On optimal matching of random samples

M. Ledoux

University of Toulouse, France

& Institut Universitaire de France

optimal matching: minimal transportation cost

between two sets of (random) points

$X_1, \ldots, X_n, \quad Y_1, \ldots, Y_n$ two samples of points in $\mathbb{R}^d$

$$\min_{\sigma} \frac{1}{n} \sum_{i=1}^{n} c(X_i, Y_{\sigma(i)})$$

σ permutation of $\{1, \ldots, n\}$

$c(\cdot, \cdot) \geq 0$ cost function

$c(x, y) = |x - y|^p, \quad 1 \leq p < \infty$

$X_1, \ldots, X_n, \quad Y_1, \ldots, Y_n$ iid random points in $\mathbb{R}^d$

order of growth in n of

$$\mathbb{E}\left(\min_\sigma \frac{1}{n} \sum_{i=1}^{n} |X_i - Y_{\sigma(i)}|^p \right)$$

dependence

- dimension d
- $1 \leq p < \infty$ (mostly $p = 1$ and $p = 2$)
- common distribution of X_i, Y_i

$X_1, \ldots, X_n, \quad Y_1, \ldots, Y_n$ iid random points in $\mathbb{R}^d$

order of growth in n of

$$\mathbb{E}\bigg(\min_\sigma \frac{1}{n} \sum_{i=1}^{n} |X_i - Y_{\sigma(i)}|^p \bigg)$$

$X_i, \ Y_i$ uniform on $[0,1]^d$

typical distance between n uniform points in $[0,1]^d \quad \asymp \frac{1}{n^{1/d}}$

expected: $$\mathbb{E}\bigg(\min_\sigma \frac{1}{n} \sum_{i=1}^{n} |X_i - Y_{\sigma(i)}|^p \bigg) \asymp \frac{1}{n^{p/d}}$$

only true when $d \geq 3$

Ajtai-Komlós-Tusnády (1984)

$X_1, \ldots, X_n, \quad Y_1, \ldots, Y_n$ independent, uniform on $[0,1]^2$

$$\mathbb{E}\left(\min_{\sigma} \frac{1}{n} \sum_{i=1}^{n} |X_i - Y_{\sigma(i)}|^p \right) \asymp \left(\frac{\log n}{n} \right)^{p/2}$$

$$1 \leq p < \infty$$

$$A \asymp B \quad \Longleftrightarrow \quad \frac{1}{C} B \leq A \leq C\, B \qquad (C \text{ independent of } n)$$

(hungarian) combinatorics on dyadic partitions

Ajtai-Komlós-Tusnády (1984)

$X_1, \ldots, X_n, \quad Y_1, \ldots, Y_n$ independent, uniform on $[0,1]^2$

$$\mathbb{E}\left(\min_\sigma \frac{1}{n} \sum_{i=1}^{n} |X_i - Y_{\sigma(i)}|^p \right) \asymp \left(\frac{\log n}{n} \right)^{p/2}$$

$$1 \leq p < \infty$$

alternate generic chaining ideas

Shor, Leighton (1989-91), Talagrand (1992-94)

$X_1, \ldots, X_n, \quad Y_1, \ldots, Y_n$ iid random points in $\mathbb{R}^d$

order of growth in n of

$$\mathbb{E}\left(\min_{\sigma} \frac{1}{n} \sum_{i=1}^{n} |X_i - Y_{\sigma(i)}|^p \right)$$

dependence

- dimension d
- $1 \leq p < \infty$ (mostly $p = 1$ and $p = 2$)
- common distribution of X_i, Y_i

$$c(x,y) = |x-y|^p, \quad 1 \le p < \infty$$

$$\mu_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i}, \quad \nu_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{Y_i}$$

$$\min_{\sigma} \frac{1}{n}\sum_{i=1}^{n} |X_i - Y_{\sigma(i)}|^p = \mathrm{W}_p^p(\mu_n, \nu_n)$$

Monge-Kantorovich metric

$$\mathrm{W}_p^p(\mu,\nu) = \inf_{\pi} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |x-y|^p \, d\pi(x,y)$$

π with respective marginals μ and ν

what is the cost of optimal matching

$$\mathbb{E}\big(\mathrm{W}_p^p(\mu_n, \nu_n)\big)$$

between two independent samples $X_1, \ldots, X_n, \quad Y_1, \ldots, Y_n$?

statistics: if the samples are iid with common law μ

what is the speed of convergence of

$$\mathbb{E}\big(\mathrm{W}_p^p(\mu_n, \mu)\big) \, ?$$

$$\mu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}$$

$$X_1, \dots, X_n \quad \text{iid in } \mathbb{R}^d \text{ with law } \mu, \qquad \mu_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i}$$

first order study of

$$\mathbb{E}\big(\mathrm{W}_p^p(\mu_n, \mu)\big) \qquad \text{or} \qquad \mathbb{E}\big(\mathrm{W}_p(\mu_n, \mu)\big)$$

- dimension d
- mostly $p = 1$ and $p = 2$ $\quad \left(\mathrm{W}_1 \leq \sqrt{\mathrm{W}_2^2}\right)$
- distribution μ

standard rates: comparison with the known uniform example specific representations of Monge-Kantorovich metrics

$$W_1(\nu, \mu) = \int_{-\infty}^{+\infty} \big| G(x) - F(x) \big| dx$$

G, F distribution functions of ν, μ on $\mathbb{R}$

quantile representation for $W_p(\nu, \mu), \quad p \geq 1$

$$W_p^p(\nu, \mu) = \int_0^1 \big| G^{-1}(t) - F^{-1}(t) \big|^p dt$$

(order statistics)

μ on $\mathbb{R}$ with distribution function F

$$\mathbb{E}\big(\mathrm{W}_1(\mu_n, \mu)\big) \asymp \frac{1}{\sqrt{n}}$$

if and only if

$$\int_{-\infty}^{+\infty} \sqrt{F(x)(1 - F(x))}\, dx < \infty$$

(for example $\int_{\mathbb{R}} |x|^q \, d\mu < \infty, \quad q > 2$)

$$\mu_n = \frac{1}{n}\sum_{i=1}^{n}\delta_{X_i}\,, \quad \nu_n = \frac{1}{n}\sum_{i=1}^{n}\delta_{Y_i}$$

$$\mathrm{W}_p^p(\mu_n,\nu_n) = \min_{\sigma}\frac{1}{n}\sum_{i=1}^{n}|X_i - Y_{\sigma(i)}|^p = \frac{1}{n}\sum_{i=1}^{n}|X_i^* - Y_i^*|^p$$

order statistics $X_1^* \leq \cdots \leq X_n^*, \quad Y_1^* \leq \cdots \leq Y_n^*$

$$\mathbb{E}\big(\mathrm{W}_p^p(\mu_n,\mu)\big) \asymp \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\big(\big|X_i^* - \mathbb{E}(X_i^*)\big|^p\big)$$

$$1 \leq p < \infty$$

$$\mathbb{E}\big(\mathrm{W}_p^p(\mu_n,\mu)\big) \asymp \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\big(\big|X_i^* - \mathbb{E}(X_i^*)\big|^p\big)$$

μ uniform on $[0,1]$

X_i^* beta $(i, n-i+1)$

$$\mathbb{E}\big(\mathrm{W}_p^p(\mu_n,\mu)\big) \asymp \frac{1}{n^{p/2}}$$

$$1 \leq p < \infty$$

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n, \mu)\big) = \frac{6}{n}$$

bipartite $$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n, \nu_n)\big) = \frac{1}{3(n+1)}$$

$$n\,\mathrm{W}_2^2(\mu_n, \mu)\big) \to \chi = \int_0^1 B^2(t)dt$$ in distribution

B Brownian bridge on $[0,1]$

$$\mathbb{E}(\chi) = \frac{1}{6}, \quad \mathrm{Var}(\chi) = \frac{1}{45}$$

μ on $\mathbb{R}$ with distribution function F, density f

$$\mathbb{E}\big(W_2^2(\mu_n, \mu)\big) \asymp \frac{1}{n}$$

if and only if

$$\int_{-\infty}^{+\infty} \frac{F(x)(1 - F(x))}{f(x)}\, dx < \infty$$

μ on $\mathbb{R}$ with distribution function F

$$\mathbb{E}\big(W_1(\mu_n, \mu)\big) \asymp \frac{1}{\sqrt{n}}$$

if and only if

$$\int_{-\infty}^{+\infty} \sqrt{F(x)(1-F(x))}\, dx < \infty$$

(for example $\int_{\mathbb{R}} |x|^q \, d\mu < \infty, \quad q > 2$)

μ on $\mathbb{R}$ with distribution function F, density f

$$\mathbb{E}\big(W_2^2(\mu_n, \mu)\big) \asymp \frac{1}{n}$$

if and only if

$$\int_{-\infty}^{+\infty} \frac{F(x)(1 - F(x))}{f(x)}\, dx < \infty$$

$$\int_{-\infty}^{+\infty} \frac{F(x)(1 - F(x))}{f(x)}\, dx = \int_0^1 \frac{t(1-t)}{I(t)^2}\, dt$$

$I(t) = f \circ F^{-1}(t)$ (isoperimetric profile)

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n, \mu)\big) \asymp \frac{1}{n}$$

if and only if

$$\int_0^1 \frac{t(1-t)}{I(t)^2}\, dt < \infty$$

μ log-concave $\qquad (d\mu = e^{-v}dx,\ v$ convex)

accurate two-sided bounds

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n, \mu)\big) \asymp \frac{1}{n} \int_{1/(n+1)}^{n/(n+1)} \frac{t(1-t)}{I(t)^2}\, dt$$

order statistics $X_1^* \leq \cdots \leq X_n^*$

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n, \mu)\big) \asymp \frac{1}{n} \sum_{i=1}^{n} \mathrm{Var}(X_i^*)$$

density of distribution of X_i^*

$$f_i(x) = n\binom{n-1}{i-1} F(x)^i \big(1 - F(x)\big)^{n-i} f(x), \qquad x \in \mathbb{R}$$

log-concave

$$\mathrm{Var}(X_i^*) \asymp \frac{1}{\sup_{x\in\mathbb{R}} f_i(x)^2} \asymp \frac{1}{n}\,\frac{t(1-t)}{I(t)^2}, \quad t = \frac{i}{n}$$

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n,\mu)\big) \asymp \frac{1}{n}\int_{1/(n+1)}^{n/(n+1)} \frac{t(1-t)}{I(t)^2}\,dt$$

- μ standard normal, $\quad I(t) \asymp t\sqrt{\log\frac{1}{t}}\,,\ t \to 0$

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n,\mu)\big) \asymp \frac{\log\log n}{n}$$

- μ exponential, $\quad I(t) \asymp t,\ t \to 0$

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n,\mu)\big) \asymp \frac{\log n}{n}$$

general 1-d investigation Bobkov-L (2016)

μ uniform on $[0,1]$

$$\mathbb{E}\big(\mathrm{W}_p^p(\mu_n,\mu)\big) \asymp \frac{1}{n^{p/2}} \qquad 1 \leq p < \infty$$

μ standard normal on $\mathbb{R}$

$$\mathbb{E}\big(\mathrm{W}_p^p(\mu_n,\mu)\big) \asymp \begin{cases} \frac{1}{n^{p/2}} & \text{if } \; 1 \leq p < 2 \\ \frac{\log\log n}{n} & \text{if } \; p = 2 \\ \frac{1}{n(\log n)^{p/2}} & \text{if } \; p > 2 \end{cases}$$

W_2 more sensitive to distribution than W_1

μ uniform on $[0,1]^d$

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n,\mu)\big) \asymp \frac{1}{n} \qquad d = 1$$

AKT $$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n,\mu)\big) \asymp \frac{\log n}{n} \qquad d = 2$$

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n,\mu)\big) \asymp \frac{1}{n^{2/d}} \qquad d \geq 3$$

(of the order of the uniform spacings $\frac{1}{n^{1/d}}$)

Dereich-Scheutzow-Schottstedt (2013)

Fournier-Guillin (2015)

general μ, enough moments

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n, \mu)\big) = O\bigg(\frac{1}{n^{2/d}}\bigg) \qquad d \geq 4 \quad (d = 3?)$$

dyadic partitions (local irregularities)

also results for W_p

general bounds

μ enough moments

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n,\mu)\big) \;=\; O\bigg(\frac{1}{\sqrt{n}}\bigg)$$

Ajtai-Komlós-Tusnády theorem

μ uniform on $[0,1]^2$

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n,\mu)\big) \;\asymp\; \frac{\log n}{n}$$

μ enough moments $\quad \left(\int_{\mathbb{R}^2} |x|^q \, d\mu < \infty, \quad q > 1 \right)$

$$\mathbb{E}\big(\mathrm{W}_1(\mu_n, \mu)\big) = O\left(\sqrt{\frac{\log n}{n}} \right)$$

generic chaining methodology Talagrand (1992), Yukich (1992)

Ajtai-Komlós-Tusnády theorem

μ uniform on $[0,1]^2$

$$\mathbb{E}\big(\mathrm{W}_1(\mu_n, \mu)\big) \asymp \sqrt{\frac{\log n}{n}}$$

some (unknown) limits may exist

$$\lim_{n\to\infty} n^{1/d}\, \mathbb{E}\big(\mathrm{W}_1(\mu_n,\mu)\big) = \gamma_d \qquad d \geq 3$$

$$\lim_{n\to\infty} n^{2/d}\, \mathbb{E}\big(\mathrm{W}_2^2(\mu_n,\mu)\big) = \tau_d \qquad d \geq 5$$

modified subadditivity arguments

Dobric-Yukich (1995), Boutet de Monvel-Martin (2002) W_1

Barthe-Bordenave (2013),

Dereich-Scheutzow-Schottstedt (2013) $\mathrm{W}_p, \quad p < \frac{d}{2}$

(absolutely continuous distributions)

$d = 2$

Ajtai-Komlós-Tusnády (1984)

μ uniform on $[0,1]^2$

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n,\mu)\big) \asymp \frac{\log n}{n}$$

$A \asymp B \iff \frac{1}{C}\,B \leq A \leq C\,B$ (C independent of n)

Ambrosio-Stra-Trevisan (2016)

μ uniform on $[0,1]^2$

$$\lim_{n\to\infty} \frac{n}{\log n}\, \mathbb{E}\big(\mathrm{W}_2^2(\mu_n, \mu)\big) \,=\, \frac{1}{4\pi}$$

bipartite

$$\lim_{n\to\infty} \frac{n}{\log n}\, \mathbb{E}\big(\mathrm{W}_2^2(\mu_n, \nu_n)\big) \,=\, \frac{1}{2\pi}$$

Ambrosio-Stra-Trevisan (2016)

(M, g) compact Riemannian manifold, dimension 2

μ (normalized) Riemannian volume element

$$\lim_{n\to\infty} \frac{n}{\log n}\, \mathbb{E}\big(\mathrm{W}_2^2(\mu_n, \mu)\big) \,=\, \frac{\mathrm{vol}(M)}{4\pi}$$

2-sphere/torus Holden-Peres-Zhai (2017) gravitational allocation

bipartite Ambrosio-Glaudo (2018)

pde ansatz by Caracciolo, Lucibello, Parisi, Sicuro (2014)

$$T = \nabla\psi : \rho_0 \mapsto \rho_1$$

Monge-Ampère equation

$$\rho_1(\nabla\psi)\,\det\nabla^2\psi = \rho_0$$

linearization as $\rho_i \sim 1, \qquad \psi \sim \frac{1}{2}|x|^2 + f$

Poisson equation

$$-\Delta f = \rho_1 - \rho_0$$

W_2 approximated by H^{-1} Sobolev norm upper bound

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n,\mu)\big) \;=\; \mathbb{E}\bigg(\inf_{\pi}\int_M\int_M \rho(x,y)^2\,d\pi(x,y)\bigg) \;=\; O\bigg(\frac{\log n}{n}\bigg)$$

- regularization

- energy estimate

(Sobolev-type inequality, heat kernel estimates)

(M, g) compact Riemannian manifold, dimension d

(weighted manifold, RCD space, Markov triple)

$p_t(x,y),\ t > 0,\ x,y \in M$ heat kernel $(t = t(n) \to 0)$

$$\mu_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i} \ \to\ d\mu_n^t = \frac{1}{n}\sum_{i=1}^{n} p_t(X_i, \cdot)d\mu$$

standard convexity of W_2

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n,\mu)\big) \le D_t + \mathbb{E}\big(\mathrm{W}_2^2(\mu_n^t,\mu)\big)$$

dispersion contribution

$$D_t = \int_M \int_M \rho(x,y)^2\, p_t(x,y)\, d\mu(x) d\mu(y), \quad t > 0$$

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n^t, \mu)\big)$$

density $\qquad d\mu_n^t = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} p_t(X_i, \cdot)d\mu$

$$f = f(y) = \frac{1}{n}\sum_{i=1}^{n} p_t(X_i, y)$$

f random densities

law of large numbers, $\quad \int_M p_t(\cdot, y)d\mu = 1$

$$f \approx 1$$

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n^t, \mu)\big)$$

density $\qquad d\mu_n^t = \dfrac{1}{n}\sum_{i=1}^{n} p_t(X_i, \cdot)\, d\mu$

central limit theorem heuristics

$$f = f(y) = \frac{1}{n}\sum_{i=1}^{n} p_t(X_i, y) \sim 1 + \frac{1}{\sqrt{n}}\, G(y)$$

$$G = G(t, y), \quad t > 0,\ y \in M$$

Gaussian Free Field

infinitemisals

$$f = 1 + \varepsilon g, \quad \int_M g \, d\mu = 0$$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \, \mathrm{W}_2^2(f\mu, \mu) \; = \; \int_M \big| \nabla((-\Delta)^{-1} g) \big|^2 \, d\mu$$

$$d\nu \; = f \, d\mu$$

$$\mathrm{W}_2^2(\nu, \mu) \; \leq \; 4 \, \| f - 1 \|^2_{\mathrm{H}^{-1}(\mu)}$$

dual Sobolev norm $\quad \int_M g \, d\mu \; = \; 0$

$$\| g \|_{\mathrm{H}^{-1}(\mu)} \; = \; \bigg(\int_M \big| \nabla((-\Delta)^{-1} g) \big|^2 \, d\mu \bigg)^{1/2}$$

$$d\nu = f \, d\mu$$

$$W_2^2(\nu, \mu) \leq 4 \, \|f - 1\|^2_{H^{-1}(\mu)}$$

$$\frac{1}{2} W_2(\nu, \mu)^2 = \sup \left(\int_M Q_1 \varphi \, d\nu - \int_M \varphi \, d\mu \right)$$

$\varphi : M \to \mathbb{R}$ bounded continuous

$$Q_u \varphi(x) = \inf_{y \in M} \left[\varphi(y) + \frac{d(x,y)^2}{2u} \right], \quad x \in M, \; u > 0$$

Hamilton-Jacobi $\quad \frac{d}{du} Q_u \varphi = -\frac{1}{2} \, |\nabla Q_u \varphi|^2$

$$\frac{1}{2}\,W_2(\nu,\mu)^2 \;=\; \sup\left(\int_M Q_1\varphi\,d\nu - \int_M \varphi\,d\mu\right)$$

$$\theta : [0,1] \;\to\; [0,1] \;\; \text{increasing,} \qquad g = f - 1$$

$$\int_M Q_1\varphi\,d\nu - \int_M \varphi\,d\mu$$

$$= \int_0^1 \frac{d}{du}\int_M \big(1+\theta(u)g\big)Q_u\varphi\,d\mu\,du$$

$$= \int_0^1\int_M \Big[\theta'(u)g\,Q_u\varphi - \big(1+\theta(u)g\big)\frac{1}{2}|\nabla Q_u\varphi|^2\Big]d\mu\,du$$

$$= \int_0^1\int_M \Big[-\theta'(u)\nabla\big((-\Delta)^{-1}g\big)\cdot\nabla Q_u\varphi - \big(1+\theta(u)g\big)\frac{1}{2}|\nabla Q_u\varphi|^2\Big]d\mu\,du$$

$$\leq \int_0^1\int_M \frac{1}{2}\,\frac{\theta'(u)^2}{1+\theta(u)g}\,\big|\nabla\big((-\Delta)^{-1}g\big)\big|^2 d\mu\,du$$

infinitemisals

$$f = 1 + \varepsilon g, \quad \int_M g \, d\mu = 0$$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathrm{W}_2^2(f\mu, \mu) \; = \; \int_M \big|\nabla((-\Delta)^{-1} g)\big|^2 \, d\mu$$

$$d\nu \; = f \, d\mu$$

$$\mathrm{W}_2^2(\nu, \mu) \; \leq \; 4 \, \|f - 1\|^2_{\mathrm{H}^{-1}(\mu)}$$

dual Sobolev norm $\int_M g \, d\mu \; = \; 0$

$$\|g\|_{\mathrm{H}^{-1}(\mu)} \; = \; \bigg(\int_M \big|\nabla((-\Delta)^{-1} g)\big|^2 \, d\mu \bigg)^{1/2}$$

$$\|g\|^2_{\mathrm{H}^{-1}(\mu)} = \int_M \left|\nabla((-\Delta)^{-1}g)\right|^2 d\mu = \int_M g(-\Delta)^{-1}g\, d\mu$$

trace formula

$$(-\Delta)^{-1} = \int_0^\infty P_s\, ds$$

$$\int_M g(-\Delta)^{-1}g\, d\mu = 2\int_0^\infty \int_M (P_s\, g)^2 d\mu\, ds$$

$$d\nu = f\, d\mu, \quad g = f - 1$$

$$\mathrm{W}_2^2(\nu, \mu) \leq 4\, \|f - 1\|^2_{\mathrm{H}^{-1}(\mu)} = 8\int_0^\infty \int_M (P_s\, g)^2 d\mu\, ds$$

$$\mathrm{W}_2^2(\nu, \mu) \leq 8 \int_0^\infty \int_M (P_s g)^2 \, d\mu \, ds$$

density $\quad d\nu = f \, d\mu$

$$g = g(y) = f(y) - 1 = \frac{1}{n} \sum_{i=1}^{n} \big[p_t(X_i, y) - 1 \big]$$

$$P_s g = \frac{1}{n} \sum_{i=1}^{n} \big[p_{t+s}(X_i, y) - 1 \big]$$

average of the sample $\quad$ (CLT heuristics)

$$\mathbb{E}\big((P_s g)^2 \big) = \frac{1}{n} \mathbb{E}\Big(\big[p_{t+s}(X_1, y) - 1 \big]^2 \Big) = \frac{1}{n} \big[p_{2(t+s)}(y, y) - 1 \big]$$

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n^t, \mu) \big) \leq \frac{4}{n} \int_{2t}^\infty \int_M \big[p_s(y, y) - 1 \big] \, d\mu(y) \, ds$$

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n,\mu)\big) \;\leq\; D_t \,+\, \mathbb{E}\big(\mathrm{W}_2^2(\mu_n^t,\mu)\big), \quad t > 0$$

dispersion

$$D_t \;=\; \int_M \int_M \rho(x,y)^2\, p_t(x,y)\, d\mu(x) d\mu(y)$$

energy (trace) estimate

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n^t,\mu)\big) \;\leq\; \frac{4}{n} \int_{2t}^{\infty} \int_M \big[p_s(y,y) - 1\big]\, d\mu(y)\, ds$$

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n, \mu)\big) \;\leq\; D_t + \frac{4}{n} \int_{2t}^{\infty} \int_M \big[p_s(y,y) - 1\big]\, d\mu(y)\, ds$$

(M, g) compact Riemannian manifold, dimension d

$$D_t \;=\; \int_M \int_M \rho(x,y)^2\, p_t(x,y) d\mu(x) d\mu(y) \;\leq\; Ct$$

heat kernel bounds $\quad p_s(y,y) \;\leq\; \dfrac{C}{s^{d/2}} \qquad 0 < s \leq 1$

optimization $\quad t \sim \dfrac{1}{n^{2/d}} \quad \Big(\dfrac{\log n}{n} \quad \text{if} \quad d = 2\Big)$

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n, \mu)\big) = \begin{cases} O\big(\frac{1}{n}\big) & \text{if} \quad d = 1 \\ O\Big(\frac{\log n}{n}\Big) & \text{if} \quad d = 2 \\ O\Big(\frac{1}{n^{2/d}}\Big) & \text{if} \quad d \geq 3 \end{cases}$$

heat kernel bounds

$$p_s(y, y) \leq \frac{C}{s^{d/2}} \qquad 0 < s \leq 1$$

reflect the dimensional rates

similar (optimal) conclusions for $\mathbb{E}\big(\mathrm{W}_p^p(\mu_n,\mu)\big)$

μ uniform on $[0,1]^d$

$$\mathbb{E}\big(\mathrm{W}_p^p(\mu_n,\mu)\big) \asymp \begin{cases} \frac{1}{n^{p/2}} & \text{if} \quad d = 1 \\ \left(\frac{\log n}{n}\right)^{p/2} & \text{if} \quad d = 2 \\ \frac{1}{n^{p/d}} & \text{if} \quad d \geq 3 \end{cases}$$

$1 \leq p < \infty$

Ambrosio-Stra-Trevisan (2016)

(M, g) compact Riemannian manifold, dimension 2

μ Riemannian volume element $(\mathrm{vol}(M) = 1)$

$$\lim_{n\to\infty} \frac{n}{\log n}\, \mathbb{E}\big(\mathrm{W}_2^2(\mu_n, \mu)\big) \,=\, \frac{1}{4\pi}$$

more precise arguments

common heat kernel asymptotics

$$\lim_{s\to 0} 4\pi s \int_M p_s(x,x)\,d\mu \,=\, 1$$

Ambrosio-Stra-Trevisan (2016)

(M, g) compact Riemannian manifold, dimension 2

μ Riemannian volume element $(\mathrm{vol}(M) = 1)$

$$\lim_{n\to\infty} \frac{n}{\log n} \, \mathbb{E}\big(\mathrm{W}_2^2(\mu_n, \mu)\big) = \frac{1}{4\pi}$$

only for $p = 2$, $d = 2$

pde ansatz by Caracciolo, Lucibello, Parisi, Sicuro (2014)

$$\lim_{n\to\infty} \left(\frac{n}{\log n} \mathbb{E}\big(W_2^2(\mu_n,\mu)\big) - \frac{1}{4\pi} \right) \log n \;=\; \xi \in \mathbb{R}$$

finer estimates Ambrosio, Glaudo (2018)

$$\liminf_{n\to\infty} \left(\frac{n}{\log n} \mathbb{E}\big(W_2^2(\mu_n,\mu)\big) - \frac{1}{4\pi} \right) \frac{\log n}{\log\log n} \;>\; -\infty$$

$$\limsup_{n\to\infty} \left(\frac{n}{\log n} \mathbb{E}\big(W_2^2(\mu_n,\mu)\big) - \frac{1}{4\pi} \right) \left(\frac{\log n}{\log\log n} \right)^{1/2} \;<\; \infty$$

concentration arguments

$$\frac{n}{\log n} \mathrm{W}_2^2(\mu_n, \mu) \to \frac{1}{4\pi} \qquad \text{in probability}$$

further conjecture

$$n\Big[\mathrm{W}_2^2(\mu_n, \mu) - \mathbb{E}\big(\mathrm{W}_2^2(\mu_n, \mu)\big)\Big] \to \chi \qquad \text{in distribution}$$

χ (recentered) chi-square type distribution (L^2-norm of GFF)

$$n\,\mathrm{W}_2^2(\mu_n, \mu) - \frac{1}{4\pi}\log n \to \xi + \chi$$

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n, \mu)\big) \;=\; \frac{6}{n}$$

bipartite $$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n, \nu_n)\big) \;=\; \frac{1}{3(n+1)}$$

$$n\,\mathrm{W}_2^2(\mu_n, \mu)\big) \;\to\; \chi \;=\; \int_0^1 B^2(t)dt$$ in distribution

B Brownian bridge on $[0, 1]$

$$\mathbb{E}(\chi) \;=\; \frac{1}{6}\,, \quad \mathrm{Var}(\chi) \;=\; \frac{1}{45}$$

$X_1, \ldots, X_n$ independent

with standard Gaussian law μ in $\mathbb{R}^2$

order of

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n, \mu)\big) \,?$$

μ uniform on $[0,1]$

$$\mathbb{E}\big(\mathrm{W}_p^p(\mu_n,\mu)\big) \asymp \frac{1}{n^{p/2}} \qquad 1 \leq p < \infty$$

μ standard normal on $\mathbb{R}$

$$\mathbb{E}\big(\mathrm{W}_p^p(\mu_n,\mu)\big) \asymp \begin{cases} \frac{1}{n^{p/2}} & \text{if} \quad 1 \leq p < 2 \\ \frac{\log\log n}{n} & \text{if} \quad p = 2 \\ \frac{1}{n(\log n)^{p/2}} & \text{if} \quad p > 2 \end{cases}$$

W_2 more sensitive to distribution than W_1

μ standard Gaussian on $\mathbb{R}^d$

pde-transportation approach

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n, \mu)\big) \le D_t + \frac{4}{n} \int_{2t}^{\infty} \int_{\mathbb{R}^d} \big[p_s(y,y) - 1\big]\, d\mu(y)\, ds$$

$p_s(x,y), \quad s > 0, \ x, y \in \mathbb{R}^d$ Mehler kernel

$$p_s(x,y) = \frac{1}{(1 - e^{-2s})^{d/2}} \exp\bigg(- \frac{e^{-2s}}{1 - e^{-2s}} \big[|x|^2 + |y|^2 - 2\, e^s\, x \cdot y \big] \bigg)$$

no uniform bounds

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n,\mu)\big) \;\leq\; D_t + \frac{4}{n}\int_{2t}^{\infty}\int_{\mathbb{R}^d}\big[p_s(y,y)-1\big]\,d\mu(y)\,ds$$

$$D_t \;=\; \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|x-y|^2\,p_t(x,y)\,d\mu(x)d\mu(y) \;\leq\; 2dt$$

$$\int_{\mathbb{R}^d} p_s(y,y)\,d\mu(y) \;=\; \frac{1}{(1-e^{-s})^d} \;\sim\; \frac{1}{s^d}$$

optimization in t

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n,\mu)\big) \;=\; O\Big(\frac{1}{n^{1/d}}\Big) \qquad \text{if} \quad d\geq 2$$

μ^R normalized restriction of μ to the ball $B = B(0, R)$, $R \sim \sqrt{\log n}$

$Z_1, \ldots, Z_n$ independent with law μ^R

$$X_i^R = \begin{cases} X_i & \text{if} \quad |X_i| \leq R \\ Z_i & \text{if} \quad |X_i| > R \end{cases}$$

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n, \mu_n^R)\big) = O\Big(\frac{1}{n}\Big), \qquad \mu_n^R = \frac{1}{n}\sum_{i=1}^n \delta_{X_i^R}$$

$$\int_{\mathbb{R}^d} p_s(y,y)\, d\mu^R(y) = \frac{1}{(1-e^{-s})^{d/2}}\, \frac{\mu(\theta B)}{\theta^d}$$

$$\theta \sim s^{1/2} \quad \text{as} \quad s \to 0$$

$$\frac{\mu(\theta B)}{\theta^d} \sim \lambda(B) \sim R^d \quad \text{as} \quad \theta R \leq 1$$

μ standard Gaussian on $\mathbb{R}^d$

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n,\mu)\big) = \begin{cases} O\Big(\frac{\log\log n}{n}\Big) & \text{if} \quad d = 1 \\ O\Big(\frac{(\log n)^2}{n}\Big) & \text{if} \quad d = 2 \\ O\Big(\frac{\log n}{n^{2/d}}\Big) & \text{if} \quad d \geq 3 \end{cases}$$

good enough to cover $d = 1$

extra factor $R^2 = \log n$ for $d \geq 2$

λ uniform on $[0,1]^2$

$U_1, \ldots, U_n$ independent with law λ, $\quad \nu_n = \frac{1}{n}\sum_{i=1}^n \delta_{U_i}$

$$\lambda = \Phi^{\otimes 2}(\mu)$$

$$\Phi(x) = \int_{-\infty}^{x} e^{-u^2/2}\frac{du}{\sqrt{2\pi}}, \quad x \in \mathbb{R}, \qquad \|\Phi\|_{\mathrm{Lip}} \leq 1$$

$$\mathbb{E}\big(W_2^2(\mu_n, \mu)\big) \geq \mathbb{E}\big(W_2^2(\nu_n, \lambda)\big)$$

AKT $\quad \mathbb{E}\big(W_2^2(\nu_n, \lambda)\big) \asymp \frac{\log n}{n}$

μ standard Gaussian on $\mathbb{R}^2$

$$\frac{1}{C}\,\frac{\log n}{n} \;\leq\; \mathbb{E}\big(\mathrm{W}_2^2(\mu_n,\mu)\big) \;\leq\; C\,\frac{(\log n)^2}{n}$$

some support for the left-hand side

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n, \mu)\big) \;=\; O\bigg(\frac{\log n}{n^{2/d}}\bigg) \qquad d \;\geq\; 3$$

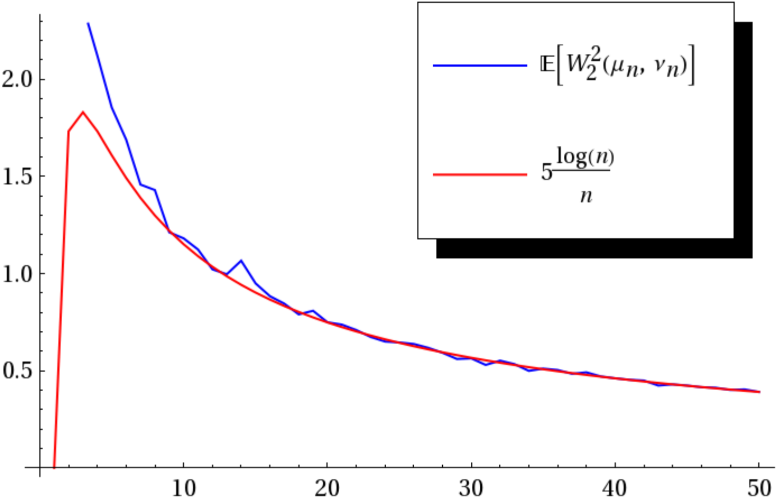
general moment bounds

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n, \mu)\big) \;=\; O\bigg(\frac{1}{n^{2/d}}\bigg) \qquad d \;\geq\; 4$$

same pde-transportation methodology

$$\mathbb{E}\big(\mathrm{W}_p^p(\mu_n, \mu)\big) \;\asymp\; \Big(\frac{\log n}{n}\Big)^{p/2} \qquad d = 2, \quad 1 \;\leq\; p \;<\; 2$$

(same as for uniform)


2.0
1.5
1.0
0.5
10
20
30
40
50
$\mathbb{E}\left[W_2^2(\mu_n, \nu_n)\right]$
$5\frac{\log(n)}{n}$

F. Stra (2016)

μ standard Gaussian on $\mathbb{R}^2$

$$\frac{1}{C}\,\frac{\log n}{n} \;\leq\; \mathbb{E}\big(\mathrm{W}_2^2(\mu_n, \mu)\big) \;\leq\; C\,\frac{(\log n)^2}{n}$$

M. Talagrand (2018)

$$\mathbb{E}\big(\mathrm{W}_2^2(\mu_n, \mu)\big) \;\asymp\; \frac{(\log n)^2}{n}$$

AKT method + scaling argument

(also by the pde approach – limit?)

Thank you for your attention

Vershik (2013) wrote a historic essay explaining why it is more fair to fix the name "Kantorovich distance" for all metrics like W_p (calling them Kantorovich power metrics).

Some general topological properties of W_1 were studied in 1970 by Dobrushin, who re-introduced this metric with reference to [Vasershtein 1969] ; apparently, that is why the name "Wasserstein distance" has become rather traditional.

As Vershik writes, "Leonid Vasershtein is a famous mathematician specializing in algebraic K-theory and other areas of algebra and analysis, and ... he is absolutely not guilty of this distortion of terminology, which occurs primarily in Western literature".