SEMIGROUP PROOFS OF THE ISOPERIMETRIC INEQUALITY IN EUCLIDEAN AND GAUSS SPACE

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ABSTRACT. — *This paper is an exposition of some of the semigroup tools which may be used to investigate the isoperimetric inequality in Euclidean and Gauss space. Inspired by the work of N. Varopoulos in his functional approach to isoperimetric inequalities on groups and manifolds, we will observe here, in particular, that the classical isoperimetric inequality in* $\mathbb{R}^n$ *is equivalent to saying that the* L^2*-norm of the heat semigroup acting on characteristic functions of sets increases under isoperimetric rearrangement. We then check the corresponding property in Gauss space and, following the approach of B. Maurey and G. Pisier to the concentration of measure phenomenon, we survey how the various properties of the Ornstein-Uhlenbeck semigroup such as the commutation property or hypercontractivity can yield in a simple way both the concentration phenomenon and (a form of) the isoperimetric inequality itself for Gauss measures.*

1. Introduction and the classical isoperimetric inequality in $\mathbb{R}^n$

The classical isoperimetric inequality in $\mathbb{R}^n$ (see e.g. [Ha], [Os], [B-Z]) states that among all subsets A with fixed (finite) volume $\mathrm{vol}_n(A)$ and smooth boundary ∂A, Euclidean balls minimise the surface measure of the boundary. In other words, whenever $\mathrm{vol}_n(A) = \mathrm{vol}_n(B)$ where B is a ball with some radius r (and $n > 1$),

$$\mathrm{vol}_{n-1}(\partial A) \geq \mathrm{vol}_{n-1}(\partial B). \tag{1.1}$$

Now, $\mathrm{vol}_{n-1}(\partial B) = nr^{n-1}\omega_n$ where ω_n is the volume of the ball of radius 1 so that (1.1) may be expressed equivalently as

$$\mathrm{vol}_{n-1}(\partial A) \geq n\omega_n^{1/n}\mathrm{vol}_n(A)^{(n-1)/n}. \tag{1.2}$$

The function $n\omega_n^{1/n}x^{(n-1)/n}$ on $\mathbb{R}^+$ is the *isoperimetric function* of the classical isoperimetric problem on $\mathbb{R}^n$. Euclidean balls are the *extremal* sets and achieve equality in (1.2).

It is well-known that (1.2) may be expressed equivalently on functions by means of the coarea formula (cf. [Fe], [Maz2], [Os]) : for every C^∞ compactly supported function f on $\mathbb{R}^n$,

$$\int |\nabla f|\,dx = \int_0^\infty \mathrm{vol}_{n-1}(C_s)\,ds$$

where $C_s = \{x \in \mathbb{R}^n; |f(x)| = s\}$, so that (1.2) together with integration by parts yields

$$n\omega^{1/n}\|f\|_{n/n-1} \le \big\||\nabla f|\big\|_1 \tag{1.3}$$

for every C^∞ compactly supported function f on $\mathbb{R}^n$ with gradient ∇f. This inequality is equivalent to (1.2) by letting f approximate the characterisitic function I_A of a set A whose boundary ∂A is smooth enough so that $\int |\nabla f|\,dx$ approaches $\mathrm{vol}_{n-1}(\partial A)$. *For simplicity, smoothness properties of boundaries will always be understood in this way throughout this paper.* Inequality (1.3) is due independently to E. Gagliardo [Ga] and L. Nirenberg [Ni] with a nice inductive proof on the dimension. This proof, however, does not seem to yield the optimal constant, and therefore the extremal character of balls. The connection between (1.2) and (1.3) through the coarea formula seems to be due to H. Federer and W. H. Fleming [F-F] and V. G. Maz'ya [Maz1] (cf. [Os]).

Inequality (1.3) of course belongs to the family of Sobolev inequalities. Replacing f (positive) by f^α for some appropriate α easily yields after an application of Hölder's inequality that, for every C^∞ compactly supported function f on $\mathbb{R}^n$,

$$\|f\|_q \le C(n,p,q)\big\||\nabla|\big\|_p \tag{1.4}$$

with $1 \le p < n$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$ and $C(n,p,q) > 0$ a constant only depending on n, p, q. The family of inequalities (1.4) with $1 < p < n$ goes back to S. L. Sobolev [So], the inequality for $p = 1$ (*which implies the others*) having thus been established later on. Of particular interest is the value $p = 2$ which may be expressed equivalently by integration by parts as $(n > 2)$

$$\|f\|_{2n/n-2} \le C\int |\nabla f|^2\,dx = C\int f(-\Delta f)\,dx \tag{1.5}$$

where Δ is the usual Laplacian on $\mathbb{R}^n$. As developed in an abstract setting by N. Varopoulos [Va1] (see [C-SC-V]), this Dirichlet type inequality (1.5) is closely related to the behaviour of the heat semigroup $T_t = \mathrm{e}^{-t\Delta}$, $t \ge 0$, as $\|T_t f\|_\infty \le Ct^{-n/2}\|f\|_1$, $t > 0$.

As an introduction, our first task in this work will be to illustrate, in this concrete setting, some aspects of the semigroup ideas of N. Varopoulos, and to show how these can yield, in a very simple way, (a form of) the isoperimetric inequality, actually the inequality (1.2) (or (1.3)) with a worse constant. We will work with the (probabilistic) integral representation of the heat semigroup $T_t = \mathrm{e}^{-t\Delta}$, $t \ge 0$, given by

$$T_t f(x) = \int_{\mathbb{R}^n} f\big(x + \sqrt{2t}\,y\big)\,d\gamma_n(y), \quad x \in \mathbb{R}^n, \quad f \in \mathrm{L}^1(dx),$$

where γ_n is the canonical Gaussian measure on $\mathbb{R}^n$ with density with respect to Lebesgue measure $\varphi_n(x) = (2\pi)^{-n/2}\exp(-|x|^2/2)$, $x \in \mathbb{R}^n$.

The following proposition is crucial for the understanding of the general principle of proof. Set, for Borel subsets A, B in $\mathbb{R}^n$, and $t \geq 0$, $K_t^T(A,B) = \int_B T_t(I_A)\,dx$ where I_A is the indicator function of the set A. A^c denotes below the complement of A.

PROPOSITION 1.1. — *For every subset A of finite volume in $\mathbb{R}^n$ with smooth boundary ∂A and every $t \geq 0$,*

$$K_t^T(A, A^c) \leq \left(\frac{t}{\pi}\right)^{1/2} \mathrm{vol}_{n-1}(\partial A).$$

Proof. Let f, g be smooth functions on $\mathbb{R}^n$. For every $t \geq 0$, we can write

$$\begin{aligned}\int g\,(T_t f - f)\,dx &= \int_0^t \left(\int g\,\Delta T_s f\,dx\right) ds \\ &= -\int_0^t \left(\int \langle \nabla T_s g, \nabla f\rangle\,dx\right) ds.\end{aligned}$$

Now, by integration by parts,

$$\nabla T_s g = \frac{1}{\sqrt{2s}} \int_{\mathbb{R}^n} y\, g\big(x + \sqrt{2s}\,y\big)\, d\gamma_n(y)\,.$$

Hence

$$\int g\,(T_t f - f)\,dx = -\int_0^t \frac{1}{\sqrt{2s}} \iint \langle \nabla f(x), y\rangle\, g\big(x + \sqrt{2s}\,y\big)\, dx d\gamma_n(y) ds.$$

This identity of course extends to $g = I_{A^c}$. Since

$$\iint \langle \nabla f(x), y\rangle\, dx d\gamma_n(y) = 0,$$

we see that, for every $s \geq 0$,

$$\begin{aligned}-\iint \langle \nabla f(x), y\rangle\, I_{A^c}\big(x + \sqrt{2s}\,y\big)\, dx d\gamma_n(y) &\leq \iint \big(\langle \nabla f(x), y\rangle\big)^-\, dx d\gamma_n(y) \\ &= \frac{1}{2}\iint \big|\langle \nabla f(x), y\rangle\big| dx d\gamma_n(y) \\ &= \frac{1}{\sqrt{2\pi}} \int |\nabla f|\,dx\end{aligned}$$

by partial integration with respect to $d\gamma_n(y)$. The conclusion follows since, by letting f approximate I_A, $\int |\nabla f|\,dx$ approaches $\mathrm{vol}_{n-1}(\partial A)$. The proof of Proposition 1.1 is complete.

Proposition 1.1 is sharp since it may be tested on balls. Namely, if B is an Euclidean ball, one may check that

$$\lim_{t\to 0}\Big(\frac{\pi}{t}\Big)^{1/2} K_t^T(B,B^c) = \mathrm{vol}_{n-1}(\partial B). \tag{1.6}$$

By translation invariance and homogeneity, we may assume that B is the unit ball of center the origin and radius 1. Then, for $t>0$,

$$K_t^T(B,B^c) = \int_{|x|>1} \gamma_n\big(y\in\mathbb{R}^n;\, \big|x+\sqrt{2t}\,y\big|\le 1\big)\,dx.$$

Using polar coordinates and the rotational invariance of γ_n,

$$\begin{aligned} K_t^T(B,B^c) &= \int_1^\infty \int_{\omega\in\partial B} \rho^{n-1}\gamma_n\big(y;\, \big|\rho\omega+\sqrt{2t}\,y\big|\le 1\big)\,d\rho d\omega \\ &= \mathrm{vol}_{n-1}(\partial B)\int_1^\infty \rho^{n-1}\gamma_1\otimes\gamma_{n-1}\big((y_1,\tilde y);\, \big|\rho+\sqrt{2t}y_1\big|^2+2t|\tilde y|^2\le 1\big)\,d\rho \end{aligned}$$

where $y=(y_1,\tilde y)$, $y_1\in\mathbb{R}$, $\tilde y\in\mathbb{R}^{n-1}$. We then use Fubini's theorem to write

$$K_t^T(B,B^c) = \mathrm{vol}_{n-1}(\partial B)\int J_t(y_1,\tilde y)\,d\gamma_1(y_1)d\gamma_{n-1}(\tilde y)$$

where

$$J_t(y_1,\tilde y) = I_{\{2t|\tilde y|^2\le 1;\sqrt{2t}y_1\le\sqrt{1-2t|\tilde y|^2}-1\}}\int_1^\infty \rho^{n-1} I_{\{|\rho+\sqrt{2t}y_1|^2\le 1-2t|\tilde y|^2\}}\,d\rho.$$

By a simple integration of the preceding, it is easily seen that for almost all y_1, $\tilde y$

$$\lim_{t\to 0}\frac{1}{\sqrt t}\,J_t(y_1,\tilde y) = -\sqrt2\,y_1 I_{\{y_1\le 0\}}$$

so that, by dominated convergence,

$$\lim_{t\to 0}\frac{1}{\sqrt t}\,K_t^T(B,B^c) = -\,\mathrm{vol}_{n-1}(\partial B)\int_{-\infty}^0 \sqrt2\,y_1\,d\gamma_1(y_1) = \frac{1}{\sqrt\pi}\,\mathrm{vol}_{n-1}(\partial B)$$

which is the claim (1.6).

Together with (1.6), the isoperimetric inequality (1.2) will now follow from Proposition 1.1 if we have that, for every $t\ge 0$ and every Borel subset A of $\mathbb{R}^n$, $K_t^T(A,A)\le K_t^T(B,B)$ whenever B is a ball with the same volume as A, or in other words, since $K_t^T(A,A) = \|T_{t/2}(I_A)\|_2^2$, if

$$\big\|T_t(I_A)\big\|_2 \le \big\|T_t(I_B)\big\|_2, \quad t\ge 0. \tag{1.7}$$

Indeed, under such a property, by Proposition 1.1, for every $t > 0$,

$$\mathrm{vol}_{n-1}(\partial A) \geq \left(\frac{\pi}{t}\right)^{1/2} K_t^T(A, A^c) \geq \left(\frac{\pi}{t}\right)^{1/2} K_t^T(B, B^c),$$

and when $t \to 0$, $\mathrm{vol}_{n-1}(\partial A) \geq \mathrm{vol}_{n-1}(\partial B)$ by (1.6).

Inequality (1.7) was actually established by A. Baernstein and B. A. Taylor [B-T] through delicate rearrangement inequalities of isoperimetric nature (see also [Ba]). While we noticed its *equivalence* with isoperimetry, one may wonder for an independent simpler proof of (1.7).

If one does not mind bad constants, one can actually deduce (a form of) isoperimetry from Proposition 1.1 in an elementary way. Note from the inequality $\|T_t f\|_\infty \leq Ct^{-n/2}\|f\|_1$, $t > 0$, that, by interpolation, $\|T_t f\|_2 \leq Ct^{-n/4}\|f\|_1$, $t > 0$ for every f in $\mathrm{L}^1(dx)$ and some possibly different constant C still only depending on n. Hence, by Proposition 1.1, for every subset A in $\mathbb{R}^n$ with fijnite volume and smooth boundary ∂A, and every $t > 0$,

$$\begin{aligned}
\mathrm{vol}_{n-1}(\partial A) &\geq \left(\frac{\pi}{t}\right)^{1/2} K_t^T(A, A^c) \\
&= \left(\frac{\pi}{t}\right)^{1/2} \left[\mathrm{vol}_n(A) - \left\|T_{t/2}(I_A)\right\|_2^2\right] \\
&\geq \left(\frac{\pi}{t}\right)^{1/2} \left[\mathrm{vol}_n(A) - C\left(\frac{t}{2}\right)^{-n/2} \mathrm{vol}_n(A)^2\right].
\end{aligned}$$

Optimising over $t > 0$ then yields

$$\mathrm{vol}_{n-1}(\partial A) \geq C' \mathrm{vol}_n(A)^{(n-1)/n}$$

hence (1.2), with however a worse constant. This easy proof could appear to be even simpler than the one by E. Gagliardo and L. Nirenberg.

The purpose of this work will be to develop throughout the next sections the same approach in the setting of the Gaussian isoperimetric inequality. As in this classical Euclidean case, we will follow closely the semigroup techniques of the work by N. Varopoulos [Va1], [Va2], [C-SC-V] (and the references therein) in his functional approach to geometric inequalities and heat kernel estimates on groups and manifolds. We work out these techniques with the Ornstein-Uhlenbeck semigroup, using some general results such as hypercontractivity, as well as tools developed by G. Pisier in [Pi1], [Pi2] (on the concentration phenomenon and Gaussian Riesz transforms). While classical Sobolev inequalities provide one of the main abstract tools in the classical case, we only substitute here logarithmic Sobolev inequalities and hypercontractivity. Although we do not attack the question here, this overall approach may possibly be developed similarly for the Ornstein-Uhlenbeck in backward as studied in [Bo5].

The present exposition does not present any real new results. It only would like to emphasise some simple semigroup ideas in the study of the geometry of (Euclidean and)

Gauss space. We hope furthermore that some of these ideas could be used in abstract settings. While true isoperimetric techniques such as symmetrisation always seem to yield best constants and therefore characterise extremal sets, we thought of some interest to investigate the Gaussian isoperimetric inequality through the Ornstein-Uhlenbeck semigroup whose central rôle in the Analysis of Wiener space need not be demonstrated anymore. In the next section, we briefly recall the isoperimetric inequality in Gauss space in its various formulations and applications, while in section 3, we survey the Ornstein-Uhlenbeck semigroup $(P_t)_{t\geq 0}$ and some of its classical properties such as hypercontractivity, for which we provide an elementary proof. In the last section, we show how to use the Ornstein-Uhlenbeck semigroup $(P_t)_{t\geq 0}$ to establish both the concentration of measure phenomenon for γ_n and a version of the isoperimetric inequality itself via the analogue of Proposition 1.1.

2. The isoperimetric inequality in Gauss space

In the rest of this work, we will work with the canonical Gaussian distribution γ_n on $\mathbb{R}^n$ with density with respect to Lebesgue measure

$$\varphi_n(x) = (2\pi)^{-n/2} \exp\bigl(-|x|^2/2\bigr), \quad x \in \mathbb{R}^n.$$

The measure γ_n is thus the product measure of the one-dimensional canonical Gaussian measure on each coordinate. As Lebesgue measure, the Gaussian measure γ_n satisfies an isoperimetric property, which, avoiding firstly surface measure considerations, may easily be described as follows. Let A be a Borel set in $\mathbb{R}^n$ and let H be a *half-space* $H = \{x \in \mathbb{R}^n; \langle x, u\rangle \leq a\}$, $|u| = 1$, $a \in [-\infty, +\infty]$, such that $\gamma_n(A) = \gamma_n(H)$. Then, for any real number $r \geq 0$,

$$\gamma_n(A_r) \geq \gamma_n(H_r) \tag{2.1}$$

where A_r is defined to be the Euclidean neighbourhood of order $r \geq 0$ of A, that is $A_r = \{x \in \mathbb{R}^n; d(x, A) \leq r\}$ with d the Euclidean distance on $\mathbb{R}^n$ or equivalently $A_r = A + B(0,r) = \{x + y\,; x \in A, y \in B(0,r)\}$ with $B(0,r)$ the (closed) Euclidean ball with center the origin and radius r. Hence, while balls are the extremal sets in the classical case, half-spaces play this rôle in this Gaussian setting. By rotational invariance, and since γ_n is a product measure, the Gaussian measure of the half-space H is computed in dimension 1 as $\gamma_n(H) = \Phi(a)$ where

$$\Phi(t) = \int_{-\infty}^{t} \varphi_1(x)\,dx = (2\pi)^{-1/2} \int_{-\infty}^{t} \mathrm{e}^{-x^2/2}\,dx, \quad t \geq 0.$$

Furthermore, since the neighbourhood H_r of H is again a half-space and $\gamma_n(H_r) = \Phi(a+r)$. Therefore, (2.1) may be stated equivalently as

$$\gamma_n(A_r) \geq \Phi(a + r) \quad \text{for every } r \geq 0 \tag{2.2}$$

whenever $\gamma_n(A) = \Phi(a)$. In this form, the Gaussian isoperimetric inequality appears to be dimension-free and indeed easily extends to infinite dimensional Gaussian measures. More precisely, let μ be a (centered) Gaussian measure on a real separable Banach space E with reproducing kernel Hilbert space $\mathcal{H}$ (the Cameron-Martin space for Wiener measure for example). That is, the abstract Wiener space factorisation

$$E^* \xrightarrow{i} \mathrm{L}^2(\mu) \xrightarrow{i^*} E$$

where i is the canonical injection map defines a hilbertian subspace $\mathcal{H} = i^*(\mathrm{L}^2(\mu))$ of E. Then, given a Borel set A in E with $\mu(A) = \Phi(a)$, the isoperimetric inequality for the Gaussian measure μ reads as

$$\mu_*\big(A + B_{\mathcal{H}}(0,r)\big) \geq \Phi(a+r) \quad \text{for every } r \geq 0 \tag{2.3}$$

where $B_{\mathcal{H}}(0,r)$ is the ball in $\mathcal{H}$ with center the origin and radius r and where μ_* is inner measure (the use of inner measure is necessary since $A + B_{\mathcal{H}}(0,r)$ need not always be measurable). Inequality (2.3) is deduced from (2.2) in a standard way by a finite dimensional approximation (see for example [Bo1], [Fa1]) ; the stricking property is of course that in infinite dimension $\mu(\mathcal{H}) = 0$. Although it is this infinite dimensional version which might appear to be the most useful in applications, for example in the Analysis of Wiener space, *it is really the finite dimensional version which one has to establish first.* Therefore, we only concentrate in this paper on this finite dimensional version. In particular, all the isoperimetric like inequalities which we will establish for γ_n with simple semigroup tools may easily be extended to infinite dimensional Gaussian measures and can therefore be used in applications as simpler minded inequalities.

Inequality (2.1) thus expresses the extremal character of half-spaces in the isoperimetric problem for Gaussian measures. It was established independently by C. Borell [Bo1] and V. N. Sudakov and B. S. Tsirel'son [S-T] on the basis of the isoperimetric inequality on the sphere [Lé], [Sch] and a limiting argument known as Poincaré's lemma (cf. [MK]). An intrisic proof using Gaussian isoperimetric symmetrisation was then provided by A. Ehrhard [Eh1] with applications to Gaussian Dirichlet integrals [Eh2], [Bo4].

The Gaussian isoperimetric inequality has been proved useful and extremely powerful in various and rather distinct contexts such as, for example, tail estimates of Gaussian seminorms [Bo1], [Ta1] and Wiener chaos [Bo2], [Bo3], [L-T], large deviations [Che], [G-K], [BA-L], [Led3], Banach space Geometry (in particular the study of almost spherical sections of convex bodies) [F-L-M], [M-S], [Pi1], [Pi3], and hypercontractivity and logarithmic Sobolev inequalities [Eh3], [Led2]. In these applications, the Gaussian isoperimetric inequality is used in two rather distinct ways depending on whether the isoperimetric enlargement A_r of a set A is considered for the *large* values of r or the *small* values.

In the first three mentioned applications, it is used for the large values of r in the form of what has been called the *concentration of measure phenomenon* [G-M], [M-S]. This phenomenon expresses here that if A is a Borel subset of $\mathbb{R}^n$ such that $\gamma_n(A) \geq \frac{1}{2}$, then one may take $a = 0$ in (2.2) and get that for every $r \geq 0$,

$$\gamma_n(A_r) \geq \Phi(r) \geq 1 - \tfrac{1}{2}\exp(-r^2/2). \tag{2.4}$$

In other words, starting from a set A with $\gamma_n(A) \geq \frac{1}{2}$, its enlargement A_r gets very rapidly a mass close to 1 when r increases to infinity.

Let us mention here that the inequality (2.4) has been recently surprisingly improved by M. Talagrand [Ta2] (see also [Ma]) who showed that for several sets A, the Euclidean enlargement may not be the optimal one in an inequality such as (2.4). M. Talagrand also studies in [Ta4] improved versions of the concentration inequality (2.4) which depend on the geometry of A rather that on its measure only (with applications, in particular, to balls of the Wiener space, which may be considered as cubes for the canonical product measure [Ta3]).

The inequality (2.4) may be translated equivalently on functions and is often most useful in this form in applications. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz map on $\mathbb{R}^n$ with Lipschitz norm

$$\|f\|_{\mathrm{Lip}} = \sup\left\{ \frac{|f(x) - f(y)|}{|x-y|}\,;\, x \neq y \text{ in } \mathbb{R}^n \right\},$$

and let M_f be a median of f for γ_n i.e.

$$\gamma_n(f \geq M_f) \geq \tfrac{1}{2} \quad \text{and} \quad \gamma_n(f \leq M_f) \geq \tfrac{1}{2}\,.$$

By applying (2.4) to $A = \{f \geq M_f\}$ and $A = \{f \leq M_f\}$ we get that for every $r \geq 0$,

$$\gamma_n\big(|f - M_f| > r\big) \leq \exp\left(-\frac{r^2}{2\|f\|^2_{\mathrm{Lip}}}\right). \tag{2.5}$$

Conversely, given a set A one may take in (2.5) the Lipschitz map $f(x) = d(x, A)$ and obtain, after some elementary considerations, an inequality such as (2.4) (or at least close enough to (2.4) — see Section 4).

In the applications to hypercontractivity and logarithmic Sobolev inequalities, the Gaussian isoperimetric inequality is used in its infinitesimal formulation connecting the "Gaussian volume" of a set to the "Gaussian length" of its boundary, which is really isoperimetry, the concentration phenomenon being only a mild corollary. More precisely, given a Borel subset A of $\mathbb{R}^n$, define (following [B-Z], [Fe], [Os], [Eh2]) the (Gaussian) Minkowski content of its boundary ∂A as

$$\mathcal{O}_{n-1}(\partial A) = \liminf_{r \to 0} \frac{1}{r}\big[\gamma_n(A_r) - \gamma_n(A)\big].$$

The isoperimetric inequality (2.1) (for the *small* values of $r \geq 0$) then expresses that if H is a half-space with the same measure as A, then

$$\mathcal{O}_{n-1}(\partial A) \geq \mathcal{O}_{n-1}(\partial H), \tag{2.6}$$

that is the analogue of (1.1). Now, one may easily compute (in dimension one) the Minkowski content of a half-space H as

$$\mathcal{O}_{n-1}(\partial H) = \liminf_{r\to 0} \frac{1}{r}\big[\Phi(a+r) - \Phi(a)\big] = \varphi_1(a)$$

where $\Phi(a) = \gamma_n(H) = \gamma_n(A)$. Hence, denoting by Φ^{-1} the inverse function of Φ, we get that, for every Borel set A in $\mathbb{R}^n$,

$$\mathcal{O}_{n-1}(\partial A) \geq \varphi_1 \circ \Phi^{-1}\big(\gamma_n(A)\big). \tag{2.7}$$

The function $\varphi_1 \circ \Phi^{-1}$ is the *isoperimetric function* of the Gauss space $(\mathbb{R}^n, \gamma_n)$. It may be compared to the function $n\omega^{1/n}x^{(n-1)/n}$ of the classical isoperimetric inequality in $\mathbb{R}^n$. The function $\varphi_1 \circ \Phi^{-1}$ is still concave ; it is defined on $[0,1]$, is symmetric with respect to the vertical line going through $\frac{1}{2}$ with a maximum equal to $(2\pi)^{-1/2}$ there, and its behaviour at the origin (or at 1 by symmetry) is governed by the equivalence

$$\lim_{x\to 0} \frac{\varphi_1 \circ \Phi^{-1}(x)}{x(2\log(1/x))^{1/2}} = 1. \tag{2.8}$$

This can easily be established by noticing that the derivative of $\varphi_1 \circ \Phi^{-1}$ is $-\Phi^{-1}$ and by comparing $\Phi^{-1}(x)$ to $(2\log(1/x))^{1/2}$.

As in the classical case, (2.7) may be expressed equivalently on functions by means, again, of the coarea formula (see [Eh2], [Led3]). Writing for a smooth function f on $\mathbb{R}^n$ with gradient ∇f that

$$\int |\nabla f|\, d\gamma_n = \int_0^\infty \bigg(\int_{C_s} \varphi_n(x)\, d\mathcal{H}_{n-1}(x)\bigg) ds$$

where $C_s = \{x \in \mathbb{R}^n; |f(x)| = s\}$ and where $d\mathcal{H}_{n-1}$ is the Hausdorff measure of dimension $n-1$ on C_s, we deduce from (2.7) that

$$\int |\nabla f|\, d\gamma_n \geq \int_0^\infty \varphi_1 \circ \Phi^{-1} \gamma_n\big(|f| \geq s\big)\, ds. \tag{2.9}$$

When f is a smooth function approximating the indicator function of a set A, we of course recover (2.7) from (2.9), at least for subsets A with smooth boundary. Due to the equivalence (2.8), it follows in particular from (2.9) (and integration y parts) that a smooth function f satisfying $\int |\nabla f|\, d\gamma_n < \infty$ is such that $\int |f|(\log(1+|f|))^{1/2} d\gamma_n < \infty$. In analogy with (1.3), such a property belongs to the family of Sobolev inequalities, *but here of logarithmic type.*

In the past years, it has been realised that rather elementary arguments may be used to establish the concentration properties (2.4) or (2.5). This observation is due B. Maurey and G. Pisier [Pi1] and the main idea involves the Ornstein-Uhlenbeck or Hermite semigroup with respect to the Gauss measure γ_n defined by the representation

$$P_t f(x) = \int_{\mathbb{R}^n} f\big(\mathrm{e}^{-t}x + (1-\mathrm{e}^{-2t})^{1/2}\big)\, d\gamma_n(y), \quad t \geq 0,$$

for f in $\mathrm{L}^1(\gamma_n)$. One of the crucial underlying properties of the semigroup $(P_t)_{t\geq 0}$ used in these proofs is the *commutation* property $\nabla P_t f = \mathrm{e}^{-t} P_t(\nabla f)$, in particular for the *large* values of the time t.

One of the purposes of this work will be to show, in the same spirit as what we presented in the classical Euclidean case, that the behaviour of the Ornstein-Uhlenbeck semigroup $(P_t)_{t\geq 0}$ for the *small* values of the time t together with its *hypercontractivity* property may properly be combined to yield the infinitesimal version (2.7) of the isoperimetric inequality. More precisely, we will show, *with these tools*, that there exists a small enough numerical constant $0 < c < 1$ such that for every measurable subset A with smooth boundary,

$$\mathcal{O}_{n-1}(\partial A) \geq c\,\varphi_1 \circ \Phi^{-1}\big(\gamma_n(A)\big).$$

We doubt that this approach can lead to the exact constant $c = 1$. The line of reasoning will thus follow the one of the classical case presented in the first section via tha analogue of Proposition 1.1, simply replacing then the classical heat semigroup estimates and Sobolev inequalities on $\mathbb{R}^n$ by the hypercontractivity property and logarithmic Sobolev inequalities of the Ornstein-Uhlenbeck semigroup which we now describe.

3. The Ornstein-Uhlenbeck semigroup and hypercontractivity

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion starting from the origin with values in $\mathbb{R}^n$. Consider the stochastic differential equation

$$dX_t = \sqrt{2}\,dB_t - X_t\,dt$$

with initial condition $X_0 = x$, whose solution is

$$X_t = \mathrm{e}^{-t}\bigg(x + \sqrt{2}\int_0^t \mathrm{e}^s\,dB_s\bigg), \quad t \geq 0.$$

Since $\int_0^t \mathrm{e}^s\,dB_s$ has the same distribution as $B_{\mathrm{e}^{2t}-1}$, the Markov semigroup $(P_t)_{t\geq 0}$ of $(X_t)_{t\geq 0}$ is given by

$$P_t f(x) = \mathbb{E}\big(f(\mathrm{e}^{-t} + \mathrm{e}^{-t} B_{\mathrm{e}^{2t}-1})\big) = \int_{\mathbb{R}^n} f\big(\mathrm{e}^{-t} + (1-\mathrm{e}^{-2t})^{1/2} y)\big) d\gamma_n(y) \tag{3.1}$$

for any f in $\mathrm{L}^1(\gamma_n)$ and x in $\mathbb{R}^n$, thus defining the *Ornstein-Uhlenbeck or Hermite semigroup* with respect to the Gaussian measure γ_n. $(P_t)_{t\geq 0}$ is a Markovian semigroup of contractions on all $\mathrm{L}^p(\gamma_n)$-spaces, $1 \leq p \leq \infty$, symmetric and invariant with respect to γ_n, and with generator L which acts on each smooth function f on $\mathbb{R}^n$ as

$$Lf(x) = \Delta f(x) - \langle x, \nabla f(x)\rangle.$$

The generator L satifies the integration by parts formula with respect to γ_n

$$\int f(-Lg)\, d\gamma_n = \int \langle \nabla f, \nabla g \rangle\, d\gamma_n$$

for every smooth functions f, g on $\mathbb{R}^n$.

One of the remarkable properties of the Ornstein-Uhlenbeck semigroup is its *hypercontractivity* property discovered by E. Nelson [Ne] : whenever $1 < p < q < \infty$ and $t > 0$ satisfy $\mathrm{e}^t \geq [(q-1)/(p-1)]^{1/2}$, then, for all functions f in $\mathrm{L}^p(\gamma_n)$,

$$\|P_t f\|_q \leq \|f\|_p \tag{3.2}$$

where (now) $\|\cdot\|_p$ is the norm in $\mathrm{L}^p(\gamma_n)$. In other words, P_t maps $\mathrm{L}^p(\gamma_n)$ in $\mathrm{L}^q(\gamma_n)$ $(q > p)$ with *norm one*. Many simple proofs of (3.2) have been given in the litterature, mainly based on its equivalent formulation as logarithmic Sobolev inequalities due to L. Gross [Gr]. Fix $p = 2$ and let $q(t) = 1 + \mathrm{e}^{2t}$, $t \geq 0$. Given a smooth function f, set $\Psi(t) = \|P_t f\|_{q(t)}$, $t \geq 0$. Under the hypercontractivity property (2.2), $\Psi(t) \leq \Psi(0)$ for every $t \geq 0$ and thus $\Psi'(0) \leq 0$. Performing this differentiation, we see that

$$\int f^2 \log|f|\, d\gamma_n - \int f^2 d\gamma_n \, \log\left(\int f^2 d\gamma_n\right)^{1/2} \leq \int |\nabla f|^2\, d\gamma_n \tag{3.3}$$

which in turn implies (3.2) by applying it to $P_t f$ $(f > 0)$ for every t. The inequality (3.3) is called a *logarithm Sobolev inequality*. One may note, with respect to the classical Sobolev inequalities on $\mathbb{R}^n$, that it is only of logarithmic type, with however constants independent of the dimension, a characterisitic feature of Gaussian measures.

Simple proofs of (3.3) may be found in e.g. [Nev], [A-C], [B-E]... The one which we present now for completeness already appeared in [Led4] and only relies (see also [B-E]) on the observation, immediately drawn from the representation (3.1), that $\nabla P_t f = \mathrm{e}^{-t} P_t(\nabla f)$ (of course exploited for the large values of t). Namely, to establish (3.3), replacing f (positive) by $\sqrt{f}$, it is enough to show that for every smooth positive function f on $\mathbb{R}^n$,

$$\int f \log f\, d\gamma_n - \int f\, d\gamma_n \, \log\left(\int f\, d\gamma_n\right) \leq \frac{1}{2}\int \frac{1}{f}\, |\nabla f|^2\, d\gamma_n. \tag{3.4}$$

To this aim, we can write by the semigroup properties and integration by parts that

$$\begin{aligned}
\int f \log f\, d\gamma_n - \int f\, d\gamma_n \, \log\left(\int f\, d\gamma_n\right) &= -\int_0^\infty \left(\frac{d}{dt}\int P_t f \, \log P_t f\, d\gamma_n\right) dt \\
&= -\int_0^\infty \left(\int LP_t f \, \log P_t f\, d\gamma_n\right) dt \\
&= \int_0^\infty \left(\int \langle \nabla P_t f, \nabla(\log P_t f)\rangle\, d\gamma_n\right) dt \\
&= \int_0^\infty \left(\int \frac{1}{P_t f}\, |\nabla P_t f|^2\, d\gamma_n\right) dt.
\end{aligned}$$

Now, set

$$F(t) = \int \frac{1}{P_t f} \, |\nabla P_t f|^2 \, d\gamma_n \quad t \geq 0.$$

The commutation property $\nabla P_t f = \mathrm{e}^{-t} P_t(\nabla f)$ and Cauchy-Schwarz inequality on the integral representation of P_t show that, for every $t \geq 0$,

$$\begin{aligned} F(t) &= \mathrm{e}^{-2t} \sum_{i=1}^{k} \int \frac{1}{P_t f} \left(P_t \frac{\partial f}{\partial x_i} \right)^2 d\gamma_n \\ &\leq \mathrm{e}^{-2t} \sum_{i=1}^{k} \int P_t \left(\frac{1}{f} \left(\frac{\partial f}{\partial x_i} \right)^2 \right) d\gamma_n = \mathrm{e}^{-2t} \int \frac{1}{f} \, |\nabla f|^2 \, d\gamma_n \end{aligned}$$

which immediately yields (3.4). *Therefore, hypercontractivity is established in this way.*

While our aim is to investigate isoperimetric inequalities via semigroup techniques, it is however of some interest to notice that the Gaussian isoperimetric inequality (2.7) or (2.9) may be used to establish the logarithmic Sobolev inequality (3.3) and therefore hypercontractivity. This was noticed in [Led2] in analogy with the classical case discussed in the first section. Let f be a smooth positive function on $\mathbb{R}^n$ with $\|f\|_2 = 1$. Apply then (2.9) to $g = f^2(\log(1+f^2))^{1/2}$. Using (2.8), one obtains after some elementary, although cumbersome, computations that for every $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ only depending on ε such that

$$\int f^2 \log(1+f^2) \, d\gamma_n \leq (1+\varepsilon) \left(\int |\nabla f|^2 d\gamma_n \right)^{1/2} \left(\int f^2 \log(1+f^2) \, d\gamma_n + 2 \right)^{1/2} + C(\varepsilon).$$

It follows that

$$\begin{aligned} 2 \int f^2 \log f \, d\gamma_n &\leq \int f^2 \log(1+f^2) \, d\gamma_n \\ &\leq 2(1+\varepsilon)^4 \int |\nabla f|^2 d\gamma_n + 2(1+\varepsilon)^2 \left(\int |\nabla f|^2 d\gamma_n \right)^{1/2} + C'(\varepsilon) \end{aligned}$$

where $C'(\varepsilon) = (1+\varepsilon)C(\varepsilon)/\varepsilon$. To get rid of the extra terms on the left hand side of this inequality, we use a tensorisation argument due to A. Ehrhard [Eh3] : this inequality namely holds with constants independent of the dimension n. Therefore, applying it to $f^{\otimes k}$ in $(\mathbb{R}^n)^k = \mathbb{R}^{nk}$ yields

$$k \int f^2 \log f \, d\gamma_n \leq k(1+\varepsilon)^4 \int |\nabla f|^2 d\gamma_n + \sqrt{k} \, (1+\varepsilon)^2 \left(\int |\nabla f|^2 d\gamma_n \right)^{1/2} + C'(\varepsilon).$$

Divide then by k, let k tend to infinity and then ε to zero and we obtain (3.3) by homogeneity.

4. Semigroup proofs of the Gaussian isoperimetric inequality

Our first task, in this last section, will be to show how the concentration of measure phenomenon (2.4) or (2.5) may be analysed with the Ornstein-Uhlenbeck semigroup. This observation goes back to B. Maurey and G. Pisier ; more precisely, the next lemma is due to B. Maurey (see [Pi1, p. 181]). We follow here the proof oulined in greater generality (Riemannian manifolds with non-negative Ricci curvature e.g.) in [Led5].

PROPOSITION 4.1. — *Let f be a Lipschitz map on $\mathbb{R}^n$ with $\|f\|_{\mathrm{Lip}} \le 1$ and $\int f\,d\gamma_n = 0$. Then, for every real number λ,*

$$\int \exp(\lambda f)\,d\gamma_n \le \exp\Big(\frac{\lambda^2}{2}\Big).$$

Before turning to the proof of Proposition 4.1, let us briefly indicate how it relates to concentration. Namely, by Chebyshev's inequality, it immediately follows that for every Lipschitz map f and every $r \ge 0$,

$$\gamma_n\big(|f - \textstyle\int f\,d\gamma_n| \ge r\big) \le 2\exp\Big(-\frac{r^2}{2\|f\|_{\mathrm{Lip}}^2}\Big). \tag{4.1}$$

This property is almost identical to (2.5) although it is not completely understood how one can go directly from the median to the expectation or conversely, *preserving the best constant* 2 in the exponent. However, (4.1) has a similar meaning on sets. Let indeed (see [Led1]) A be a Borel set in $\mathbb{R}^n$ with $\gamma_n(A) > 0$. For every $u > 0$, let f_u be the Lipschitz map defined by

$$f_u(x) = \min\big(d(x,A), u\big), \quad x \in \mathbb{R}^n.$$

Clealy $\|f_u\|_{\mathrm{Lip}} \le 1$ and $\int f_u\,d\gamma_n \le u(1 - \gamma_n(A))$ so that (4.1) applied to f_u for $r = u\gamma_n(A)$ already yields, for every $u > 0$,

$$\begin{aligned}\gamma_n(x \in \mathbb{R}^n;\, x \notin A_u) &\le \gamma_n(f_u \ge u)\\ &\le \gamma_n\big(f_u \ge \textstyle\int f_u\,d\gamma_n + u\gamma_n(A)\big) \le 2\exp\big(-u^2\gamma_n(A)^2/2\big).\end{aligned}$$

But now, we may improve with this inequality our previous estimate on $\int f_u\,d\gamma_n$ and get that

$$\int f_u\,d\gamma_n \le \int_0^u \gamma_n(x;\, x \notin A_v)\,dv \le \int_0^\infty \min\big(1-\gamma_n(A),\, 2\exp(-v^2\gamma_n(A)^2/2\big)\,dv.$$

Denoting by $\delta(\gamma_n(A))$ the right hand side of this inequality, (4.1) applied to f_u for $r = u - \delta(\gamma_n(A)) \ge 0$ then yields

$$\gamma_n(x \in \mathbb{R}^n;\, x \notin A_u) \le 2\exp\Big(-\frac{u^2}{2} + u\,\delta\big(\gamma_n(A)\big)\Big).$$

This inequality is as good as (2.4) in applications, in particular if we notice furthermore that $\delta(\gamma_n(A)) \to 0$ when $\gamma_n(A) \to 1$. It also immediately extends to infinite dimensional Gaussian measures, replacing A_u by $A + B_{\mathcal{H}}(0,u)$.

Now, we prove Proposition 4.1.

Proof of Proposition 4.1. As for the proof of the logarithmic Sobolev inequality in the previous section, we simply write by the semigroup properties and the integration by parts formula for the operator L that, for every $t \geq 0$,

$$\begin{aligned} G(t) = \int \exp(\lambda P_t f)\, d\gamma_n &= 1 - \int_t^\infty G'(s)\, ds \\ &= 1 - \lambda \int_t^\infty \Big(\int LP_s f \exp(\lambda P_s f)\, d\gamma_n \Big) ds \\ &= 1 + \lambda^2 \int_t^\infty \Big(\int |\nabla P_s f|^2 \exp(\lambda P_s f) d\gamma_n \Big) ds. \end{aligned}$$

Now $\nabla P_s f = \mathrm{e}^{-s} P_s(\nabla f)$ and $|\nabla f| \leq 1$ almost everywhere since $\|f\|_{\mathrm{Lip}} \leq 1$. It follows that, for every $t \geq 0$,

$$G(t) \leq 1 + \lambda^2 \int_t^\infty \mathrm{e}^{-2s} G(s)\, ds.$$

Let $H(t)$ be the logarithm of the right hand side of this inequality. Then $H'(t) \geq -\lambda^2 \mathrm{e}^{-2t}$, $t \geq 0$. Therefore

$$\log G(0) \leq H(0) = -\int_0^\infty H'(t)\, dt \leq \frac{\lambda^2}{2}$$

which is the conclusion. Proposition 4.1 is established.

Finally, we turn to the isoperimetric inequality itself. The next proposition, implicit in [Pi1, p. 180], is the first step towards our goal and is the Gaussian analogue of Proposition 1.1. Given Borel sets A, B in $\mathbb{R}^n$ and $t \geq 0$, we set

$$K_t^P(A,B) = \int_A P_t(I_B)\, d\gamma_n.$$

Note that $K_t^P(A,A) = \|P_{t/2}(I_A)\|_2^2$. The notation K_t^P is used in analogy with that of a kernel. Large deviation estimates of the kernel $K_t^P(A,B)$ for the Wiener measure when $d(A,B) > 0$ are developed in [Fa2]. As in the classical case, we will simply agree that "smooth" for the boundary ∂A of a set A means that $\int |\nabla f|\, d\gamma_n$ approaches the Gaussian length $\mathcal{O}_{n-1}(\partial A)$ of the boundary of A when f is a smooth function on $\mathbb{R}^n$ which approaches the indicator function of A.

PROPOSITION 4.2. — *For every Borel set A in $\mathbb{R}^n$ with smooth boundary ∂A and every $t \geq 0$,*

$$K_t^P(A, A^c) \leq (2\pi)^{-1/2} \arccos(\mathrm{e}^{-t})\, \mathcal{O}_{n-1}(\partial A).$$

Proof. It is similar to the proof of Proposition 1.1. Let f, g be smooth functions on $\mathbb{R}^n$. For every $t \geq 0$, we can write

$$\begin{aligned}\int g\,(P_t f - f)\, d\gamma_n &= \int_0^t \Big(\int g\, LP_s f\, d\gamma_n\Big) ds \\ &= -\int_0^t \Big(\langle \nabla P_s g, \nabla f\rangle\, d\gamma_n\Big) ds.\end{aligned}$$

Now, by an integration by parts on the representation of P_s using the Gaussian density,

$$\nabla P_s g = \frac{\mathrm{e}^{-s}}{(1-\mathrm{e}^{-2s})^{1/2}} \int_{\mathbb{R}^n} y\, g\big(\mathrm{e}^{-s}x + (1-\mathrm{e}^{-2s})^{1/2} y\big)\, d\gamma_n(y)\,.$$

Hence

$$\begin{aligned}&\int g\,(P_t f - f)\, d\gamma_n \\ &\quad = -\int_0^t \frac{\mathrm{e}^{-s}}{(1-\mathrm{e}^{-2s})^{1/2}} \iint \langle \nabla f(x), y\rangle\, g\big(\mathrm{e}^{-s}x + (1-\mathrm{e}^{-2s})^{1/2} y\big)\, d\gamma_n(x) d\gamma_n(y)\, ds.\end{aligned}$$

This identity of course extends to $g = I_{A^c}$. Since

$$\iint \langle \nabla f(x), y\rangle\, d\gamma_n(x) d\gamma_n(y) = 0\,,$$

we see that, for every $s \geq 0$,

$$\begin{aligned}-\iint \langle \nabla f(x), y\rangle I_{A^c}&\big(\mathrm{e}^{-s}x + (1-\mathrm{e}^{-2s})^{1/2} y\big)\, d\gamma_n(x) d\gamma_n(y) \\ &\leq \iint \big(\langle \nabla f(x), y\rangle\big)^{-} d\gamma_n(x) d\gamma_n(y) \\ &= \frac{1}{2} \iint \big|\langle \nabla f(x), y\rangle\big| d\gamma_n(x) d\gamma_n(y) \\ &= \frac{1}{\sqrt{2\pi}} \int |\nabla f|\, d\gamma_n\,.\end{aligned}$$

The conclusion follows by letting f approximate I_A since $\int |\nabla f|\, d\gamma_n$ will then approach $\mathcal{O}_{n-1}(\partial A)$ when ∂A is smooth enough. The proof is complete.

The inequality of the proposition is sharp in many respects. When $t \to \infty$, it reads as

$$\mathcal{O}_{n-1}(\partial A) \geq 2\sqrt{\frac{2}{\pi}}\, \mu(A)\big(1-\mu(A)\big)\,, \tag{4.2}$$

that is, when $\mu(A) = \frac{1}{2}$, the maximun of the isoperimetric function $\varphi_1 \circ \Phi^{-1}(x)$ at $x = \frac{1}{2}$. Inequality (4.2) may actually be interpreted as Cheeger's isoperimetric constant [Ch] of the Gauss space $(\mathbb{R}^n, \gamma_n)$. It is responsible for the optimal factor $\pi/2$ which appears in the vector valued inequalities of [Pi1]. Indeed, one may integrate (4.2) by the coarea formula (cf. [Ya]) to get that for every smooth function f with mean zero,

$$\int |f|\, d\gamma_n \leq \sqrt{\frac{\pi}{2}} \int |\nabla f|\, d\gamma_n,$$

an inequality which is easily seen to be best possible (take $n = 1$ and f on $\mathbb{R}$ be defined by $f(x) = x/\varepsilon$ for $|x| \leq \varepsilon$, $f(x) = x/|x|$ elsewhere, and let $\varepsilon \to 0$).

Proposition 4.2 may also be tested on half-spaces, as we did with balls in the first section. Namely, if we let $H = \{x \in \mathbb{R}^n; \langle x, u\rangle \leq a\}$, $|u| = 1$, $a \in \mathbb{R}$, it is easily checked (reduce to dimension one by rotational invariance and use polar coordinates) that

$$\begin{aligned} K_t^P(H, H^c) &= \frac{1}{2\pi} \iint_{\mathbb{R}^2} \mathrm{e}^{-x^2/2} \mathrm{e}^{-y^2/2} I_{\{x \leq a,\, \mathrm{e}^{-t}x + (1-\mathrm{e}^{-2t})^{1/2} y > a\}} dxdy \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} r\mathrm{e}^{-r^2/2} I_{\{r\sin(\varphi) \leq a,\, r\sin(\varphi+\theta) > a\}} d\varphi dr \\ &= \frac{\theta}{2\pi}\, \mathrm{e}^{-a^2/2} - \frac{1}{2\pi} \int_{|a|}^{|a|/\sin((\pi-\theta)/2)} \big(2\arcsin\big(|a|/r\big) + \theta - \pi\big) r\mathrm{e}^{-r^2/2} dr \end{aligned}$$

where $\theta = \arccos(\mathrm{e}^{-t})$. The absolute value of the second term of the latter may be bounded by

$$\frac{\theta}{2\pi}\big(\mathrm{e}^{-a^2/2} - \mathrm{e}^{-a^2/2\cos^2(\theta/2)}\big) \leq \frac{\theta}{2\pi} \cdot \frac{a^2}{2} \tan^2(\theta/2)\, \mathrm{e}^{-a^2/2} \leq \frac{\theta^3}{2\pi}\, a^2\, \mathrm{e}^{-a^2/2}$$

at least for all θ small enough. In particular, since $\theta = \arccos(\mathrm{e}^{-t})$ and thus $\theta \sim \sqrt{2t}$ when $t \to 0$, it follows that

$$K_t^P(H, H^c) = (2\pi)^{-1/2} \arccos(\mathrm{e}^{-t})\, \mathcal{O}_{n-1}(\partial H) + o\big(\sqrt{t}\big) \quad \text{for } t \to 0. \tag{4.3}$$

On the basis of Proposition 4.2, we now would need lower estimates of the functional $K_t^P(A, A^c)$ for the small values of t. The typical isoperimetric approach would be to use a symmetrisation result of C. Borell [Bo4], which is the analogue of (1.7), asserting that if H is a half-space with the same measure as A, then for every $t \geq 0$,

$$K_t^P(A, A) \leq K_t^P(H, H).$$

Hence $K_t^P(A, A^c) \geq K_t^P(H, H^c)$ and we would conclude from Proposition 4.2 and (4.3) that

$$\mathcal{O}_{n-1}(\partial A) \geq \mathcal{O}_{n-1}(\partial H),$$

i.e. (2.6). In particular, and as in the classical case, isoperimetry is therefore *equivalent* to saying that for every Borel subset A

$$\big\|P_t(I_A)\big\|_2 \leq \big\|P_t(I_H)\big\|_2, \quad t \geq 0, \tag{4.4}$$

when H is a half-space with the same measure as A. This inequality is thus established in [Bo4], extending ideas of [Eh2] and based on techniques developed in the classical case in [Ba] and [B-T], via the Gaussian isoperimetric inequality. It is moreover established, as (1.7) in [B-T], for functions and not only indicator functions. It might be that a simple direct approach to (4.4) (and (1.7) as we mentioned in Section 1) is possible ; it would then be an ideal complement to the simple approach presented here.

Our approach to bound $K_t^P(A,A)$ will be to use *hypercontractivity* as the corresponding semigroup estimate in this Gaussian setting. Namely, we simply write for a Borel set A with (smooth) boundary ∂A in $\mathbb{R}^n$ and $p(t) = 1 + \mathrm{e}^{-t}$ that

$$K_t^P(A,A) = \big\| P_{t/2}(I_A)\big\|_2^2 \le \big\| I_A \big\|_{p(t)}^2, \quad t \ge 0. \tag{4.5}$$

Hence

$$K_t^P(A,A^c) \ge \gamma_n(A)\big[1 - \gamma_n(A)^{(2/p(t))-1}\big].$$

Therefore, combined with Proposition 4.2,

$$\mathcal{O}_{n-1}(\partial A) \ge (2\pi)^{1/2}\gamma_n(A) \sup_{t>0} \Big[\big(\arccos(\mathrm{e}^{-t})\big)^{-1}\big(1 - \gamma_n(A)^{(2/p(t))-1}\big)\Big].$$

Setting $\theta = \arccos(\mathrm{e}^{-t}) \in (0, \frac{\pi}{2}]$, we need evaluate

$$\sup_{0<\theta\le\frac{\pi}{2}} \frac{1}{\theta}\left[1 - \exp\left(-\frac{1-\cos\theta}{1+\cos\theta}\,\log\frac{1}{\gamma_n(A)}\right)\right].$$

To this aim, we can note for example that

$$\frac{1-\cos\theta}{1+\cos\theta} \ge \frac{\theta^2}{2\pi}\,,$$

and choosing thus θ of the form

$$\theta = (2\pi)^{1/2}\left(\log\frac{1}{\gamma_n(A)}\right)^{-1/2}$$

provided that $\gamma_n(A) \le \exp(-8/\pi)$, we find that

$$\mathcal{O}_{n-1}(\partial A) \ge \left(1 - \frac{1}{\mathrm{e}}\right)\gamma_n(A)\left(\log\frac{1}{\gamma_n(A)}\right)^{1/2}.$$

Due to the equivalence (2.8), there exists then $\delta > 0$ such that when $\gamma_n(A) \le \delta$,

$$\mathcal{O}_{n-1}(\partial A) \ge \frac{1}{3}\,\varphi_1 \circ \Phi^{-1}\big(\gamma_n(A)\big).$$

When $\delta < \gamma_n(A) \le 1/2$, we can always use (4.2) to get

$$\mathcal{O}_{n-1}(\partial A) \ge \sqrt{\frac{\pi}{2}}\,\gamma_n(A) \ge c(\delta)\,\varphi_1 \circ \Phi^{-1}\big(\gamma_n(A)\big)$$

for some $c(\delta) > 0$. These two inequalities, together with symmetry, yield that, for some numerical constant $0 < c < 1$ and all measurable subsets A in $\mathbb{R}^n$ with smooth boundary,

$$\mathcal{O}_{n-1}(\partial A) \geq c\,\varphi_1 \circ \Phi^{-1}\big(\gamma_n(A)\big), \tag{4.6}$$

that is, a form of the Gaussian isoperimetric inequality. One may try to tighten the preceding computations to reach the value $c = 1$ in (4.6). This however does not seem likely and it is certainly in the hypercontractive estimate (4.5) that a good deal of the best constant is lost. One may wonder why this is the case ; it seems that hypercontractivity, while an equality on exponential functions, is perhaps not that sharp on indicator functions. This would have to be understood in connection with (4.4). Note finally that one may integrate back (4.6) to obtain the analogue of (2.1), that is, if $\gamma_n(A) = \Phi(a)$, for every $r \geq 0$,

$$\gamma_n(A_r) \geq \Phi(a + cr).$$

It is likely that the preceding approach has some interesting consequences in more abstract settings. In particular, one could imagine to study, with these tools, the hypercontractivity constant of a compact Riemannian manifold, or rather to investigate isoperimetric properties implied by hypercontractivity (see [Led6]). This would be in analogy with the isoperimetric inequalities obtained in [Va1], [Va2], [C-SC-V] via the dimension of the underlying heat semigroup.

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