathbb{R}^n$.

In order to produce spectral gap inequalities, it is thus of interest to study lower bounds on κ. To this task, note that by simple calculus (using invariance of L in the form $\int \mathrm{L}\varphi dm = 0$),

$$\begin{aligned}\int (\mathrm{L}f)^2 dm &= -\int \langle \nabla \mathrm{L}f, \nabla f\rangle dm \\ &= \int \Big(\sum_{i,j=1}^n (\partial_{ij} f)^2 + \langle \mathrm{Hess}\,(U)\nabla f, \nabla f\rangle \Big) dm.\end{aligned} \tag{1.8}$$

The characterization of Proposition 1.3 thus reads

$$\kappa \int |\nabla f|^2 dm \leq \int \Big(\sum_{i,j=1}^n (\partial_{ij} f)^2 + \langle \mathrm{Hess}\,(U)\nabla f, \nabla f\rangle \Big) dm \tag{1.9}$$

for every smooth f.

Convexity conditions on U, extending the Gaussian example, lead then to simple criteria ensuring the validity of (1.9).

Corollary 1.4. *Let $dm = \frac{1}{Z}\,\mathrm{e}^{-U} dx$ where, as symmetric matrices,* $\mathrm{Hess}\,(U)(x) \geq c\,\mathrm{Id}$ *for some $c > 0$ uniform in $x \in \mathbb{R}^n$. By (1.9), $\kappa \geq c$ so that*

$$\mathrm{SG}(m) \geq c.$$

This convexity result goes back to A. Lichnerowicz in Riemannian geometry (cf. [G-H-L]), and also follows from the deeper Brascamp-Lieb inequality [B-L].

Proposition 1.3 has been developed similarly for logarithmic Sobolev inequalities by D. Bakry and M. Emery [Ba-E] in terms of the so-called Γ_2 operator. Let, for a smooth function f on $\mathbb{R}^n$,

$$\Gamma_2(f) = \frac{1}{2}\,\mathrm{L}\big(\langle \nabla f, \nabla f\rangle\big) - \langle \nabla f, \nabla \mathrm{L}f\rangle = \sum_{i,j=1}^n (\partial_{ij} f)^2 + \langle \mathrm{Hess}\,(U)\nabla f, \nabla f\rangle.$$

Note that $\int \Gamma_2(f) dm = \int (\mathrm{L}f)^2 dm$. Arguing almost as for the variance, for a smooth positive function f on $\mathbb{R}^n$,

$$\mathrm{Ent}_m(f) = \int_0^\infty \frac{d}{dt}\Big(\int P_t f \log P_t f dm \Big) dt = \int_0^\infty \Big(\int \frac{1}{P_t f}\, |\nabla P_t f|^2 dm \Big) dt.$$

Set now $F(t) = \int \frac{1}{P_t f}\, |\nabla P_t f|^2 dm$, $t \geq 0$. After several use of the integration by parts formula (1.6), and by definition of Γ_2, it may be shown that

$$F'(t) = -2 \int P_t f \Gamma_2(\log P_t f) dm.$$

Assume now that for some $\kappa > 0$ and every f,

$$\kappa \int f |\nabla \log f|^2 dm \leq \int f \Gamma_2(\log f) dm.$$

Since $F(t) = \int \frac{1}{P_t f} |\nabla P_t f|^2 dm = \int P_t f |\nabla \log P_t f|^2 dm$, it then follows that $-F'(t) \geq 2\kappa F(t)$ for every $t \geq 0$ so that $F(t) \leq \mathrm{e}^{-2\kappa t} F(0)$. Therefore,

$$\mathrm{Ent}_m(f) \leq \int_0^\infty \mathrm{e}^{-2\kappa t} F(0) dt = \frac{1}{2\kappa} \int \frac{1}{f} |\nabla f|^2 \, dm.$$

Hence, changing f into f^2, $\mathrm{LS}(m) \geq \kappa$. We may thus state

Proposition 1.5. *If for some $\kappa > 0$ and every f,*

$$\kappa \int f |\nabla \log f|^2 dm \leq \int f \Gamma_2(\log f) dm, \tag{1.10}$$

then $\mathrm{LS}(m) \geq \kappa$.

The only, however main, difference with spectral gap is that here $\mathrm{LS}(m)$ is not characterized in general by κ of (1.10) as shown by the following example communicated to us by B. Helffer. Let $dm = \frac{1}{z} \mathrm{e}^{-u} dx$ be the probability measure on $\mathbb{R}$ with

$$u(x) = x^4 - \beta x^2, \quad x \in \mathbb{R}, \tag{1.11}$$

where $\beta > 0$. Although u is not uniformly strictly convex, it is clearly convex at infinity so that, by Corollary 1.7 below, $\mathrm{LS}(m) > 0$. However, if we let $f(x) = \mathrm{e}^{-\beta x^2}$, $x \in \mathbb{R}$, it is easily seen that

$$\int f \Gamma_2(\log f) dm = 4\beta^2 \int \big[1 + (12x^2 - 2\beta)x^2\big] \mathrm{e}^{-x^4} \frac{dx}{z} < 0$$

for β large enough so that (1.10) certainly fails (more generally, see [B-H2]).

The same convexity condition as in Corollary 1.4 however leads to the logarithmic Sobolev inequality.

Corollary 1.6. *Let $dm = \frac{1}{Z} \mathrm{e}^{-U} dx$ where, as symmetric matrices,* $\mathrm{Hess}\,(U)(x) \geq c\,\mathrm{Id}$ *for some $c > 0$ uniform in $x \in \mathbb{R}^n$. By the definition of Γ_2 applied to* $\log f$, *(1.10) holds with $\kappa = c$ so that*

$$\mathrm{LS}(m) \geq c.$$

It might be important to recall at this stage that the condition $\mathrm{Hess}\,(U) \geq c\,\mathrm{Id}$ for some $c > 0$ may be used in a slightly different way in proofs of spectral gap and logarithmic Sobolev inequalities. Inspired by results in Riemannian geometry and the stochastic calculus of variation, it may be shown indeed (cf. [Ba1], [Ba2]) under the condition $\mathrm{Hess}\,(U) \geq c\,\mathrm{Id}$ that, for every smooth function f and every $t \geq 0$,

$$|\nabla P_t f|^2 \leq \mathrm{e}^{-2ct} P_t\big(|\nabla f|^2\big) \tag{1.12}$$

(at each point). Under this condition, by invariance,

$$\int |\nabla P_t f|^2 dm \leq \mathrm{e}^{-2ct} \int P_t\big(|\nabla f|^2\big) dm = \mathrm{e}^{-2ct} \int |\nabla f|^2 dm$$

so that, by (1.7), $\mathrm{Var}_m(f) \leq \frac{1}{c}\int |\nabla f|^2 dm$ whenever $c > 0$. The proof of (1.12) is a variation on the principle leading to Propositions 1.3 and 1.5. Indeed, fix $t > 0$ and define, for every $s \leq t$, $G(s) = \mathrm{e}^{-2cs} P_s(|\nabla P_{t-s} f|^2)$. Then, by the definition of Γ_2,

$$G'(s) = 2\,\mathrm{e}^{-2cs} P_s\Big(\Gamma_2(P_{t-s}f) - c\,|\nabla P_{t-s} f|^2\Big) \geq 0$$

from which the result follows. This argument may be used similarly for logarithmic Sobolev inequalities but requires the strengthening of (1.12) into $|\nabla P_t f| \leq \mathrm{e}^{-ct} P_t(|\nabla f|)$. We refer to [Ba2], [Le2] for details.

It follows from the perturbation result of Lemma 1.2 together with Corollaries 1.4 and 1.6 that whenever $dm = \frac{1}{Z}\,\mathrm{e}^{-U} dx$ is such that $U = V + W$ with $\mathrm{Hess}\,(V)(x) \geq c\,\mathrm{Id}$ for some $c > 0$ uniformly in $x \in \mathbb{R}^n$ and W is bounded (such a potential will be called below strictly convex at infinity), then the probability measure m satisfies both a spectral gap and a logarithmic Sobolev inequality. Note however that by example (1.11), strict convexity at infinity may fail criterion (1.10) of Proposition 1.5.

Corollary 1.7. *Let $dm = \frac{1}{Z}\,\mathrm{e}^{-U} dx$ where $U = V + W$ with $\mathrm{Hess}\,(V) \geq c\,\mathrm{Id}$ for some $c > 0$ and $\|W\|_\infty < \infty$. Then*

$$\mathrm{SG}(m) \geq \mathrm{LS}(m) \geq c\,\mathrm{e}^{-4\|W\|_\infty} > 0. \tag{1.13}$$

One odd feature of this perturbation argument is that it yields rather poor constants as functions of the dimension. Typically in $\mathbb{R}^n$, the cost would be exponential in n.

In other directions, it was shown recently by S. Bobkov [Bo2] that whenever $\mathrm{Hess}\,(U) \geq 0$, $\mathrm{SG}(m) > 0$, but again dependence in the dimension is poor. Furthermore, if $\mathrm{Hess}\,(U) \geq c\,\mathrm{Id}$ for some $c \in \mathbb{R}$, F.-Y. Wang [Wa] and S. Aida [Ai] (see also [Le1]) showed that whenever m is integrable enough in the sense that

$$\int \mathrm{e}^{\alpha|x|^2} dm(x) < \infty$$

for some $\alpha > 2\max(0, -c)$, then $\mathrm{LS}(m) > 0$ depending on the value of the preceding integral. Thus again, this result is rather useless for dimension free estimates.

As is pointed out in [Ro], the class of potentials strictly convex at infinity contains the class of potentials $U = V + W$, $\mathrm{Hess}\,(V) \geq c\,\mathrm{Id}$ for some $c > 0$ and W Lipschitz. To check it, let γ_σ be the Gaussian density $(2\pi\sigma^2)^{-n/2}\,\mathrm{e}^{-|x|^2/2\sigma^2}$, $\sigma > 0$, on $\mathbb{R}^n$ and write

$$U = (V + W * \gamma_\sigma) + (W - W * \gamma_\sigma).$$

It is easily seen that for every $\alpha \in \mathbb{R}^n$,

$$\big|\langle \mathrm{Hess}\,(W * \gamma_\sigma)\alpha, \alpha\rangle\big| \leq K\sigma^{-2}|\alpha|^2$$

where K is the Lipschitz constant of W whereas

$$\|W * \gamma_\sigma - W\|_\infty \leq K\sqrt{n}\,\sigma.$$

Provided σ is large enough so that $K < c\sigma^2$, the claim follows. (The preceding argument was kindly communicated to us by L. Miclo.)

To conclude this recall section, and in order to motivate our investigation, let us consider the following simple example that concentrates most of the questions we will deal with next. Let, on the real line $\mathbb{R}$, $d\mu = \frac{1}{z}\,\mathrm{e}^{-u}dx$ where u is strictly convex at infinity, that is $u = v + w$ with $v'' \geq c > 0$, and w bounded. A typical such example is $u(x) = x^4 - \beta x^2$, $\beta > 0$. As we have seen in Corollary 1.7, μ satisfies both a spectral gap and a logarithmic Sobolev inequality. On $\mathbb{R}^n$, consider then the probability measure

$$dQ(x) = \frac{1}{Z}\,\mathrm{e}^{-U(x)}dx$$

with

$$U(x) = \sum_{i=1}^{n} u(x_i) + J\sum_{i=1}^{n} x_i x_{i+1}, \quad x \in \mathbb{R}^n,$$

where $J \in \mathbb{R}$ and $x_{n+1} = x_1$. We would like to known whether Q satisfies a Poincaré or a logarithmic Sobolev inequality with constants independent of n, at least if J is small enough for example. The preceding general results allow us to conclude in two cases. If $J = 0$, Q is the n-fold product measure μ^n of μ for which, by Lemma 1.1, both Poincaré and logarithmic Sobolev inequalities hold with constants independent of n. If $w = 0$, then it is not difficult to see that, at every $x \in \mathbb{R}^n$, and for every $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$,

$$\langle \mathrm{Hess}\,(U)(x)\alpha, \alpha\rangle = \sum_{i=1}^{n} \alpha_i^2 u''(x_i) + 2J\sum_{i=1}^{n} \alpha_i\alpha_{i+1} \geq \big(c - 2|J|\big)|\alpha|^2$$

that is thus strictly positive as soon as J is small enough (with respect to $c > 0$). Therefore, $\mathrm{Hess}\,(U) \geq c'\,\mathrm{Id}$ for some $c' > 0$ so that, by Corollaries 1.4 and 1.6, Q satisfies a Poincaré and a logarithmic Sobolev inequality independently of the dimension. The main trouble now comes from the fact that if the two situations are mixed, none of the preceding general arguments may be used to conclude, and a rather delicate analysis is needed to take into account the perturbation in the product. This is the problem we investigate below.

2. General correlation inequalities

In this section, we make use of the preceding semigroup tools to describe some correlation inequalities that will be crucial in the analysis of logarithmic Sobolev inequalities for spin systems. We start with general L^2 correlation inequalities drawn for the paper [He2]. We take again the notation of the preceding section.

Proposition 1.3 may be adapted to estimates on correlations by a simple change of metric. If m is a measure on $(E, \mathcal{E})$, denote by $\mathrm{Cor}_m(f, g)$ the correlation (or covariance)

$$\mathrm{Cor}_m(f, g) = \int fg\,dm - \int f\,dm \int g\,dm$$

of the square integrable functions f and g. The correlation may also be written by duplication

$$\mathrm{Cor}_m(f,g) = \frac{1}{2}\iint \big[f(x)-f(y)\big]\big[g(x)-g(y)\big]dm(x)dm(y).$$

Let $dm = \frac{1}{Z}\,\mathrm{e}^{-U}dx$ be as in Section 1 and denote by $(P_t)_{t\geq 0}$ the semigroup with generator $\mathrm{L} = \Delta - \langle \nabla U, \nabla\rangle$. As for (1.7), for smooth functions f, g on $\mathbb{R}^n$,

$$\begin{aligned}\mathrm{Cor}_m(f,g) &= \int \big(f - \textstyle\int f dm\big) g dm \\ &= -\int_0^\infty \Big(\int g \mathrm{L} P_t f dm\Big) dt \\ &= \int_0^\infty \Big(\int \langle \nabla P_t f, \nabla g\rangle dm\Big) dt\end{aligned}$$

where we used integration by parts (1.6) in the last step. This formula is the semigroup version of the correlation representation put forward in [H-S], [He1] via the Witten Laplacian on forms $\mathrm{L}^{\otimes n} + \mathrm{Hess}\,(U)$. Now, let D be an invertible $n\times n$ diagonal matrix with diagonal $(d_i)_{1\leq i\leq n}$. We may clearly write

$$\begin{aligned}\mathrm{Cor}_m(f,g) &= \int_0^\infty \Big(\int \langle D\nabla P_t f, D^{-1}\nabla g\rangle dm\Big) dt \\ &\leq \int_0^\infty \Big(\int |D\nabla P_t f|^2 dm\Big)^{1/2} \Big(\int |D^{-1}\nabla g|^2 dm\Big)^{1/2} dt.\end{aligned}$$

We analyze $F(t) = \int |D\nabla P_t f|^2 dm$, $t\geq 0$, as for the spectral gap in Section 1. We have

$$F'(t) = -2\int \mathrm{L} P_t f\, \mathrm{L}^D P_t f dm$$

where

$$L^D f = \sum_{i=1}^n d_i^2 \partial_{ii} f - \sum_{i=1}^n d_i^2 \partial_i U \partial_i f.$$

If for some $\kappa > 0$ and every f,

$$\kappa \int |D\nabla f|^2 dm \leq \int \mathrm{L} f\, \mathrm{L}^D f dm, \tag{2.1}$$

then $-F'(t) \leq 2\kappa F(t)$ for every $t\geq 0$ so that $F(t) \leq \mathrm{e}^{-2\kappa t} F(0)$. Hence we conclude to the following result.

Proposition 2.1. *If (2.1) holds for some diagonal matrix D and some $\kappa > 0$, for every smooth functions f and g on $\mathbb{R}^n$,*

$$\kappa\, \mathrm{Cor}_m(f,g) \leq \Big(\int |D\nabla f|^2 dm\Big)^{1/2} \Big(\int |D^{-1}\nabla g|^2 dm\Big)^{1/2}.$$

As in (1.8), it is useful to interpret (2.1) with the help of the Hessian of U as

$$\int \mathrm{L}f\,\mathrm{L}^D f dm = \int \Big(\sum_{i,j=1}^n d_i^2 (\partial_{ij} f)^2 + \sum_{i,j=1}^n d_i^2 \partial_{ij} U \partial_i f \partial_i f \Big) dm. \tag{2.2}$$

In particular, $\kappa \geq c^D$ whenever

$$D\,\mathrm{Hess}\,(U) D^{-1} \geq c^D\,\mathrm{Id}. \tag{2.3}$$

We turn to our second correlation inequality, put forward in [S-Z1], [S-Z2] and adapted to the unbounded case in [B-H1] (see also [Yo3]). It will prove useful in the inductive proof of logarithmic Sobolev inequalities. Although we will only use this result in dimension one below, we state it in $\mathbb{R}^n$ for possible independent interest.

Proposition 2.2. *Assume $dm = \frac{1}{Z}\,\mathrm{e}^{-U} dx$ satisfies the logarithmic Sobolev inequality with constant $\rho > 0$. Then, there is a constant $C > 0$ only depending on $\rho > 0$ such that such that for all smooth functions f, g on $\mathbb{R}^n$,*

$$\mathrm{Cor}_m(f^2, g) \leq 2C\,\|\nabla g\|_\infty \Big(\int f^2 dm \Big)^{1/2} \Big(\int |\nabla f|^2 dm \Big)^{1/2}.$$

The proposition of course applies when $U = V + W$, $\mathrm{Hess}\,(V) \geq c\,\mathrm{Id}$ for some $c > 0$ and $\|W\|_\infty < \infty$ since then, by (1.13), $\mathrm{LS}(m) \geq c\,\mathrm{e}^{-4\|W\|_\infty} > 0$.

Proof. We may assume by homogeneity that $\|\nabla g\|_\infty \leq 1$. By duplication and the Cauchy-Schwarz inequality,

$$\begin{aligned}
\mathrm{Cor}_m(f^2, g) &= \frac{1}{2} \iint \big[f(x) - f(y)\big]\big[f(x) + f(y)\big]\big[g(x) - g(y)\big] dm(x) dm(y) \\
&\leq \Big(\frac{1}{2} \iint \big|f(x) - f(y)\big|^2 dm(x) dm(y) \Big)^{1/2} \\
&\quad \times \Big(\frac{1}{2} \iint \big|f(x) + f(y)\big|^2 \big|g(x) - g(y)\big|^2 dm(x) dm(y) \Big)^{1/2} \\
&\leq \mathrm{Var}_m(f)^{1/2} \Big(2 \iint f^2(x) \big|g(x) - g(y)\big|^2 dm(x) dm(y) \Big)^{1/2}.
\end{aligned}$$

Now, for $a, b \geq 0$, $ab \leq a \log a + \mathrm{e}^b$, so that, for every $\varepsilon > 0$,

$$\begin{aligned}
&\iint f^2(x) \big|g(x) - g(y)\big|^2 dm(x) dm(y) \\
&\qquad\qquad \leq \varepsilon\,\mathrm{Ent}_m(f^2) + \varepsilon \int f^2 dm \iint \mathrm{e}^{|g(x)-g(y)|^2/\varepsilon} dm(x) dm(y).
\end{aligned}$$

Since $\mathrm{LS}(m) \geq \rho$, by the Herbst inequality as in [A-M-S] (see [Le1], p. 151), whenever $\varepsilon\rho > 4$,

$$\iint \mathrm{e}^{|g(x)-g(y)|^2/\varepsilon} dm(x) dm(y) \leq \frac{1}{\sqrt{1 - 4/\varepsilon\rho}}\,.$$

Summarizing, for every $\varepsilon\rho > 4$ and some $C(\varepsilon) > 0$ only depending on ε and ρ,

$$\begin{aligned}\mathrm{Cor}_m(f^2, g) &\leq \mathrm{Var}_m(f)^{1/2}\Big(\varepsilon\,\mathrm{Ent}_m(f^2) + C(\varepsilon)\int f^2 dm\Big)^{1/2} \\ &\leq \Big(\int f^2 dm\Big)^{1/2}\Big(\varepsilon\,\mathrm{Ent}_m(f^2) + C(\varepsilon)\mathrm{Var}_m(f)\Big)^{1/2}.\end{aligned}$$

Since m satisfies $\mathrm{SG}(m) \geq \mathrm{LS}(m) \geq \rho > 0$, the conclusion follows. Proposition 2.2 is established. □

As is clear, the proof of Proposition 2.2 actually shows that for every $\varepsilon > 0$ such that $\varepsilon\rho > 4$, there exists $C(\varepsilon) > 0$ only depending on ε and ρ such that for all smooth functions f, g on $\mathbb{R}^n$,

$$\mathrm{Cor}_m(f^2, g) \leq \|\nabla g\|_\infty \Big(\int f^2 dm\Big)^{1/2}\Big(\varepsilon\,\mathrm{Ent}_m(f^2) + C(\varepsilon)\mathrm{Var}(f)\Big)^{1/2}, \tag{2.4}$$

an inequality of independent interest in the perturbative regime [Yo3], [B-H2].

In the spirit of Proposition 2.2, one may establish by related tools stronger L^1 correlation bounds. More precisely, one can show, mostly on the basis of the material developed in Section 1, that if $dm = \frac{1}{Z}\,\mathrm{e}^{-U}dx$ where $U = V + W$, $\mathrm{Hess}\,(V) \geq c\,\mathrm{Id}$, $c > 0$, $\|W\|_\infty < \infty$ and $\|\nabla W\|_\infty < \infty$, then, for some constant $C > 0$ only depending on c, $\|W\|_\infty$ and $\|\nabla W\|_\infty$, and for all smooth functions f, g on $\mathbb{R}^n$,

$$\mathrm{Cor}_m(f, g) \leq C\,\|\nabla g\|_\infty \int |\nabla f| dm. \tag{2.5}$$

Applied to f^2 instead of f, it yields a stronger conclusion than Proposition 2.2 of possible independent applications (see the final comments after Theorem 6.3).

3. Spectral gaps for some families of potentials

Let u be a smooth function on $\mathbb{R}$ such that $z = \int \mathrm{e}^{-u}dx < \infty$ and denote by μ the probability measure on the Borel sets of $\mathbb{R}$ defined by

$$d\mu(x) = \frac{1}{z}\,\mathrm{e}^{-u(x)}dx.$$

Let now H be a smooth potential on $\mathbb{R}^n$ such that $Z = \int \mathrm{e}^{-H}d\mu^n < \infty$ and consider the probability measure

$$dQ = \frac{1}{Z}\,\mathrm{e}^{-H}d\mu^n. \tag{3.1}$$

In the notation of Section 1,

$$dQ = \frac{1}{Z'}\,\mathrm{e}^{-U}dx$$

with

$$U(x) = \sum_{i=1}^n u(x_i) + H(x), \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n,$$

(and $Z' = z^n Z$).

Whenever $i_1, \dots, i_k$ are distinct in $\{1, \dots, n\}$, we denote below by $Q^{x_{i_1}, \dots, x_{i_k}}$ the conditional measure on $\mathbb{R}^{n-k}$ given $x_{i_1}, \dots, x_{i_k}$ defined by

$$dQ^{x_{i_1}, \dots, x_{i_k}}(x_j : j \neq i_1, \dots, i_k) = \frac{1}{Z^{x_{i_1}, \dots, x_{i_k}}} \, \mathrm{e}^{-H(x)} \prod_{j \neq i_1, \dots, i_k} d\mu(x_j)$$

where

$$Z^{x_{i_1}, \dots, x_{i_k}} = \int \mathrm{e}^{-H(x)} \prod_{j \neq i_1, \dots, i_k} d\mu(x_j).$$

These should actually only be considered for almost every $(x_{i_1}, \dots, x_{i_k}) \in \mathbb{R}^k$. We will ignore below the negligeable sets involved in this definition.

In this section, we describe, following [He2], conditions on H in order that Q satisfies a Poincaré inequality. The following proposition has been observed by B. Helffer [He2] by means of his Witten Laplacian approach. The proof is elementary.

Proposition 3.1. *Assume that for some $h = h_Q$ and $\bar{h} = \bar{h}_Q$ in $\mathbb{R}$, $\mathrm{Hess}\,(H)(x) \geq h\,\mathrm{Id}$ and $\max_{1 \leq i \leq n} \partial_{ii} H(x) \leq \bar{h}$ uniformly in $x \in \mathbb{R}^n$. Let*

$$s = s_Q = \inf \mathrm{SG}\big(Q^{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n}\big)$$

where the infimum is running over all $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ in $\mathbb{R}$ and $1 \leq i \leq n$. Then

$$\mathrm{SG}(Q) \geq s + h - \bar{h}.$$

Proof. By Proposition 1.3 and (1.9) of Section 1, it is enough to show that

$$\int (\mathrm{L}f)^2 dQ = \int \Big(\sum_{i,j=1}^n (\partial_{ij} f)^2 + \langle \mathrm{Hess}\,(U) \nabla f, \nabla f \rangle \Big) dQ \geq \big(s + h - \bar{h}\big) \int |\nabla f|^2 dQ$$

where we recall that $U(x) = \sum_{i=1}^n u(x_i) + H(x)$. For a smooth function f on $\mathbb{R}^n$,

$$\begin{aligned}
\sum_{i,j=1}^n (\partial_{ij} f)^2 &+ \langle \mathrm{Hess}\,(U) \nabla f, \nabla f \rangle \\
&= \sum_{i,j=1}^n (\partial_{ij} f)^2 + \sum_{i=1}^n u''(x_i) (\partial_i f)^2 + \langle \mathrm{Hess}\,(H) \nabla f, \nabla f \rangle \\
&\geq \sum_{i=1}^n (\partial_{ii} f)^2 + \sum_{i=1}^n u''(x_i) (\partial_i f)^2 + h |\nabla f|^2.
\end{aligned}$$

Now, for every $i = 1, \ldots, n$,

$$\begin{aligned}
&\int \big[(\partial_{ii} f)^2 + u''(x_i)(\partial_i f)^2\big] dQ \\
&\quad \geq \int \big[(\partial_{ii} f)^2 + \big(u''(x_i) + \partial_{ii} H\big)(\partial_i f)^2\big] dQ - \bar{h} \int (\partial_i f)^2 dQ \\
&\quad = \int \bigg(\int \big[(\partial_{ii} f)^2 + \big(u''(x_i) + \partial_{ii} H\big)(\partial_i f)^2\big] dQ^{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n} \bigg) dQ \\
&\qquad\qquad - \bar{h} \int (\partial_i f)^2 dQ.
\end{aligned}$$

The one-dimensional measure $Q^{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n}$ has a spectral gap bounded below by s. By Proposition 1.3, it thus also satisfy the corresponding integral criterion (1.9) with $\kappa = s$. Now, the definition of $Q^{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n}$ shows that

$$dQ^{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n}(x_i) = \frac{1}{z Z^{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n}} \, \mathrm{e}^{-u(x_i) - H(x)} dx_i \tag{3.2}$$

so that (1.9) applied to $Q^{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n}$ yields that

$$\begin{aligned}
&\int \big[(\partial_{ii} f)^2 + \big(u''(x_i) + \partial_{ii} H\big)(\partial_i f)^2\big] dQ^{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n} \\
&\qquad\qquad \geq s \int (\partial_i f)^2 dQ^{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n}.
\end{aligned}$$

Proposition 3.1 is established. □

Proposition 2.1 may be used in the same way to produce correlation bounds. If D is an invertible $n \times n$ diagonal matrix, let $h^D \in \mathbb{R}$ be such that

$$D \operatorname{Hess}(H) D^{-1} \geq h^D \operatorname{Id}.$$

Together with Proposition 2.1 and (2.2), note that

$$\begin{aligned}
&\sum_{i,j=1}^n d_i^2 (\partial_{ij} f)^2 + \sum_{i,j=1}^n d_i^2 \partial_{ij} U \partial_i f \partial_i f \\
&\qquad = \sum_{i,j=1}^n d_i^2 (\partial_{ij} f)^2 + \sum_{i=1}^n d_i^2 u''(x_i)(\partial_i f)^2 + \sum_{i,j=1}^n d_i^2 \partial_{ij} H \partial_i f \partial_i f \\
&\qquad \geq \sum_{i=1}^n d_i^2 \big[(\partial_{ii} f)^2 + u''(x_i)(\partial_i f)^2\big] + h^D \sum_{i=1}^n d_i^2 (\partial_i f)^2.
\end{aligned}$$

One then argue as in the proof of Proposition 3.1 to conclude to the following result of B. Helffer [He2].

Proposition 3.2. *In the notation of Proposition 3.1, for every smooth functions f, g on $\mathbb{R}^n$,*

$$(s + h^D - \bar{h}) \operatorname{Cor}_Q(f, g) \leq \bigg(\int |D^{-1} \nabla f|^2 dQ \bigg)^{1/2} \bigg(\int |D \nabla g|^2 dQ \bigg)^{1/2}.$$

Note that Proposition 3.2 includes the case $f = g$ of Proposition 3.1 with optimal constant. A similar result holds with $h^{D^{-1}}$.

Typical applications of Propositions 3.1 and 3.2 are the following. Assume for example that

$$H(x) = \langle Ax, x\rangle + \langle B, x\rangle$$

where A is an $n \times n$ matrix with zero diagonal and $B \in \mathbb{R}^n$. Then $\mathrm{Hess}(H) = A + {}^tA$ so that h is the infimum of the eigenvalues of the symmetric matrix $A + {}^tA$ while $\partial_{ii}H = 0$ for every i. Furthermore $s = \inf_{\theta\in\mathbb{R}} \mathrm{SG}(\mu_\theta)$ where, for every $\theta \in \mathbb{R}$,

$$d\mu_\theta(x) = \frac{1}{z_\theta}\, \mathrm{e}^{\theta x} d\mu(x).$$

In another direction, assume that u is strictly convex at infinity, that is $u = v+w$, for some $c > 0$, $u''(x) \geq c$ uniformly in x and $\|w\|_\infty < \infty$. Provided that for some $c'' < c$,

$$\partial_{ii}H(x) \geq -c'' \tag{3.3}$$

for every $x \in \mathbb{R}^n$ and $i = 1, \ldots, n$, then

$$s \geq (c - c'')\, \mathrm{e}^{-4\|w\|_\infty}. \tag{3.4}$$

Indeed, by (3.2), along the i-th coordinate,

$$u(x_i) + H(x) = v(x_i) + H(x) + w(x_i)$$

with $v''(x_i) + \partial_{ii}H(x) \geq c - c'' > 0$ and $\|w\|_\infty < \infty$. The claim thus follows from (1.13). In particular, if $\max_{1\leq i\leq n} \|\partial_{ii}H\|_\infty \leq c'' < c$,

$$\mathrm{SG}(Q) \geq s + h - \bar{h} \geq (c - c'')\, \mathrm{e}^{-4\|w\|_\infty} + h - c''.$$

Some examples with non-convex phase have been constructed recently by I. Gentil and C. Roberto [G-R] using perturbations via Hardy inequalities.

4. Marginal distributions

Due to example (1.11), we cannot hope for Proposition 3.1 to hold similarly for logarithmic Sobolev inequalities. We thus have to turn back to the induction method for product measures. In particular, we need to apply a logarithmic Sobolev inequality at each step. To this task, we describe following [B-H1] the marginals of our probability measure Q. It will be enough to consider one-dimensional marginals.

Let Q be as defined by (3.1). Denote by Q_i its marginals on the i-th coordinate, $i = 1, \ldots, n$. Q_i is a probability measure on $\mathbb{R}$ with density e^{-H_i} with respect to Lebesgue measure given by $H_i(x_i) = u(x_i) - K_i(x_i)$ where

$$K_i(x_i) = \log\left(\frac{1}{zZ}\int \mathrm{e}^{-H(x)} \prod_{j\neq i} d\mu(x_j)\right), \quad x_i \in \mathbb{R}.$$

In order to show that Q_i satisfies a Poincaré or logarithmic Sobolev inequality, we will use the convexity criteria on H_i developed in Section 1. To this task, let us describe the second derivative of H_i. Denote by Q^{x_i} the probability Q conditionally on x_i. It is easy to check that

$$K_i''(x_i) = \mathrm{Var}_{Q^{x_i}}(\partial_i H) - \int \partial_{ii} H dQ^{x_i}. \tag{4.1}$$

By the definition of the spectral gap,

$$\mathrm{SG}(Q^{x_i})\,\mathrm{Var}_{Q^{x_i}}(\partial_i H) \leq \int |\nabla \partial_i H|^2 dQ^{x_i} \tag{4.2}$$

where the gradient ∇ is acting on the coordinates $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$.

In order to make use of (4.2), we follow the observation of [B-H1] by imposing convexity condition on the one-dimensional phase measure μ. Assume namely that u is strictly convex at infinity, that is, for some $c > 0$, $u = v + w$ with $v'' \geq c > 0$, $\|w\|_\infty < \infty$. Assume furthermore that for some c', c'', $c' + c'' < c$,

$$\int |\nabla \partial_i H|^2 dQ^{x_i} \leq c'\,\mathrm{SG}(Q^{x_i}) \quad \text{and} \quad \int \partial_{ii} H dQ^{x_i} \geq -c''$$

uniformly in $x_i \in \mathbb{R}$. Then, provided that $\mathrm{SG}(Q^{x_i}) > 0$, $H_i = (v - K_i) + w$ where, by (4.1) and (4.2),

$$(v - K_i)''(x_i) \geq c - c' - c''.$$

Hence, by (1.13), $\mathrm{LS}(Q_i) \geq (c - c' - c'')\,\mathrm{e}^{-4\|w\|_\infty}$.

We may summarize these conclusions in the following statement.

Proposition 4.1. *Assume that u is convex at infinity, that is $u = v + w$, $v'' \geq c > 0$, $\|w\|_\infty < \infty$. If for some c', c'', $c' + c'' < c$,*

$$\int |\nabla \partial_i H|^2 dQ^{x_i} \leq c'\,\mathrm{SG}(Q^{x_i}) \quad \text{and} \quad \int \partial_{ii} H dQ^{x_i} \geq -c'' \tag{4.3}$$

uniformly in $x_i \in \mathbb{R}$, $i = 1, \ldots, n$, the one-dimensional marginal Q_i of Q has density e^{-H_i} with respect to Lebesgue measure on $\mathbb{R}$ where $H_i = v_i + w$, $v_i'' \geq c - c' - c'' > 0$, $\|w\|_\infty < \infty$. In particular, $\mathrm{SG}(Q_i) \geq \mathrm{LS}(Q_i) \geq (c - c' - c'')\,\mathrm{e}^{-4\|w\|_\infty}$.

In the setting of Proposition 4.1, we may apply furthermore Proposition 2.2 to the marginals Q_i. Assume thus that (4.3) of Proposition 4.1 holds. We write below $f = f(x_i)$ to indicate that a smooth function f is actually a one-variable function only depending on the i-th coordinate, $i = 1, \ldots, n$. Let $f = f(x_i)$ and g be smooth functions on $\mathbb{R}^n$. To apply Proposition 2.2, observe first that

$$\mathrm{Cor}_Q(f, g) = \mathrm{Cor}_{Q_i}(f, G)$$

where $G(x_i) = \int g dQ^{x_i}$ (conditional expectation under Q of g given x_i). We thus deduce from Proposition 2.2 that for some constant $C > 0$ only depending on $c - c' - c'' > 0$ and $\|w\|_\infty < \infty$,

$$\mathrm{Cor}_Q(f^2, g) \leq 2C\,\|G'\|_\infty \Big(\int f^2 dQ\Big)^{1/2} \Big(\int f'^2 dQ\Big)^{1/2}. \tag{4.4}$$

In the following, we will show that $\|G'\|_\infty$ can be made small in several instances. To that purpose, note that

$$G'(x_i) = \int \partial_i g dQ^{x_i} - \mathrm{Cor}_{Q^{x_i}}(g, \partial_i H) \tag{4.5}$$

and $G(x_i) = g(x_i)$ if $g = g(x_i)$. If g does not depend on x_i, we will see below how the L^2 bounds of Proposition 3.2 on the correlations $\mathrm{Cor}_{Q^{x_i}}(g, \partial_i H)$ will ensure that $\|G'\|_\infty$ is small as a function of the distance between the supports of f and g.

5. Logarithmic Sobolev inequalities

In this section, we investigate the logarithmic Sobolev inequality with the preceding tools. Consider the probability measure Q of (3.1) defined by $dQ = \frac{1}{Z}\,\mathrm{e}^{-H} d\mu^n$. We follow in a natural way the proof of Lemma 1.1 and perform a Markov tensorization (the so-called martingale method [L-Y]).

Given a smooth function f on $\mathbb{R}^n$, define, for $k = 1, \ldots, n$, f_k on $\mathbb{R}^k$ as the square root of the conditional expectation of f^2 given $x_1, \ldots, x_k$ under the law Q. Since $f_n^2 = f^2$ and $f_0^2 = \int f^2 dQ$,

$$\mathrm{Ent}_Q(f^2) = \sum_{k=1}^n \Big[\int f_k^2 \log f_k^2 dQ - \int f_{k-1}^2 \log f_{k-1}^2 dQ \Big].$$

Now, f_{k-1}^2 is also the conditional expectation of f_k^2 given $x_1, \ldots, x_{k-1}$ so that it may be represented as

$$f_{k-1}^2(x_1, \ldots, x_{k-1}) = \int f_k^2(x_1, \ldots, x_k) dQ^{x_1 \ldots, x_{k-1}}(x_k, \ldots, x_n) \tag{5.1}$$

where we recall that $Q^{x_1, \ldots, x_{k-1}}$ is the conditional distribution given $x_1, \ldots, x_{k-1}$. Therefore,

$$\mathrm{Ent}_Q(f^2) = \sum_{k=1}^n \int \mathrm{Ent}_{Q^{x_1, \ldots, x_{k-1}}}(f_k^2) dQ.$$

Furthermore, since f_k^2 is a function of $x_1, \ldots, x_k$, and since $Q^{x_1, \ldots, x_{k-1}}$ is a measure of the variables $x_k, \ldots, x_n$,

$$\mathrm{Ent}_{Q^{x_1, \ldots, x_{k-1}}}(f_k^2) = \mathrm{Ent}_{Q_k^{x_1, \ldots, x_{k-1}}}(f_k^2)$$

where $Q_k^{x_1, \ldots, x_{k-1}}$ is the first marginal of $Q^{x_1, \ldots, x_{k-1}}$ (marginal in the x_k coordinate).

Let u on $\mathbb{R}$ be strictly convex at infinity, $u = v + w$, $v'' \geq c > 0$, $\|w\|_\infty < \infty$. Assume that each one-dimensional marginal $Q_k^{x_1, \ldots, x_{k-1}}$ satisfy a logarithmic Sobolev inequality with constant $\rho > 0$ uniform in $x_1, \ldots, x_{k-1}$ and $k = 1, \ldots, n$. By Proposition 4.1, this is ensured in particular if, for some c', c'', $c' + c'' < c$,

$$\int |\nabla \partial_k H|^2 dQ^{x_1, \ldots, x_k} \leq c'\, \mathrm{SG}(Q^{x_1, \ldots, x_k}) \quad \text{and} \quad \int \partial_{kk} H dQ^{x_1, \ldots, x_k} \geq -c'' \tag{5.2}$$

uniformly over $x_1, \ldots, x_{k-1}$ and $k = 1, \ldots, n$, with $\rho = (c - c' - c'')\,\mathrm{e}^{-4\|w\|_\infty}$. In (5.2), the gradient ∇ is a priori acting on the coordinates $x_{k+1}, \ldots, x_n$. In this case therefore,

$$\rho\,\mathrm{Ent}_Q(f^2) \leq 2\sum_{k=1}^n \int |\partial_k f_k|^2 dQ. \tag{5.3}$$

In the next step, we evaluate the partial derivatives $\partial_k f_k$. As a substitute to (1.5), we now have, for every $1 \leq k \leq \ell < n$,

$$2f_\ell \partial_k f_\ell = \partial_k f_\ell^2 = \int 2f_{\ell+1}\partial_k f_{\ell+1} dQ^{x_1,\ldots,x_\ell} - \mathrm{Cor}_{Q^{x_1,\ldots,x_\ell}}\big(f_{\ell+1}^2, \partial_k H\big). \tag{5.4}$$

This formula displays the importance of correlation bounds to investigate logarithmic Sobolev inequalities.

Next, we control the correlation terms in (5.4) together with (4.4) above. Under (5.2), we may apply (4.4) to each $Q^{x_1,\ldots,x_\ell}$ to see that, uniformly in $x_1, \ldots, x_\ell$,

$$\big|\mathrm{Cor}_{Q^{x_1,\ldots,x_\ell}}\big(f_{\ell+1}^2, \partial_k H\big)\big| \leq 2C \cdot C_{k,\ell+1} f_\ell \bigg(\int |\partial_{\ell+1} f_{\ell+1}|^2 dQ^{x_1,\ldots,x_\ell}\bigg)^{1/2} \tag{5.5}$$

(recall $f_\ell^2 = \int f_{\ell+1}^2 dQ^{x_1,\ldots,x_\ell}$) where, by (4.5), for $1 \leq k \leq \ell < n$,

$$C_{k,\ell+1} = \|\partial_{\ell+1,k} H\|_\infty + \big|\sup_{x_1,\ldots,x_{\ell+1}} \mathrm{Cor}_{Q^{x_1,\ldots,x_{\ell+1}}}(\partial_k H, \partial_{\ell+1} H)\big|. \tag{5.6}$$

Now, by (5.4) and (5.5),

$$|f_\ell \partial_k f_\ell| \leq \int |f_{\ell+1}\partial_k f_{\ell+1}| dQ^{x_1,\ldots,x_\ell} + C \cdot C_{k,\ell+1} f_\ell \bigg(\int |\partial_{\ell+1} f_{\ell+1}|^2 dQ^{x_1,\ldots,x_\ell}\bigg)^{1/2}.$$

Since again $f_\ell^2 = \int f_{\ell+1}^2 dQ^{x_1,\ldots,x_\ell}$, we get from the Cauchy-Schwarz inequality that

$$|\partial_k f_\ell| \leq \bigg(\int |\partial_k f_{\ell+1}|^2 dQ^{x_1,\ldots,x_\ell}\bigg)^{1/2} + C \cdot C_{k,\ell+1}\bigg(\int |\partial_{\ell+1} f_{\ell+1}|^2 dQ^{x_1,\ldots,x_\ell}\bigg)^{1/2}.$$

By the triangle inequality in L^2 and the composition of conditional expectations, it follows by iteration that, for every $1 \leq k < n$,

$$|\partial_k f_k| \leq \bigg(\int |\partial_k f|^2 dQ^{x_1,\ldots,x_k}\bigg)^{1/2} + C\sum_{\ell=k}^{n-1} C_{k,\ell+1}\bigg(\int |\partial_{\ell+1} f_{\ell+1}|^2 dQ^{x_1,\ldots,x_k}\bigg)^{1/2}.$$

Hence (since $(a+b)^2 \leq 2a^2 + 2b^2$ and $f_n^2 = f^2$),

$$\begin{aligned}
&\sum_{k=1}^n \int |\partial_k f_k|^2 dQ \\
&\leq 2\sum_{k=1}^n \int |\partial_k f|^2 dQ + 2C^2 \sum_{k=1}^n \int \bigg(\sum_{\ell=k}^{n-1} C_{k,\ell+1}\bigg(\int |\partial_{\ell+1} f_{\ell+1}|^2 dQ^{x_1,\ldots,x_k}\bigg)^{1/2}\bigg)^2 dQ \\
&\leq 2\sum_{k=1}^n \int |\partial_k f|^2 dQ + 2C^2 \sum_{k=1}^n \sum_{j=k}^{n-1} C_{k,j+1} \sum_{\ell=k}^{n-1} C_{k,\ell+1} \int |\partial_{\ell+1} f_{\ell+1}|^2 dQ
\end{aligned}$$

where we used the Cauchy-Schwarz inequality. Now,

$$\sum_{k=1}^{n}\sum_{j=k}^{n-1} C_{k,j+1}\sum_{\ell=k}^{n-1} C_{k,\ell+1}\int |\partial_{\ell+1} f_{\ell+1}|^2 dQ$$
$$= \sum_{\ell=1}^{n-1}\Big(\sum_{k=1}^{\ell}\sum_{j=k}^{n-1} C_{k,j+1}C_{k,\ell+1}\Big)\int |\partial_{\ell+1} f_{\ell+1}|^2 dQ.$$

Provided that

$$\max_{\ell}\sum_{k=1}^{\ell}\sum_{j=k}^{n-1} C_{k,j+1}C_{k,\ell+1} \le \frac{1}{4C^2} \tag{5.7}$$

(where $C > 0$ is the constant of (4.4)), it follows that

$$\sum_{k=1}^{n}\int |\partial_k f_k|^2 dQ \le 4\sum_{k=1}^{n}\int |\partial_k f|^2 dQ.$$

Hence, under (5.7) and together with (5.3), the logarithmic Sobolev inequality for Q holds, with a constant only depending on ρ.

Proposition 5.1. *Assume that, for some c', c'', $c' + c'' < c$,*

$$\int |\nabla\partial_k H|^2 dQ^{x_1,\dots,x_k} \le c'\,\mathrm{SG}(Q^{x_1,\dots,x_k}) \quad \text{and} \quad \int \partial_{kk} H\, dQ^{x_1,\dots,x_k} \ge -c''$$

uniformly over $x_1,\dots,x_k$ and $k = 1,\dots,n$ and that the coefficients $C_{k,\ell+1}$ of (5.6) satisfy (5.7). Then, for every smooth function f on $\mathbb{R}^n$,

$$\frac{\rho}{4}\,\mathrm{Ent}_Q(f^2) \le 2\int |\nabla f|^2 dQ$$

with $\rho = (c - c' - c'')\,\mathrm{e}^{-4\|w\|_\infty}$. In other words, $\mathrm{LS}(Q) \ge \frac{1}{4}\,(c - c' - c'')\,\mathrm{e}^{-4\|w\|_\infty}$.

It will be the purpose of the next section to describe models and conditions under which the hypotheses of Proposition 5.1 may be seen to be easily satisfied.

6. Logarithmic Sobolev inequalities for spin systems

We illustrate in this section the preceding general conclusions in the context of unbounded spin systems with nearest neighbors interaction. We develop here the tools to check the conditions in Propositions 3.1 and 5.1 for these specific spin systems. For a finite subset Λ in $\mathbb{Z}^d$, $d \ge 1$, denote by μ^Λ the product measure of μ on $\mathbb{R}^\Lambda$. Given the boundary condition $\omega \in \mathbb{R}^{\mathbb{Z}^d}$, consider the probability measure $dQ = dQ^{\Lambda,\omega} = \frac{1}{Z}\,\mathrm{e}^{-H} d\mu^\Lambda$ on $\mathbb{R}^\Lambda$ with Hamiltonian

$$H(x) = H^{\Lambda,\omega}(x) = \sum_{\{p,q\}\cap\Lambda\neq\emptyset, p\sim q} J_{pq}(x_p, x_q), \quad x = (x_p)_{p\in\Lambda} \in \mathbb{R}^\Lambda. \tag{6.1}$$

In (6.1), the summation is taken on couple $(p,q)=(q,p)$ of nearest neighbors $p\sim q$ in $\mathbb{Z}^d$, and when $p\notin\Lambda$, $x_p=\omega_p$. The functions J_{pq}, $p,q\in\mathbb{Z}^d$, are symmetric smooth functions on $\mathbb{R}^2$. The typical choices for J_{pq} are $J_{pq}(x,y)=Jxy$ (cf. [Yo1]) or $J_{pq}(x,y)=V(x-y)$ (cf. [He2], [B-H1]).

We assume that the single spin phase μ has a density that is strictly convex at infinity, that is $d\mu(x)=\frac{1}{z}\,\mathrm{e}^{-u(x)}dx$ where $u=v+w$, $v''\geq c>0$ and w is bounded. The typical assumption on the functions J_{pq} in the definition (6.1) will concern the quantity

$$J=\sup_{p,q}\Big(\|\partial_{11}J_{pq}\|_\infty+\|\partial_{12}J_{pq}\|_\infty\Big). \tag{6.2}$$

We will only be concerned with the perturbative regime where the coupling parameter J is small enough.

Since u is convex at infinity, $U(x)=\sum_{r\in\Lambda}u(x_r)+H(x)$, $x=(x_r)_{r\in\Lambda}$, is convex at infinity on $\mathbb{R}^\Lambda$ as soon as J is small enough. In particular, $Z=\int\mathrm{e}^{-H}d\mu^\Lambda<\infty$ for every Λ and boundary condition ω. Furthermore, by Corollary 1.7, $\mathrm{SG}(Q^{\lambda,\omega})\geq\mathrm{LS}(Q^{\lambda,\omega})>0$ with bounds however depending on (the size of) Λ and ω. It is the purpose of this section to show that these can actually be made uniform.

We now check on this model the various conditions required in order to apply the conclusions of the preceding sections. The various details might look tedious, but are straightforward. Fix $\Lambda\subset\mathbb{Z}^d$ and $\omega\in\mathbb{R}^{\mathbb{Z}^d}$ and write sometimes for simplicity Q instead of $Q^{\Lambda,\omega}$. Conditional distributions of $Q=Q^{\Lambda,\omega}$ are of the same form $Q^{\Lambda',\omega'}$ for some $\Lambda'\subset\Lambda\subset\mathbb{Z}^d$ and $\omega'\in\mathbb{R}^{\mathbb{Z}^d}$.

We start with the spectral gap and the bounds h and $\bar{h}$ on $\mathrm{Hess}\,(H)$ and $\partial_{ii}H$ of Proposition 3.1 where $H=H^{\Lambda,\omega}$ is defined by (6.1). For $r,r'\in\Lambda$,

$$\partial_{rr}H=\sum_{p\sim r}\partial_{11}J_{rp}$$

while when $r\neq r'$,

$$\partial_{rr'}H=\partial_{12}J_{rr'}$$

if $r\sim r'$ and $\partial_{rr'}H=0$ if not. In particular,

$$\max_{r\in\Lambda}\|\partial_{rr}H\|_\infty\leq 2dJ. \tag{6.3}$$

Similarly, for $\alpha=(\alpha_r)_{r\in\Lambda}\in\mathbb{R}^\Lambda$,

$$\begin{aligned}\langle\mathrm{Hess}\,(H)\alpha,\alpha\rangle&=\sum_r\partial_{rr}H\,\alpha_r^2+\sum_{r\sim r'}\partial_{rr'}H\,\alpha_r\alpha_{r'}\\&\geq-\max_{a\in\Lambda}\|\partial_{aa}H\|_\infty\sum_r\alpha_r^2-\max_{a,b\in\Lambda}\|\partial_{ab}H\|_\infty\sum_{r\sim r'}|\alpha_r||\alpha_{r'}|\\&\geq-2dJ|\alpha|^2.\end{aligned} \tag{6.4}$$

Hence, together with (3.3) and (3.4), $s\geq(c-2dJ)\,\mathrm{e}^{-4\|w\|_\infty}$ and

$$s+h-\bar{h}\geq(c-2dJ)\,\mathrm{e}^{-4\|w\|_\infty}-4dJ.$$

As a consequence of Proposition 3.1, we may already state for this example the following result of B. Helffer [He2]. It produces uniform spectral gaps in the perturbative regime (J small).

Proposition 6.1. *For every finite subset $\Lambda \subset \mathbb{Z}^d$ and every boundary condition $\omega \in \mathbb{R}^{\mathbb{Z}^d}$,*

$$\mathrm{SG}(Q^{\Lambda,\omega}) \geq (c - 2dJ)\,\mathrm{e}^{-4\|w\|_\infty} - 4dJ.$$

In particular, there exist $J_0 > 0$ and $\lambda > 0$ small enough, only depending on $d \geq 1$, $c > 0$ and $\|w\|_\infty < \infty$, such that for every finite subset $\Lambda \subset \mathbb{Z}^d$, every boundary condition $\omega \in \mathbb{R}^{\mathbb{Z}^d}$, and every $|J| \leq J_0$,

$$\mathrm{SG}(Q^{\Lambda,\omega}) \geq \lambda.$$

In other words, the spectral gap inequality holds for the measures $Q^{\Lambda,\omega}$ uniformly over finite subsets $\Lambda \subset \mathbb{Z}^d$ and boundary conditions $\omega \in \mathbb{R}^{\mathbb{Z}^d}$ provided J is small enough.

Now, we aim to use Proposition 3.2 to deduce some L^2 correlation inequalities that will be of help later on. Again, fix Λ and ω and write $Q = Q^{\Lambda,\omega}$. Let $p, q \in \Lambda$, and denote by $d(p,q)$ the graph distance between p and q on $\mathbb{Z}^d$. Recall we write $f = f(x_p)$, $p \in \Lambda$, to express that a smooth function f on $\mathbb{R}^\Lambda$ is actually a one-variable function only depending on the coordinate x_p. Let then $f = f(x_p)$ and $g = g(x_q)$. Choose the diagonal matrix D in Proposition 3.2 with $(d_r)_{r\in\Lambda}$ given by $d_r = \mathrm{e}^{d(p,r)}$. Other choices are clearly possible at this stage, and might be helpful to carefully follow the various constants involved into the problem. (What is actually needed right now is a function σ of the distance such that $\sup_{a>0} |\frac{\sigma(a+1)}{\sigma(a)}| < \infty$.) We however only consider this one for more simplicity. Then,

$$\int |D\nabla f|^2 dQ = \int f'^2 dQ$$

while

$$\int |D^{-1}\nabla g|^2 dQ = \mathrm{e}^{-2d(p,q)} \int g'^2 dQ.$$

One has now to control h^D of Proposition 3.2 for this choice of D. But, for every $\alpha = (\alpha_r)_{r\in\Lambda} \in \mathbb{R}^\Lambda$, it is easily seen as in (6.4) that

$$\begin{aligned}
\sum_{r,r'\in\Lambda} d_r d_{r'}^{-1} \partial_{rr'} H\, \alpha_r \alpha_{r'} &= \sum_r \partial_{rr} H\, \alpha_r^2 + \sum_{r\sim r'} \mathrm{e}^{d(p,r)} \mathrm{e}^{-d(p,r')} \partial_{rr'} H\, \alpha_r \alpha_{r'} \\
&\geq -\max_{a\in\Lambda} \|\partial_{aa} H\|_\infty \sum_r \alpha_r^2 - \mathrm{e} \max_{a,b\in\Lambda} \|\partial_{ab} H\|_\infty \sum_{r\sim r'} |\alpha_r||\alpha_{r'}| \\
&\geq -2d\mathrm{e}J|\alpha|^2.
\end{aligned}$$

Therefore, if J is small enough,

$$s + h^D - \bar{h} \geq (c - 2dJ)\,\mathrm{e}^{-4\|w\|_\infty} - 2d(1+\mathrm{e})J > 0.$$

As a consequence of Proposition 3.2, we may therefore state for this example the following result of B. Helffer [He2] on correlations bounds.

Proposition 6.2. *If*

$$(c - 2dJ)\,\mathrm{e}^{-4\|w\|_\infty} - 2d(1+\mathrm{e})J \geq \theta > 0,$$

for every finite subset $\Lambda \subset \mathbb{Z}^d$, every boundary condition $\omega \in \mathbb{R}^{\mathbb{Z}^d}$, and every smooth functions f and g on $\mathbb{R}^\Lambda$, $f = f(x_p)$, $g = g(x_q)$, $p, q \in \Lambda$,

$$\mathrm{Cor}_{Q^{\Lambda,\omega}}(f,g) \leq \theta^{-1}\,\mathrm{e}^{-d(p,q)} \Big(\int f'^2 dQ^{\Lambda,\omega} \Big)^{1/2} \Big(\int g'^2 dQ^{\Lambda,\omega} \Big)^{1/2}.$$

In particular,

$$\mathrm{Cor}_{Q^{\Lambda,\omega}}(f,g) \leq \theta^{-1}\,\mathrm{e}^{-d(p,q)} \|f'\|_\infty \|g'\|_\infty.$$

This result may be stated more generally as in [He2] and [B-H1] (see also ([B-H2]) for functions with arbitrary disjoints supports. We will not use this extension below.

Finally, we investigate the logarithmic Sobolev inequality for these spin systems through Propositions 5.1. Fix $Q = Q^{\Lambda,\omega}$. Recall that since the conditional distributions of Q are given by some $Q^{\Lambda',\omega'}$, Propositions 6.1 and 6.2 apply to all the conditional distributions of Q with the same uniform bounds. Let $J_0 > 0$ be small enough so that both Proposition 6.1, for some $\lambda > 0$, and Proposition 6.2, for some $\theta > 0$, hold for every $|J| \leq J_0$. If $r \in \Lambda$ and if ∇ is the gradient acting on $\mathbb{R}^{\Lambda\setminus\{r\}}$, then

$$|\nabla \partial_r H|^2 = \sum_{r' \sim r} |\partial_{rr'} H|^2 \leq 2dJ^2.$$

Recall also (6.3). Therefore, assuming that for some c', c'' with $c' + c'' < c$, J is small enough so that

$$2d\,J^2 \leq c'\lambda \quad \text{and} \quad 2dJ \leq c'', \tag{6.5}$$

the first hypothesis in Proposition 5.1 is clearly satisfied.

We turn to the control of the coefficients $C_{k,\ell+1}$ of (5.6). Given $Q = Q^{\Lambda,\omega}$, it is necessary to fix an enumeration $i = 1, \ldots, n$ of a finite subset Λ of $\mathbb{Z}^d$ with cardinal n. To distinguish between points of the lattice and elements in the enumeration, we use the letters $p, q, r, \ldots$ for the first ones, and $k, \ell, \ldots$ for the latter ones. By Proposition 6.2 applied to $Q^{x_1,\ldots,x_{\ell+1}}$, for $1 \leq k \leq \ell < n$, and the definition of J,

$$\begin{aligned}
\big|\mathrm{Cor}_{Q^{x_1,\ldots,x_{\ell+1}}}(\partial_k H, \partial_{\ell+1} H)\big| &\leq \sum_{r\sim k} \sum_{r'\sim \ell+1} \big|\mathrm{Cor}_{Q^{x_1,\ldots,x_{\ell+1}}}(\partial_1 J_{kr}, \partial_1 J_{\ell+1,r'})\big| \\
&\leq \sum_{r\sim k} \sum_{r'\sim \ell+1} J^2 \theta^{-1}\,\mathrm{e}^{-d(r,r')} \\
&\leq (2d\mathrm{e}J)^2 \theta^{-1}\,\mathrm{e}^{-d(\ell+1,k)}.
\end{aligned}$$

Therefore, for $1 \leq k \leq \ell < n$, together with $\|\partial_{\ell+1,k} H\|_\infty \leq J$ if $\ell + 1 \sim k$ and 0 otherwise,

$$C_{k,\ell+1} \leq \big(\mathrm{e}J + (2d\mathrm{e}J)^2\theta^{-1}\big)\mathrm{e}^{-d(\ell+1,k)}.$$

It is then a simple matter to check that condition (5.7) will be fulfilled for every J small enough. Setting, for fixed k (in Λ), $I_m = \{r \in \mathbb{Z}^d; d(k,r) = m\}$,

$$\sum_{j=k}^{n-1} \mathrm{e}^{-d(j+1,k)} = \sum_{m=0}^{\infty} \mathrm{e}^{-m} \sum_{j=k}^{n-1} 1_{\{j+1 \in I_m\}} \leq \sum_{m=0}^{\infty} \mathrm{e}^{-m} \mathrm{Card}\,(I_m).$$

Therefore, for every k,

$$\sum_{j=k}^{n-1} \mathrm{e}^{-d(j+1,k)} \leq \sum_{m=0}^{\infty} 2d\,(m+1)^{d-1}\mathrm{e}^{-m} < \infty.$$

Similarly, for every ℓ,

$$\sum_{k=1}^{\ell} \mathrm{e}^{-d(\ell+1,k)} \leq \sum_{m=0}^{\infty} 2d\,(m+1)^{d-1}\mathrm{e}^{-m} < \infty.$$

One deduces that

$$\max_{\ell} \sum_{k=1}^{\ell} \sum_{j=k}^{n-1} C_{k,j+1} C_{k,\ell+1} \leq M = M(J) \tag{6.6}$$

where $M(J)$ only depends on d, λ and J. Furthermore, $M(J) \to 0$ as $J \to 0$.

To conclude, recall first $J_0 > 0$ and $\lambda, \theta > 0$ have been chosen small enough so that Propositions 6.1 and 6.2 hold uniformly in Λ, ω and $|J| \leq J_0$. For $c' + c'' < c$, choose further J_0 small enough such that (6.5) holds and such that in (6.6) $M(J) \leq \frac{1}{4C^2}$ for every $|J| \leq J_0$. Hence (5.7) is satisfied and Proposition 5.1 applies. We may thus conclude in this way to the main result of the works [Ze1], [Yo1], [He2], [B-H1], in the form presented in [B-H1], in the perturbative regime.

Theorem 6.3. *Let u be convex at infinity, $u = v + w$ with $v'' \geq c > 0$, $\|w\|_\infty < \infty$. There exist $J_0 > 0$ and $\rho > 0$ small enough, only depending on $d \geq 1$, $c > 0$ and $\|w\|_\infty < \infty$, such that for every finite subset $\Lambda \subset \mathbb{Z}^d$, every boundary condition $\omega \in \mathbb{R}^{\mathbb{Z}^d}$, and every $|J| \leq J_0$,*

$$\mathrm{LS}(Q^{\Lambda,\omega}) \geq \rho.$$

In other words, the logarithmic Sobolev inequality holds for the measures $Q^{\Lambda,\omega}$ uniformly over finite subsets $\Lambda \subset \mathbb{Z}^d$ and boundary conditions $\omega \in \mathbb{R}^{\mathbb{Z}^d}$ provided J is small enough.

We conclude by a brief discussion of possible extensions and generalizations. The preceding proof may be adapted to the compact (continuous) spin systems for

which it provides a more simple analysis. We may consider different measures on each fibers, with uniform spectral gap and logarithmic Sobolev constants. Nearest neighbor interactions may also clearly by extended to finite range interactions. The proof presented above possibly allows infinite range of exponentially decreasing interactions. In another direction, appropriate polynomial decay of the correlations in Proposition 6.2 is actually sufficient to conclude (under the assumption that the one-dimensional marginals $Q_r^{\Lambda,\omega}$ satisfy uniformly a logarithmic Sobolev inequality).

In the particular case of the dimension $d = 1$, it has been proved by B. Zegarlinski [Ze1] that if the phase is super-convex ($u'' \to \infty$), and satisfies some technical assumption, then the spectral gap and the logarithmic Sobolev inequality hold uniformly whatever the value of J. In the non-perturbative regime (J arbitrary), N. Yoshida [Yo3] (see also [B-H2]), extending [S-Z3] in the compact case, showed the formal equivalence between spectral gap, decay of correlations and logarithmic Sobolev inequalities.

The scheme of proof of Theorem 6.3, together with the L^1-bounds on the correlations (2.5), may be used exactly in the same way to prove by induction the isoperimetric inequality, in its functional form,

$$I\big(\int f dQ^{\Lambda,\omega}\big) \leq \int \sqrt{I^2(f) + C|\nabla f|^2}\, dQ^{\Lambda,\omega} \tag{6.7}$$

of [Bo1] and [Ba-L] for $Q^{\Lambda,\omega}$. This inequality strengthens the logarithmic Sobolev inequality. In (6.7), I is the Gaussian isoperimetric function defined as $I = \varphi \circ \Phi$ where Φ is the distribution function of the standard Gaussian distribution on $\mathbb{R}$ and φ its density, and f is a smooth function with values in $[0, 1]$. Indeed, (6.7) is stable by products as Poincaré and logarithmic Sobolev inequalities (cf. [Bo1], [Ba-L]), and the Markov tensorization of Section 5 together with the L^1 correlation bounds apply similarly to yield the desired claim. However, since nearest neighbor interactions produce a uniform lower bound (6.4) on $\mathrm{Hess}\,(H)$, one may also use at a cheaper price Theorem 4.1 of [Ba-L] to deduce directly the isoperimetric inequality from the logarithmic Sobolev inequality of Theorem 6.3. Inequality (6.7) for discrete spin systems is considered in [Ze2], [Fo].

Acknowledgement. I am grateful to Th. Bodineau, B. Helffer and N. Yoshida for helpful comments and to I. Gentil and C. Roberto for a careful reading of the manuscript.

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