Curvature-Dimension

Curvature-Dimension conditions (or inequalities, bounds, hypotheses, criterions...) are widely used today in a large geometric spectrum, including (weighted) Riemannian geometry, Markov diffusion operators, metric measure spaces, graphs and discrete spaces etc. While present in one form or another earlier, the Curvature-Dimension condition has been emphasized in the pioneering work of D. Bakry and M. Émery in the mid-eighties within the context of Markov diffusion generators. At the same time, Bakry-Émery's legacy has a tendency nowadays (2020) to concentrate on the terminology "Curvature-Dimension", beyond the mathematical content itself. It is therefore of some interest to place the invention of the notion within the mathematical developments, emphasizing in particular the crucial input of coupling together curvature and dimension, and justifying its use, importance, and applicability. This note only briefly recounts some main historical steps, without a detailed documented and mathematical discussion; the bibliography is focused, restricted to some main references and monographs.

Classical Riemannian geometry deals with manifolds M with dimension $n \geq 1$, and many studies, results and inequalities do involve this dimensional parameter. A prototypical illustration is Gromov's compactness theorem on families of Riemannian manifolds with an upper bound on the dimension and the diameter and a lower bound on the Ricci curvature (see [11]). The role of Ricci curvature (lower) bounds is besides an essential feature of these studies, following M. Berger's quote "la domination universelle de la courbure de Ricci", discovered by M. Gromov in the seventies, the dimension of the given manifold appearing as an ambiant parameter.

A first historical example illustrating quantitatively this picture is the famous Lichnerowicz lower bound on the first eigenvalue λ_1 of the Laplacian Δ on a positively curved manifold [13]. If $(M, \mathfrak{g})$ is a compact connected (smooth) n-dimensional ($n \geq 2$) Riemannian manifold with Ricci curvature bounded from below by $\rho > 0$, then

$$\lambda_1 \,\geq\, \frac{\rho\, n}{n-1}\,. \tag{1}$$

This is optimal on the unit n-sphere $\mathbb{S}^n$ for which $\rho = n-1$ and $\lambda_1 = n$. The proof of this inequality is based on the Bochner formula

$$\frac{1}{2}\,\Delta\big(|\nabla f|^2\big) - \nabla f \cdot \nabla(\Delta f) \,=\, \mathrm{Ric}(\nabla f, \nabla f) + \big\|\mathrm{Hess}(f)\big\|_2^2$$

holding for every smooth function $f : M \to \mathbb{R}$, where Ric is the Ricci tensor and $\|\mathrm{Hess}(f)\|_2$ denotes the Hilbert-Schmidt norm of the Hessian of f. Under a lower bound ρ on the Ricci tensor, and the Cauchy-Schwarz inequality on the Hessian,

$$\frac{1}{2}\,\Delta\big(|\nabla f|^2\big) - \nabla f \cdot \nabla(\Delta f) \,\geq\, \rho\,|\nabla f|^2 + \frac{1}{n}\,(\Delta f)^2. \tag{2}$$

This inequality applied to an eigenfunction of Δ, and integrated with respect to the Riemannian measure, then easily yields (1). The use of the (Riemannian, invariant) measure is an essential feature of the argument, also part of the Curvature-Dimension concept as described below.

The Bochner inequality (2) is at the origin of the notion of Curvature-Dimension inequality. It was introduced by D. Bakry and M. Émery in [4] in the context of diffusion generators L on some measure space $(E, \mathcal{E}, \mu)$ (acting on a suitable algebra $\mathcal{A}$ of functions on E, typically smooth compactly supported functions on a manifold), for which both the left-hand side and right-hand side of (2) may be suitably adapted. That is, the bilinear carré du champ operator

$$\Gamma(f,g) \,=\, \frac{1}{2}\,\big[\mathrm{L}(fg) - f\,\mathrm{L}g - g\,\mathrm{L}f\big]$$

for $f, g \in \mathcal{A}$, is identified with $\nabla f \cdot \nabla g$ in a Riemannian setting when $\mathrm{L} = \Delta$. In particular, the measure μ being assumed to be invariant and symmetric with respect to L, the integration by parts formula $\int_E f\,\mathrm{L}g d\mu = -\int_E \Gamma(f,g) d\mu$ holds true. In this setting, the Bochner inequality (2) is then reinterpreted into

$$\frac{1}{2}\,\mathrm{L}\big(\Gamma(f,f)\big) - \Gamma(f, \mathrm{L}f) \,\geq\, \rho\,\Gamma(f,f) + \frac{1}{n}\,(\mathrm{L}f)^2.$$

The left-hand side has actually the same form as Γ, with the product fg replaced by $\Gamma(f,g)$, and is called the iterated carré du champ $\Gamma_2(f,f)$.

On this basis, a Markov generator L is said to satisfy a Curvature-Dimension inequality $\mathrm{CD}(\rho, m)$ with $\rho \in \mathbb{R}$ and $m \in [1, \infty]$, if for every $f \in \mathcal{A}$,

$$\Gamma_2(f,f) \,\geq\, \rho\,\Gamma(f,f) + \frac{1}{m}\,(\mathrm{L}f)^2. \tag{3}$$

This definition is considered as such in the seminal paper [4] by D. Bakry and M. Émery towards hypercontractivity of Markov diffusion operators, and defined as a Curvature-Dimension inequality, or condition, in the lectures [3] (with the author's notation). In the spirit of the Bochner formula and inequality, the lower bound ρ on the curvature and the dimension m are considered as a couple *defining the Curvature-Dimension inequality* $\mathrm{CD}(\rho, m)$.

Of course, due to the Bochner inequality (2), the Laplace operator Δ on an n-dimensional Riemannian manifold with Ricci curvature bounded from below by $\rho \in \mathbb{R}$ satisfies $\mathrm{CD}(\rho, n)$ (and the latter is actually equivalent to the lower bound ρ on the Ricci curvature). But the setting allows for enough flexibility that n need not be the topological dimension of the manifold, and real values of n, even infinite, may be considered.

Two simple examples are relevant. On the interval $(-1, +1)$, let L be the second order differential operator acting on smooth functions f as

$$\mathrm{L}f\,(x) = (1 - x^2) f''(x) - nx f'(x)$$

where $n > 0$. In this example, $\Gamma(f,f) = (1-x^2)f'^2$ and the invariant (probability) measure is given by $d\mu(x) = \frac{\Gamma(n)}{2^{n-1}\Gamma(\frac{n}{2})^2}\,(1-x^2)^{\frac{n}{2}-1}dx$ on $(-1,+1)$. When n is an integer, L is known as the ultraspheric generator which is obtained as the projection of the Laplace operator of $\mathbb{S}^n$ on a diameter. It is easily checked that

$$\Gamma_2(f,f) \,=\, (n-1)\,\Gamma(f,f) + \frac{1}{n}\,(\mathrm{L}f)^2 + \Big(1-\frac{1}{n}\Big)(1-x^2)^2 f''^2,$$

so that L satisfies $\mathrm{CD}(n-1,n)$ for every $n \geq 1$ (as the unit n-sphere). But the dimension in $\mathrm{CD}(\rho,n)$ does not refer to any dimension of the underlying state space.

There is a limit of this model, suitably scaled, as $n \to \infty$. It is classical (Poincaré's lemma) that the uniform measure on the n-sphere with radius $\sqrt{n}$ converges to a Gaussian measure. In this limit, the ultraspheric generators give rise to the operator $\mathrm{L}f\,(x) = f''(x) - xf'(x)$ on $\mathbb{R}$ with invariant Gaussian measure. By tensorization, the latter may be considered on $\mathbb{R}^d$ in the form

$$\mathrm{L}f\,(x) \,=\, \Delta f(x) - x\cdot\nabla f(x).$$

This is the so-called Ornstein-Uhlenbeck operator, that satisfies a Curvature-Dimension inequality $\mathrm{CD}(1,\infty)$ with infinite dimension (and no $\mathrm{CD}(\rho,m)$ for some finite m), which is geometrically natural. Its invariant measure is the standard Gaussian measure $d\mu(x) = \frac{1}{(2\pi)^{d/2}}\,e^{-\frac{1}{2}|x|^2}dx$.

More generally, the Langevin dynamics associated to a smooth potential $V : \mathbb{R}^d \to \mathbb{R}$ has generator

$$\mathrm{L}f\,(x) = \,\Delta f(x) - \nabla V(x)\cdot\nabla f(x)$$

with invariant measure $d\mu = e^{-V}dx$, and satisfies $\mathrm{CD}(\rho,\infty)$ if and only if $\mathrm{Hess}(V)(x) \geq \rho\,\mathrm{Id}$ as symmetric matrices, uniformly in $x \in \mathbb{R}^d$.

The latter example may be extended to weighted Riemannian manifolds, that is manifolds $(M,\mathfrak{g})$ (with topological dimension n) equipped with a weighted measure μ with density e^{-V} with respect to the Riemannian volume element dx, invariant and symmetric for the second order differential operator $\mathrm{L} = \Delta - \nabla V\cdot\nabla$. The associated Γ_2 operator takes the form

$$\Gamma_2(f,f) \,=\, \mathrm{Ric}(\nabla f,\nabla f) + \big\|\mathrm{Hess}(f)\big\|_2^2 + \mathrm{Hess}(V)\nabla f\cdot\nabla f,$$

sometimes called the Bakry-Émery-Ricci tensor. A Curvature-Dimension condition $\mathrm{CD}(\rho,m)$ for L may hold with $m \geq n$, but there is no longer a best optimal choice for both ρ and m, except in particular cases. Even negative dimension may be considered [16]. In this framework, the Bakry-Émery Curvature-Dimension condition is a major tool to investigate the geometry of weighted Riemannian manifolds, and related geometric and functional inequalities.

The importance and usefulness of the Curvature-Dimension hypothesis for a diffusion operator have been witnessed in numerous applications and illustrations, a couple of them may be emphasized (see [5]). One original motivation of the work [4] by D. Bakry and M. Émery was a proof of hypercontractivity properties of the Markov semigroup $(P_t)_{t\geq 0}$ with generator L, and equivalent logarithmic Sobolev inequalities [12] for the invariant (and symmetric) probability measure μ, under a $\mathrm{CD}(\rho, m)$ condition, covering in particular the example of the uniform measure on the sphere. This has been extended later to the full scale of Sobolev inequalities with sharp constants

$$\frac{\rho\, m}{m-1} \cdot \frac{1}{p-2}\bigg[\bigg(\int_E |f|^p d\mu\bigg)^{p/2} - \int_E f^2 d\mu\bigg] \,\leq\, \int_E \Gamma(f,f) d\mu,$$

where $1 \leq p \leq \frac{2m}{m-2}$, $m > 2$, the value $p = \frac{2m}{m-2}$ being the critical Sobolev exponent, $p = 1$ corresponding to the spectral gap (Poincaré) inequality and Lichnerowicz's lower bound, and $p = 2$ understood in the limit as a logarithmic Sobolev inequality. These Sobolev inequalities in turn entail ultracontractive heat kernel bounds.

An important aspect of the Curvature condition $\mathrm{CD}(\rho, \infty)$ with infinite dimension is the equivalence with the gradient bound, or commutation,

$$\Gamma(P_t f, P_t f) \,\leq\, e^{-\rho t} P_t\big(\Gamma(f,f)\big),$$

also equivalent to the strengthened form

$$\sqrt{\Gamma(P_t f, P_t f)} \,\leq\, e^{-\rho t} P_t\big(\sqrt{\Gamma(f,f)}\,\big)$$

for all $f \in \mathcal{A}$, $t > 0$. These gradient bounds are essential tools towards a number of illustrations, including Harnack inequalities, Gaussian-type isoperimetric comparison theorem, functional and concentration inequalities.

General applications of the Curvature-Dimension condition $\mathrm{CD}(\rho, m)$ also include Riesz transforms, diameter bounds, volume comparison theorems, heat kernel and spectral estimates, topological implications, Brunn–Minkowski-type inequalities etc., cf. e.g. [21, 5] and the references therein.

After these early developments for diffusion operators, the concept of Curvature-Dimension then leached into and impregnated the metric measure space world via optimal transport. The Bakry-Émery definition is attached to a given differential operator, and uses Hilbertian calculus. While it remains of interest for discrete models with a Markov kernel, it is not adapted to spaces much beyond a smooth Riemannian setting, as for example (measured) Gromov-Hausdorff limits of Riemannian manifolds [11] and other non-Hilbertian singular spaces. In a parallel development, the theory of optimal transport started, from the mid-nineties, to elaborate general tools and ideas, following in particular the influential works by Y. Brenier [6] and F. Otto [17], that progressively led to notions of curvature bounds in metric measure spaces.

A displacement convexity property of a functional on the space of probability measures along Wasserstein geodesics was introduced and studied by R. McCann [15], and later [9]. Given a geodesic space (X, d), let $\mathcal{P}_2(X)$ be the space of Borel probability measures with a second moment equipped with the L^2 Kantorovich-Wasserstein metric

$$d_W(\nu, \nu') \,=\, \inf_\pi \bigg(\int_{X\times X} d(x,y)^2 d\pi(x,y)\bigg)^{1/2}, \tag{4}$$

where the infimum is taken over all couplings π on $X \times X$ with respective marginals ν and ν'. The infimum is achieved at optimal transference plans. If $X = \mathbb{R}^n$, given two probability measures ν_0 and ν_1 in $\mathcal{P}_2(\mathbb{R}^n)$, it holds true that for every geodesic $(\nu_t)_{t\in[0,1]}$ in $\mathcal{P}_2(\mathbb{R}^n)$ joining ν_0 and ν_1,

$$\mathrm{H}_{dx}(\nu_t) \leq (1-t)\,\mathrm{H}_{dx}(\nu_0) + t\,\mathrm{H}_{dx}(\nu_1), \quad t \in [0,1], \tag{5}$$

where $\mathrm{H}_{dx}(\nu) = \int_{\mathbb{R}^n} \log \frac{d\nu}{dx}\, d\nu$ is the entropy of ν (with respect to the Lebesgue measure).

On a weighted smooth Riemannian manifold $(M, \mathfrak{g})$ with weighted measure $dm = e^{-V}dx$, it was shown in [18] that the Bakry-Émery Curvature condition $\mathrm{CD}(K,\infty)$, $K \in \mathbb{R}$, for the operator $\mathrm{L} = \Delta - \nabla V \cdot \nabla$ is equivalent to the displacement convexity with respect to d_W in the sense that for every geodesic $(\nu_t)_{t\in[0,1]}$ in $\mathcal{P}_2(M)$ joining ν_0 and ν_1,

$$\mathrm{H}_m(\nu_t) \leq (1-t)\,\mathrm{H}_m(\nu_0) + t\,\mathrm{H}_m(\nu_1) - \frac{K}{2}\, t(1-t)\, d_W(\nu_0,\nu_1)^2, \quad t \in [0,1], \tag{6}$$

with the analogous meaning for the entropy $\mathrm{H}_m(\nu)$ of ν with respect to m.

The next step was to involve the dimension, but again, after the Bakry-Émery vision, it appeared necessary to couple curvature and dimension towards a synthetic definition of curvature lower bounds within this setting. This major step was achieved in the breakthrough contributions [14, 19, 20] by J. Lott and C. Villani and K.-T. Sturm. For $K \in \mathbb{R}$, $\widetilde{N} \in (0,\infty]$, $t \in [0,1]$, and $0 < \theta < D_{K,\widetilde{N}}$ where $D_{K,\widetilde{N}} = \frac{\pi}{\sqrt{K/\widetilde{N}}}$ if $K > 0$ and $\widetilde{N} < \infty$, $D_{K,\widetilde{N}} = \infty$ otherwise, set

$$\sigma^{(t)}_{K,\widetilde{N}}(\theta) \;=\; \frac{\sin\big(t\theta\sqrt{\frac{K}{\widetilde{N}}}\,\big)}{\sin\big(\theta\sqrt{\frac{K}{\widetilde{N}}}\,\big)} \;=\; \begin{cases} \frac{\sin\big(t\theta\sqrt{\frac{K}{\widetilde{N}}}\big)}{\sin\big(\theta\sqrt{\frac{K}{\widetilde{N}}}\big)} & \text{if} \quad K>0,\ \widetilde{N}<\infty, \\ t & \text{if} \quad K=0 \quad \text{or} \quad \widetilde{N}=\infty, \\ \frac{\sinh\big(t\theta\sqrt{\frac{-K}{\widetilde{N}}}\big)}{\sinh\big(\theta\sqrt{\frac{-K}{\widetilde{N}}}\big)} & \text{if} \quad K<0,\ \widetilde{N}<\infty, \end{cases}$$

and $\sigma^{(0)}_{K,\widetilde{N}}(\theta) = t$ and $\sigma^{(t)}_{K,\widetilde{N}}(\theta) = \infty$ for $t \geq D_{K,\widetilde{N}}$. Next, for $K \in \mathbb{R}$ and $N \in (1,\infty]$, define

$$\tau^{(t)}_{K,N}(\theta) \;=\; t^{\frac{1}{N}}\sigma^{(t)}_{K,N-1}(\theta)^{1-\frac{1}{N}}.$$

Without going into technical details, and referring to [8] for a complete discussion and comparison of various related notions and definitions, a metric (finite) measure space (X,d,m) is said to be of Curvature-Dimension $\mathrm{CD}(K,N)$ (in the modern notation) if for every couple (ν_0,ν_1) of probability measures in $\mathcal{P}_2(X)$, there exists a geodesic $(\nu_t)_{t\in[0,1]}$ in $\mathcal{P}_2(X)$ such that

$$\mathrm{H}^N_m(\nu_t) \;\geq\; \int_{X\times X}\Big[\tau^{1-t}_{K,N}\big(d(x_0,x_1)\big)p_0^{-\frac{1}{N}}(x_0) + \tau^{t}_{K,N}\big(d(x_0,x_1)\big)p_1^{-\frac{1}{N}}(x_1)\Big]d\pi(x_0,x_1) \tag{7}$$

for π an optimal transference plan between ν_0 and ν_1 such that $\nu_i = p_i m$, $i = 1,2$, where $\mathrm{H}^N_m(\nu) = \int_X p^{1-\frac{1}{N}}\, dm$ with p the density with respect to m of the absolutely continuous component of ν, is the N-entropy of ν with respect to m (supposed to be finite in the preceding).

Besides its equivalence with the $\mathrm{CD}(\rho,m)$ Curvature-Dimension (with $\rho = K$, $m = N$!) in a smooth (weighted) Riemannian setting, a major feature of this synthetic definition is its stability under measured Gromov-Hausdorff convergence of metric measure spaces. In complete analogy with the smooth setting, it also entails various geometric and analytic inequalities relating metric and measure (cf. [21]). A significant illustration is the extension of the Lévy-Gromov isoperimetric comparison theorem achieved by F. Cavalletti and A. Mondino [7], showing that the isoperimetric profile of a metric measure space with Curvature-Dimension condition $\mathrm{CD}(n-1,n)$ is bounded from below by the one of the unit n-sphere. Finsler manifolds and Alexandrov spaces also satisfy the Curvature-Dimension condition.

With respect to the original Bakry-Émery definition, the $\mathrm{CD}(K,N)$ condition in metric measure spaces is delicate to verify, and some aspects are rather close to the smooth Riemannian framework. In an intermediate class of metric measure spaces, the so-called RCD, Riemannian, or infinitesimally Hilbertian, metric measure spaces, those in which the metric L^2 energy $\int_X |\nabla f|^2 dm$ satisfies the parallelogram identity, the Curvature-Dimension condition has been shown to coincide, after the suitable definition of a diffusion operator by integration by parts $\int_X f\, \mathrm{L} f dm = -\int_X |\nabla f|^2 dm$, with the Bakry-Émery definition [1, 2, 10].

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