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DIFFERENTIAL OPERATORS AND SPECTRAL DISTRIBUTIONS OF INVARIANT ENSEMBLES FROM THE CLASSICAL ORTHOGONAL POLYNOMIALS: THE DISCRETE CASE

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Abstract. – *We examine the Charlier, Meixner, Krawtchouk and Hahn discrete orthogonal polynomial ensembles, deeply investigated by K. Johansson, using integration by parts for the underlying Markov operators, differential equations on Laplace transforms and moment equations. As for the matrix ensembles, equilibrium measures are described as limits of empirical spectral distributions. In particular, a new description of the equilibrium measures as adapted mixtures of the universal arcsine law with an independent uniform distribution is emphasized. Factorial moment identities on mean spectral measures may be used towards small deviation inequalities on the rightmost charges at the rate given by the Tracy-Widom asymptotics.*

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1. Introduction

Determinantal representations of eigenvalues are the keys for a deep understanding of both the global and local regimes of random matrix and random growth models by means of orthogonal polynomials. For $N \geq 1$, let

$$dQ(x) = dQ^N(x) = \frac{1}{Z_N} \Delta_N(x)^2 \prod_{j=1}^{N} d\mu(x_j), \quad x \in \mathbb{R}^N, \tag{1}$$

be the determinantal distribution or so-called Coulomb gas associated to a probability measure μ on the real line such that

$$Z_N = \int_{\mathbb{R}^N} \Delta_N(x)^2 \prod_{j=1}^{N} d\mu(x_j) < \infty$$

where $\Delta_N(x)$ is the Vandermonde determinant

$$\Delta_N(x) = \prod_{1 \leq i < j \leq N} (x_j - x_i).$$

The study of such determinanl distributions is classical investigated by the orthogonal polynomial method (cf. [Me], [De], [Fo], [Kö]...). Denote by P_ℓ, $\ell \in \mathbb{N}$, the orthogonal polynomials of μ (provided they exist). We agree to normalize them in $\mathrm{L}^2(\mu)$. Since, for each ℓ, P_ℓ is a polynomial of degree at most ℓ, up to some normalization constant c_N, the Vandermonde determinant $\Delta_N(x)$ may indeed be represented by

$$\Delta_N(x) = c_N \det \big(P_{\ell-1}(x_k)\big)_{1 \leq k, \ell \leq N}. \tag{2}$$

This representation allows the complete description of the marginals of Q as determinantal correlation functions in terms of the kernel $K_N(x,y) = \sum_{\ell=0}^{N-1} P_\ell(x) P_\ell(y)$. In particular, the "mean (empirical) spectral measure" $E\big(\frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}\big)$ of Q may be described by the distribution of the Cesaro mean $\frac{1}{N}\sum_{\ell=0}^{N-1} P_\ell^2 d\mu$ of the orthogonal polynomials P_ℓ of μ. Namely, by (2) and orthogonality, for every bounded measurable function $f : \mathbb{R} \to \mathbb{R}$,

$$\int \frac{1}{N} \sum_{i=1}^{N} f(x_i) dQ(x) = \int f \frac{1}{N} \sum_{\ell=0}^{N-1} P_\ell^2 d\mu. \tag{3}$$

Orthogonal polynomials therefore provide a strong tool for the understanding of the associated Coulomb gas. The probability measures $Q = Q^N$ are called accordingly orthogonal polynomial ensembles (cf. [Joha1], [Joha2], [De], [Kö]).

For the choice of Gaussian, Gamma and Beta probability measures μ, the Coulomb gas distribution (1) may be interpreted as the joint distribution of the eigenvalues of the Gaussian, Laguerre and Jacobi Unitary random matrix ensembles, with associated Hermite, Laguerre and Jacobi orthogonal polynomials. This description allows the investigation of the asymptotic behavior of the eigenvalues in both the global and local regimes (cf. [Me], [De], [Fo], [Kö]...). In particular, limit theorems for spectral distributions of eigenvalues may be studied by purely analytical methods with the help of (3) (cf. [H-T], [Le]).

Recently, discrete orthogonal polynomial ensembles have been deeply investigated by K. Johansson [Joha1], [Joha2] in connection with random growth models and the Plancherel measure on partitions. In particular, K. Johansson obtained in [Joha1] (see also [Se1]) large deviations and fluctuation properties for the rightmost charges of the Meixner orthogonal polynomial ensemble similar to the behavior of the largest eigenvalues of the random matrix models. In this line of investigation, we will be interested in this work in the rescaled mean spectral measures

$$\widehat{\mu}^N = E\bigg(\frac{1}{N}\sum_{i=1}^N \delta_{x_i/N}\bigg) \tag{4}$$

and their limiting equilibrium distributions, for the discrete Charlier, Meixner, Krawtchouk and Hahn orthogonal polynomial ensembles. The expectation in (4) is taken with respect to $Q = Q^N$ so the the coordinates $x_i = x_i^N$ actually depend on N. Following the strategy of our paper [Le] for families of orthogonal polynomials of the continuous variable, we study spectral limits using the simple tools of integration by parts for the associated Markov generators, differential equations on Laplace transforms and moment identities for the probability densities $\frac{1}{N}\sum_{\ell=0}^{N-1} P_\ell^2 d\mu$. Moment equations may be used furthermore towards sharp small deviation inequalities on the rightmost charges of the orthogonal polynomial ensembles at the Tracy-Widom rate.

In the first part of this work, we describe a general abstract setting to develop integration by parts in the study of the asymptotic properties of mean spectral measures of discrete orthogonal polynomial ensembles through the representation (3). We actually study first, by differential equations on Laplace transforms, affine transformations of distributions of the type $P_N^2 d\mu$ where P_N is the N-th orthogonal polynomial associated to μ, for the Charlier, Meixner, Krawtchouk and Hahn polynomials (with varying coefficients). As in the continuous case, the arcsine distribution appears as a universal limit law (cf. [M-N-T], [Le]). (Although we deal here with non-compact, discrete orthogonal polynomials with varying coefficients, the approach of [M-N–T] could still be of interest.) We furthermore investigate another regime leading, as limiting law, to some distribution related to the Plancherel measure.

On the basis of the asymptotics of eigenfunction measures $P_N^2 d\mu$, limiting distributions of Cesaro means $\frac{1}{N}\sum_{\ell=0}^{N-1} P_\ell^2 d\mu$ and the associated mean spectral measures $\widehat{\mu}^N$ may be described as mixtures of the arcsine law with an independent uniform distribution. This approach provides an alternate description of the limiting mean spectral measures of the Charlier, Meixner, Krawtchouk and Hahn ensembles, usually identified as equilibrium measures of weighted logarithmic potentials of orthogonal polynomials (cf. [Joha1], [Joha3], [S-T]). These equilibrium measures have been also investigated as asymptotic zero distributions of orthogonal polynomials (with varying coefficients) in [D-S] and [K-VA] (cf. also the references therein). For example, the scaled mean spectral measure $\widehat{\mu}^N$ (4) of the Coulomb gas (1) with respect to the Poisson law with parameter $\theta = \theta_N \sim hN$, $N \to \infty$, $h \geq 0$, converges weakly to the law of the random variable

$$2\sqrt{hU}\,\xi + h + U$$

where ξ has the arcsine law on $(-1, +1)$ and U is uniform on $[0, 1]$ and independent from ξ.

The second part of this paper is concerned with small deviation inequalities on righmost charges. The fluctuations of rightmost charges ("largest eigenvalues") of discrete orthogonal polynomial ensembles may be shown to be governed, as in the continuous case, by the Tracy-Widom distribution of the largest eigenvalue of the Gaussian Unitary random matrix Ensemble [T-W]. As remarkably investigated by K. Johansson [Joha1], the rightmost charge of the Meixner ensemble may actually be interpreted in terms of shape functions. Let indeed $w(i,j)$, $i,j \in \mathbb{N}$, be independent geometric random variables with parameter q, $0 < q < 1$. For $M \geq N \geq 1$, set

$$W = W(M,N) = \max \sum_{(i,j)\in\pi} w(i,j) \tag{5}$$

where the maximum runs over all up/right paths π in $\mathbb{N}^2$ from $(1,1)$ to (M,N). By combinatorial arguments, K. Johansson [Joha1] proved that $W = W(M,N)$ has a Coulomb gas representation (1) with respect to the negative binomial distribution

$$\mu\big(\{x\}\big) = \frac{(\gamma)_x}{x!}\, q^x (1-q)^\gamma, \quad x \in \mathbb{N},$$

with parameters $0 < q < 1$ and $\gamma = M - N + 1$ (with associated Meixner orthogonal polynomials) in the sense that $W = W(M,N)$ is the rightmost charge ("largest eigenvalue") of Q in such a way that, for every $t \geq 0$,

$$\mathbb{P}\big(\{W \leq t\}\big) = Q\Big(\big\{\max_{1\leq i\leq N} x_i \leq t + N - 1\big\}\Big). \tag{6}$$

As presented in [Joha1], this model is closely related to the one-dimensional totally asymmetric exclusion process. It may also be interpreted as a randomly growing Young diagram, a zero-temperature directed polymer in a random environment or a kind of first-passage site percolation model (see also [Kö]). Provided with this representation, asymptotics of Meixner polynomials enabled K. Johansson [Joha1] to show that, for $c \geq 1$, (some multiple of) the random variable

$$\frac{W\big([cN], N\big) - \omega N}{N^{1/3}},$$

where

$$\omega = \frac{(1+\sqrt{qc}\,)^2}{1-q} - 1,$$

converges to the Tracy-Widom distribution F given by

$$F(s) = \exp\bigg(-\int_s^\infty (x-s)u(x)^2 dx\bigg), \quad s \in \mathbb{R}, \tag{7}$$

where $u(x)$ is the unique solution of the Painlevé equation $u'' = 2u^3 + xu$ with the asymptotics $u(x) \sim \mathrm{Ai}\,(x)$ as $x \to \infty$. In the limit as $q \to 1$, the model covers the fluctuation of the largest eigenvalue of the Laguerre Unitary Ensemble, studied independently by I. Johnstone [John]. These fluctutation results are established using the common, at this regime, Airy asymptotics of orthogonal polynomials, as used first by C. Tracy and H. Widom themselves for the largest eigenvalue of a Gaussian unitary random matrix [T-W]. It is a challenging question to establish the same fluctutation result, with the same $(\text{mean})^{1/3}$ rate, for large families of distributions of the $w(i,j)$'s.

It was proved by T. Seppäläinen [Se1] in the simple exclusion process interpretation and by K. Johansson [Joha1] in the Coulomb gas description that the shape function $W = W([cN], N)$ satisfies a large deviation principle above the mean ω. Namely, for each $c \geq 1$ and $\varepsilon > 0$,

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}\Big(\big\{W\big([cN], N\big) \geq \omega N(1+\varepsilon)\big\}\Big) = -J(\varepsilon) \tag{8}$$

for some explicit rate function $J(\varepsilon) > 0$ relying on the equilibrium measure of the Meixner orthogonal polynomial ensemble. Together with a superadditivity argument, K. Johansson ([Joha1], Corollary 2.4) observed that the limit (8) actually yields a large deviation estimate for every N fixed. Namely, for any $N \geq 1$, $c \geq 1$ and $\varepsilon > 0$,

$$\mathbb{P}\Big(\big\{W\big([cN], N\big) \geq \omega N(1+\varepsilon)\big\}\Big) \leq \mathrm{e}^{-NJ(\varepsilon)}. \tag{9}$$

It may be checked on the explicit (but somewhat intricate) expression of the rate function J that $J(\varepsilon) \geq C^{-1}\min(\varepsilon^{3/2}, \varepsilon)$ for some constant $C > 0$ depending upon c and q. In particular, the bound (9) thus reflects the typical tail behavior of the Tracy-Widom distribution F

$$C'\,\mathrm{e}^{-s^{3/2}/C'} \leq 1 - F(s) \leq C\,\mathrm{e}^{-s^{3/2}/C}$$

for s large.

The universality of the Meixner model shows, by appropriate scalings and the explicit expression of the rate function J, that the deviation inequality (9) actually covers a number of further cases of interest. For example, as $q \to 1$, it yields a similar deviation inequality for the largest eigenvalue of the Laguerre Unitary Ensemble. As $q = \frac{\rho}{N^2}$, $N \to \infty$, $\rho > 0$, the Meixner orthogonal ensemble has been shown in

[Joha2] to converge to the ρ-Poissonization of the Plancherel measure on partitions. Since the Plancherel measure is the push-forward of the uniform distribution on the symmetric group S_n by the Robinson-Schensted-Knuth RSK-correspondence which maps a permutation $\sigma \in S_n$ to a pair of standard Young tableaux of the same shape, the length of the first row is equal to the length $L_n(\sigma)$ of the longest increasing subsequence in σ. As a consequence, in this regime, for every $t \geq 0$,

$$\lim_{N \to \infty} \mathbb{P}\Big(\big\{W(N,N) \leq t\big\}\Big) = \mathbb{P}\big(\{L_{\mathcal{N}} \leq t\}\big)$$

where $\mathcal{N}$ is an independent Poisson random variable with parameter $\rho > 0$. The orthogonal polynomial approach may then be used to produce a new proof of the important Baik-Deift-Johansson theorem [B-D-J] on the fluctuations of L_n as

$$\lim_{n \to \infty} \mathbb{P}\big(\big\{L_n \leq 2\sqrt{n} + n^{1/6}s\big\}\big) = F(s), \quad s \in \mathbb{R},$$

where F is the Tracy-Widom distribution (7) (cf. [Joha2]). See [A-D] for a general presentation on the length of the longest increasing subsequence in a random permutation. As a further application of this observation together with the explicit form of the rate function J, the deviation inequality (9) also implies that, for every $n \geq 1$ and every $\varepsilon > 0$,

$$\mathbb{P}\Big(\big\{L_n \geq 2\sqrt{n}\,(1+\varepsilon)\big\}\Big) \leq C \exp\bigg(-\frac{1}{C}\sqrt{n}\,\min\big(\varepsilon^{3/2}, \varepsilon\big)\bigg) \tag{10}$$

where $C > 0$ is numerical in accordance with the Baik-Deift-Johansson theorem. The emphasis of inequalities (9) and (10) lies in their validity for every fixed N or n. They also match the upper tail moderate deviation theorems of [L-M] and [B-D-ML-M-Z].

In the last section of this work, we provide a simplified approach to the preceding non-asymptotic deviation inequalities relying on moment identities, avoiding both the delicate large deviation asymptotics and the random growth representations leading to the crucial superadditivity argument. With respect to the continuous setting developed in the companion paper [Le] where recurrence equations are used to this task, we work out here (factorial) moment identities for the Charlier, Meixner and Krawtchouk ensembles by the integration by parts formula. These moment equations may then be used towards small deviation inequalities at fixed size on the rightmost charges $\max_{1 \leq i \leq N} x_i$ of the Coulomb gas Q associated to these families of orthogonal polynomials, covering thus, by elementary means, the above tail inequalities (9) and (10).

2. Abstract Markov operator framework

In this section, we describe a convenient setting to develop integration by parts methods for Markov operators to reach differential equations on Laplace transforms for the discrete orthogonal polynomial ensembles described in the introduction. Asymptotic distributions are then obtained by identifying the limiting differential equations.

If f is a function on $\mathbb{N}$, set, for $x \in \mathbb{N}$, $\Delta f(x) = f(x+1) - f(x)$. By convention, we agree that $f(-1) = 0$.

Given a probability measure μ on $\mathbb{N}$, assume that it satisfies for all, say, finitely supported functions f on $\mathbb{N}$, the integration by parts formula

$$\int A f d\mu = \int B f(x+1) d\mu = \int B(f + \Delta f) d\mu \tag{11}$$

for some functions A and B on $\mathbb{N}$.

Assume furthermore that μ admits a finite or countable family (P_N) of (normalized in $\mathrm{L}^2(\mu)$) orthogonal polynomials satisfying, for each N and x, the difference equations

$$BP_N(x+1) = (A + B + C_N)P_N(x) - AP_N(x-1) \tag{12}$$

for some constants C_N.

This setting conveniently includes the basic examples we have in mind. We refer throughout this work to the general references [Sz], [Ch]..., and to [K-S], for the basic formulas and identities on orthogonal polynomials used below.

a) The Poisson-Charlier polynomials. Let $\mu = \mu^\theta$ be the Poisson measure on $\mathbb{N}$ with parameter $\theta > 0$, that is

$$\mu^\theta\big(\{x\}\big) = \mathrm{e}^{-\theta} \, \frac{\theta^x}{x!}, \quad x \in \mathbb{N}.$$

Then $A = x$, $B = \theta$ and $C_N = -N$.

b) The Meixner polynomials. Let $\mu = \mu_q^\gamma$ be the so-called negative binomial distribution on $\mathbb{N}$ with parameters $0 < q < 1$ and $\gamma > 0$, that is

$$\mu_q^\gamma\big(\{x\}\big) = \frac{(\gamma)_x}{x!} \, q^x (1-q)^\gamma, \quad x \in \mathbb{N},$$

where $(\gamma)_x = \gamma(\gamma+1)\cdots(\gamma+x-1)$, $x \geq 1$, $(\gamma)_0 = 1$. If $\gamma = 1$, μ is just the geometric distribution with parameter q. Then $A = x$, $B = q(x+\gamma)$ and $C_N = -(1-q)N$.

c) The Krawtchouk polynomials. Let $\mu = \mu_p^K$ be the binomial distribution on $\{0, 1, \ldots, K\}$ with parameter of success $0 < p < 1$ given by

$$\mu_p^K\big(\{x\}\big) = \binom{K}{x} p^x (1-p)^{K-x}, \quad x = 0, 1 \ldots, K.$$

Then $A = (1-p)x$, $B = p(K-x)$ and $C_N = -N$.

d) The Hahn polynomials. Let $\mu = \mu^K_{\alpha,\beta}$, $\alpha, \beta > -1$ or $\alpha, \beta < -K$, be the probability measure on $\{0, 1, \ldots, K\}$ defined by

$$\mu^K_{\alpha,\beta}(\{x\}) = \binom{\alpha + x}{x}\binom{\beta + K - x}{K - x}\frac{K!}{(\alpha + \beta + 2)_K}\,, \quad x = 0, 1 \ldots, K. \tag{13}$$

Then $A = x(K - x + \beta + 1)$, $B = (K - x)(x + \alpha + 1)$ and $C_N = -N(\alpha + \beta + 1 + N)$.

The preceding examples turn out to be in complete analogy with the continuous examples of the Hermite, Laguerre and Jacobi ensembles, leading to analogous integration by parts formulas (cf. [Le]). In particular, it is classical (cf. [Ch], [K-S]...) that the Poisson-Charlier and Laguerre models may be obtained in the limit from the Meixner one, and the Hermite ensemble is a limit of the Laguerre ensemble. The Meixner model is in turn a limit of the Hahn model, which also includes the Jacobi polynomials. All the results presented in the next sections will be consistent with these limits.

3. Differential equations

We use in this section the preceding general framework to derive differential equations on Laplace transforms for the classical discrete orthogonal polynomials. The next statement is the corresponding discrete version of the differential equations for Laplace transforms presented in [Le], and inspired from [H-T].

In the preceding general setting, let N be fixed and consider the Laplace transform of the measure $P_N^2 d\mu$ defined by

$$\varphi(\lambda) = \varphi_N(\lambda) = \int \mathrm{e}^{\lambda x} P_N^2 d\mu \tag{14}$$

assumed to be well-defined on some open domain of the complex variable λ. For a given polynomial

$$R(x) = a_n x^n + \cdots + a_1 x + a_0$$

of the integer variable x, denote by

$$\mathcal{R}(\phi) = a_n \phi^{(n)} + \cdots + a_1 \phi' + a_0 \phi$$

the corresponding differential operator acting on smooth functions ϕ. With some abuse, we write $\Delta\mathcal{R}$ for the differential operator associated to ΔR.

In the next statement, we assume for simplicity that A and B are polynomials of degree at most 2 in the variable x. This assumption covers the previous main examples, but the general principle of the proof may be pushed to reach any finite degree.

Theorem 3.1. *Assume that A and B are polynomials of the variable $x \in \mathbb{N}$ of degree at most 2, and denote by $\mathcal{T}$ the differential operator $\mathcal{A} + \mathcal{B} + C_N\mathcal{I}$. Then $\varphi = \varphi_N$ is solution of the differential equation*

$$\begin{aligned}\mathrm{e}^\lambda \mathcal{B}(\mathcal{A} + \Delta\mathcal{A})\varphi - \mathcal{B}(\mathrm{e}^{-\lambda}\mathcal{A}\varphi) \\ = \mathrm{th}\Big(\frac{\lambda}{2}\Big)\mathcal{T}^2\varphi + \frac{1}{2}\,\mathrm{ch}^{-2}\Big(\frac{\lambda}{2}\Big)\mathcal{T}\Delta\mathcal{T}\varphi - \frac{d}{4}\,\mathrm{e}^{\lambda/2}\mathrm{ch}^{-3}\Big(\frac{\lambda}{2}\Big)\mathcal{T}\varphi\end{aligned}$$

where $d = \Delta^2 A + \Delta^2 B$ is constant.

Proof. By the integration by parts formula (11),

$$\mathcal{A}\varphi = \mathrm{e}^\lambda \int B\mathrm{e}^{\lambda x}\, P_N^2(x+1)d\mu.$$

Therefore, by (12),

$$\begin{aligned}\mathcal{B}(\mathrm{e}^{-\lambda}\mathcal{A}\varphi) &= \int B^2\mathrm{e}^{\lambda x} P_N^2(x+1)d\mu \\ &= \int \big[TP_N(x) - AP_N(x-1)\big]^2\mathrm{e}^{\lambda x}d\mu \\ &= \int T^2\mathrm{e}^{\lambda x}P_N^2 d\mu + \int A^2\mathrm{e}^{\lambda x}P_N^2(x-1)d\mu \\ &\quad - 2\int AT\,\mathrm{e}^{\lambda x}P_N(x)P_N(x-1)d\mu\end{aligned} \tag{15}$$

where $T = A + B + C_N$. By the integration by parts formula (11) again,

$$\int A^2\mathrm{e}^{\lambda x}P_N^2(x-1)d\mu = \mathrm{e}^\lambda \int B(A+\Delta A)\,\mathrm{e}^{\lambda x}P_N^2 d\mu.$$

In the same way,

$$\begin{aligned}&\int AT\,\mathrm{e}^{\lambda x}P_N(x)P_N(x-1)d\mu \\ &\qquad = \mathrm{e}^\lambda \int B(T+\Delta T)\,\mathrm{e}^{\lambda x}P_N(x+1)P_N(x)d\mu \\ &\qquad = \mathrm{e}^\lambda \int (T+\Delta T)\,\mathrm{e}^{\lambda x}\big[TP_N(x) - AP_N(x-1)\big]P_N(x)d\mu \\ &\qquad = \mathrm{e}^\lambda \int T(T+\Delta T)\,\mathrm{e}^{\lambda x}P_N^2 d\mu - \mathrm{e}^\lambda \int A(T+\Delta T)\,\mathrm{e}^{\lambda x}P_N(x)P_N(x-1)d\mu.\end{aligned}$$

Hence,

$$\begin{aligned}&(\mathrm{e}^\lambda + 1)\int AT\,\mathrm{e}^{\lambda x}P_N(x)P_N(x-1)d\mu \\ &\qquad = \mathrm{e}^\lambda \int T(T+\Delta T)\,\mathrm{e}^{\lambda x}P_N^2 d\mu - \mathrm{e}^\lambda \int A\Delta T\,\mathrm{e}^{\lambda x}P_N(x)P_N(x-1)d\mu.\end{aligned}$$

We make a repeated use of (11) and (12) to get that

$$\begin{aligned}
&\int A\Delta T\,\mathrm{e}^{\lambda x} P_N(x)P_N(x-1)d\mu \\
&\qquad = \mathrm{e}^{\lambda}\int B(\Delta T+\Delta^2 T)\,\mathrm{e}^{\lambda x} P_N(x+1)P_N(x)d\mu \\
&\qquad = \mathrm{e}^{\lambda}\int (\Delta T+\Delta^2 T)\,\mathrm{e}^{\lambda x}\big[TP_N(x)-AP_N(x-1)\big]P_N(x)d\mu \\
&\qquad = \mathrm{e}^{\lambda}\int T(\Delta T+\Delta^2 T)\,\mathrm{e}^{\lambda x} P_N^2 d\mu - \mathrm{e}^{\lambda}\int A(\Delta T+\Delta^2 T)\,\mathrm{e}^{\lambda x}\, P_N(x)P_N(x-1)d\mu.
\end{aligned}$$

Since $\Delta^2 T = d$, it follows that

$$\begin{aligned}
&(\mathrm{e}^{\lambda}+1)\int A\Delta T\,\mathrm{e}^{\lambda x} P_N(x)P_N(x-1)d\mu \\
&\qquad\qquad = \mathrm{e}^{\lambda}\int T(\Delta T+\Delta^2 T)\,\mathrm{e}^{\lambda x} P_N^2 d\mu - d\,\mathrm{e}^{\lambda}\int A\,\mathrm{e}^{\lambda x} P_N(x)P_N(x-1)d\mu.
\end{aligned}$$

As a last step, apply again (11) and (12) to get

$$\begin{aligned}
\int A\,\mathrm{e}^{\lambda x} P_N(x)P_N(x-1)d\mu &= \mathrm{e}^{\lambda}\int B\,\mathrm{e}^{\lambda x} P_N(x+1)P_N(x)d\mu \\
&= \mathrm{e}^{\lambda}\int \mathrm{e}^{\lambda x}\big[TP_N(x)-AP_N(x-1)\big]P_N(x)d\mu \\
&= \mathrm{e}^{\lambda}\int T\,\mathrm{e}^{\lambda x} P_N^2 d\mu - \mathrm{e}^{\lambda}\int A\,\mathrm{e}^{\lambda x} P_N(x)P_N(x-1)d\mu
\end{aligned}$$

so that

$$(\mathrm{e}^{\lambda}+1)\int A\,\mathrm{e}^{\lambda x} P_N(x)P_N(x-1)d\mu = \mathrm{e}^{\lambda}\int T\,\mathrm{e}^{\lambda x} P_N^2 d\mu.$$

Collecting all the identities in (15) yields the conclusion. The proof of Theorem 3.1 is complete. □

4. Equilibrium measures

Provided with this general result, we analyze the limiting distributions of affine images of $P_N^2 d\mu$ by the asymptotic differential equations as $N \to \infty$. The main regime is concerned with the limiting arcsine law on $(-1,+1)$ with density $\frac{1}{\pi\sqrt{1-x^2}}$ with respect to Lebesgue measure (and we denote throughout this work by ξ a random variable with this distribution). We also examine, following [Joha2], another regime with limiting law the symmetric discrete probability measure π_ρ on $\mathbb{Z}$ with Laplace transform $J_1(4\sqrt{\rho}\,\mathrm{sh}(\frac{\lambda}{2}))$ where J_1 is the Laplace transform of ξ. It may be shown that

$$\pi_\rho = \sum_{k\in\mathbb{Z}}\sum_{n\geq|k|} \frac{(-1)^{n-k}\rho^n}{(n!)^2}\binom{2n}{n+k}\delta_k$$

when $\rho > 0$, and $\pi_0 = \delta_0$. While this distribution is certainly known, and related to Bessel functions, we have not been able to properly identify it.

On the basis of this analysis, we deduce the corresponding asymptotic behavior of the Cesaro averages $\frac{1}{N}\sum_{\ell=0}^{N-1} P_\ell^2 d\mu$ for the various ensembles addressed in Section 2, as appropriate mixtures of affine transformations of the arcsine law with an independent uniform distribution. By (3), the results are interpreted as equilibrium measures of the (rescaled) mean spectral measures $\widehat{\mu}^N$ for the corresponding Coulomb gas (1). This description provides, as in the continuous case [Le], a new view at equilibrium measures of the classical orthogonal polynomial ensembles (with varying coefficients). For characterizations as minimizers of logarithmic potentials of the orthogonal polynomials cf. [Joha1], [Joha3], [S-T]. Alternate descriptions as asymptotic zero distributions of orthogonal polynomials are provided in [D-S], [K-VA].

We turn to our examples of interest.

a) The Poisson-Charlier polynomials. Let thus $\mu = \mu^\theta$ be the Poisson measure of parameter $\theta > 0$, and denote by P_N, $N \in \mathbb{N}$, the normalized (in $\mathrm{L}^2(\mu)$) Poisson-Charlier orthogonal polynomials for μ^θ. Part ii) has to be compared to Theorem 1.6 of [Joha2].

Proposition 4.1. *Let $X_N = X_N^\theta$ be a random variable on $\mathbb{N}$ with distribution $P_N^2 d\mu^\theta$. Then $\mathbb{E}(X_N) = \theta + N$.*

i) If $\theta N \to \infty$ as $N \to \infty$, then

$$\frac{X_N - (\theta + N)}{2\sqrt{\theta N}}$$

converges weakly to the arcsine law on $(-1, +1)$. In particular, if $\theta = \theta_N \sim hN$, $N \to \infty$, $h \geq 0$, X_N/N converges weakly to $u\xi + v$ where $u^2 = 4h$ and $v = h + 1$.

ii) If $\theta N \to \rho \geq 0$, $N \to \infty$, then $X_N - (\theta + N)$ converges weakly to π_ρ.

Proof. By the recurrence relation

$$xP_N = -\sqrt{\theta}\,\sqrt{N+1}\,P_{N+1} + (\theta + N)P_N - \sqrt{\theta}\,\sqrt{N}\,P_{N-1}$$

of the Poisson-Charlier orthogonal polynomials, it immediately follows that

$$\mathbb{E}(X_N) = \int xP_N^2 d\mu^\theta = \theta + N.$$

According to Theorem 3.1, $\varphi = \varphi_N$ the Laplace transform (14) of $P_N^2 d\mu^\theta$ satisfies the differential equation in $\lambda \in \mathbb{C}$,

$$a_2(\lambda)\varphi'' + a_1(\lambda)\varphi' + a_0(\lambda)\varphi = 0$$

where

$$a_2(\lambda) = \operatorname{th}\Big(\frac{\lambda}{2}\Big),$$
$$a_1(\lambda) = 2(\theta - N)\operatorname{th}\Big(\frac{\lambda}{2}\Big) + \frac{1}{2}\operatorname{ch}^{-2}\Big(\frac{\lambda}{2}\Big) - 2\theta\operatorname{sh}(\lambda),$$
$$a_0(\lambda) = (\theta - N)^2\operatorname{th}\Big(\frac{\lambda}{2}\Big) + \frac{1}{2}(\theta - N)\operatorname{ch}^{-2}\Big(\frac{\lambda}{2}\Big) - \theta\,\mathrm{e}^{\lambda}.$$

Let $v_N = \theta + N$ and denote by $\psi(\lambda) = \psi_N(\lambda) = \mathrm{e}^{-\lambda v_N}\varphi_N(\lambda)$ the Laplace transform of the centered random variable $X_N - v_N$. Then ψ solves the differential equation

$$a_2(\lambda)\psi'' + \big[2v_N a_2(\lambda) + a_1(\lambda)\big]\psi' + \big[v_N^2 a_2(\lambda) + v_N a_1(\lambda) + a_0(\lambda)\big]\psi = 0. \tag{16}$$

We have

$$2v_N a_2(\lambda) + a_1(\lambda) = 4\theta\operatorname{th}\Big(\frac{\lambda}{2}\Big) + \frac{1}{2}\operatorname{ch}^{-2}\Big(\frac{\lambda}{2}\Big) - 2\theta\operatorname{sh}(\lambda)$$

and

$$v_N^2 a_2(\lambda) + v_N a_1(\lambda) + a_0(\lambda) = 4\theta^2\operatorname{th}\Big(\frac{\lambda}{2}\Big) + \theta\operatorname{ch}^{-2}\Big(\frac{\lambda}{2}\Big) - 2\theta(\theta + N)\operatorname{sh}(\lambda) - \theta\,\mathrm{e}^{\lambda}.$$

As $\theta N \to \rho \geq 0$, the limiting differential equation is given by

$$\operatorname{th}\Big(\frac{\lambda}{2}\Big)\Psi'' + \frac{1}{2}\operatorname{ch}^{-2}\Big(\frac{\lambda}{2}\Big)\Psi' - 2\rho\operatorname{sh}(\lambda)\Psi = 0,$$

that is

$$\operatorname{sh}(\lambda)\Psi'' + \Psi' - 4\rho\operatorname{sh}(\lambda)\operatorname{ch}^2\Big(\frac{\lambda}{2}\Big)\Psi = 0. \tag{17}$$

The Laplace transform J_1 of the arcsine law on $(-1, +1)$ solves the differential equation $\lambda J_1'' + J_1' - \lambda J_1 = 0$ (cf. [Le]). By a change of variables, $J_1(4\sqrt{\rho}\operatorname{sh}(\frac{\lambda}{2}))$ satisfies (17). Now, it is easily checked that $J_1(4\sqrt{\rho}\operatorname{sh}(\frac{\lambda}{2}))$ is the Laplace transform of π_ρ. To justify the weak convergence, the moment recurrence relations for the Poisson-Charlier polynomials (see for example Lemma 5.1 below) may be used to show that, for example, $\sup_N \mathbb{E}([X_N^{\theta_N} - v_N]^4) < \infty$, $\theta = \theta_N \sim hN$, $h \geq 0$. Extracting a weakly convergent subsequence, along the imaginary axis, the Fourier transform ψ, as well as its first and second derivative, converge pointwise as $N \to \infty$ to, respectively, Ψ, Ψ' and Ψ''. Part ii) of the proposition thus follows.

When $\theta N \to \infty$, change λ into $\varepsilon\lambda$, $\varepsilon \to 0$, to get that $\psi^{(\varepsilon)}(\lambda) = \psi(\varepsilon\lambda)$ solves the differential equation

$$\lambda{\psi^{(\varepsilon)}}'' + {\psi^{(\varepsilon)}}' - \big[(4N + 2) + \theta\varepsilon^2\lambda^2\big]\theta\varepsilon^2\lambda\psi^{(\varepsilon)} + o(\varepsilon) = 0$$

where $o(\varepsilon)$ means tending to 0 as $\theta\varepsilon^2 \sim 1$. As $\varepsilon = (4\theta N)^{-1/2} \to 0$, $N \to \infty$,

$$\lambda\Psi'' + \Psi' - \lambda\Psi = 0,$$

that is the differential equation satisfied by the Laplace transform J_1 of the arcsine distribution on $(-1,+1)$. We conclude as in the previous case. Proposition 4.1 is thus established. $\square$

As a corollary, we describe the limiting equilibrium distribution of the mean spectral measure $\widehat{\mu}^N = E\big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i/N}\big)$ of the Poisson orthogonal polynomial ensembles as a mixture of the arcsine law with an independent uniform distribution. The result is in complete analogy with the continuous setting [Le].

Corollary 4.2. *Let Q^N be the Coulomb gas distribution (1) associated to the Poisson measure μ^θ with parameter $\theta > 0$. As $\theta = \theta_N \sim hN$, $N \to \infty$, $h \geq 0$, $\widehat{\mu}^N$ converges weakly to $2\sqrt{hU}\xi + h + U$ where ξ has the arcsine law on $(-1,+1)$ and U is uniform on $[0,1]$ and independent from ξ.*

It is worthwhile mentioning that $\sqrt{U}\,\xi$ is distributed according to the semicircle law on $(-1,+1)$ so that the law of $2\sqrt{hU}\xi + h + U$ may be considered as a generalized semicircle distribution. In the classical limit (cf. [Ch], [K-S] from the Poisson-Charlier ensemble to the Hermite ensemble, we effectively recover the semicircle law. Its density may be obtained explicitely as

$$\frac{1}{2} - \frac{1}{\pi}\arcsin\left(\frac{x+h-1}{2\sqrt{hx}}\right)$$

on the interval $[(\sqrt{h}-1)^2, (\sqrt{h}+1)^2]$ provided that $h > 1$, whereas when $h \leq 1$ there is an extra piece equal to 1 on the interval $[0, (\sqrt{h}-1)^2]$ (compare [K-VA]).

Proof. We argue as in the proof of Proposition 4.2 of [Le]. Let X_N^θ be random variables with law $(P_N^\theta)^2 d\mu^\theta$ where μ^θ is the Poisson measure with parameter $\theta > 0$ and P_N^θ is the N-th Poisson-Charlier orthogonal polynomial for μ^θ. For $f : \mathbb{R} \to \mathbb{R}$ bounded and continuous, write

$$\begin{aligned}\int f\Big(\frac{x}{N}\Big)\frac{1}{N}\sum_{\ell=1}^{N-1}(P_\ell^{\theta_N})^2 d\mu^{\theta_N} &= \frac{1}{N}\sum_{\ell=1}^{N-1}\int f\Big(\frac{\ell}{N}\cdot\frac{x}{\ell}\Big)(P_\ell^{\theta_N})^2 d\mu^{\theta_N} \\ &= \int_{1/N}^1 \mathbb{E}\bigg(f\bigg(U_N(t)\cdot\frac{X^{\theta_N}_{NU_N(t)}}{NU_N(t)}\bigg)\bigg)dt\end{aligned}$$

where $U_N(t) = \ell/N$ for $\ell/N < t \leq (\ell+1)/N$, $\ell = 0, 1, \ldots, N-1$ $(U_N(0) = 0)$. Since $U_N(t) \to t$, $t \in [0,1]$, and by Proposition 4.1 i), X_N^θ/N converges weakly to $2\sqrt{h}\,\xi + h + 1$ as $\theta = \theta_N \sim hN$, $N \to \infty$, $h \geq 0$, it follows that

$$\begin{aligned}\lim_{N\to\infty}\int f\Big(\frac{x}{N}\Big)\frac{1}{N}\sum_{\ell=0}^{N-1}(P_\ell^{\theta_N})^2 d\mu^{\theta_N} &= \mathbb{E}\bigg(f\bigg(U\bigg[2\sqrt{\frac{h}{U}}\,\xi + \frac{h}{U} + 1\bigg]\bigg)\bigg) \\ &= \mathbb{E}\Big(f\Big(2\sqrt{hU}\,\xi + h + U\Big)\Big)\end{aligned}$$

where U is uniform on $[0,1]$ and independent from ξ. Together with (3), the claims follows. The proof of Corollary 4.2 is complete. $\square$

b) The Meixner polynomials. Here $\mu = \mu_q^\gamma$ denotes the negative binomial law with parameters $0 < q < 1$ and $\gamma > 0$, and P_N, $N \in \mathbb{N}$, are the normalized (in $\mathrm{L}^2(\mu)$) Meixner orthogonal polynomials for μ_q^γ. As for the Poisson-Charlier case, we deduce the following result.

Proposition 4.3. *Let X_N be a random variable on $\mathbb{N}$ with distribution $P_N^2 d\mu_q^\gamma$. Then $\mathbb{E}(X_N) = v_N$ where*

$$v_N = \frac{N + q(\gamma + N)}{1-q}\,.$$

i) If $\gamma = \gamma_N \sim c'N$, $N \to \infty$, $c' \geq 0$, and q is fixed, then X_N/N converges weakly to $u\xi + v$ where

$$u^2 = \frac{4q(c'+1)}{(1-q)^2} \quad \text{and} \quad v = \frac{1+q(c'+1)}{1-q}\,.$$

ii) If $\gamma = \gamma_N \sim c'N$, $N \to \infty$, $c' \geq 0$, and $qN^2 \to \rho \geq 0$, $N \to \infty$,then $X_N - v_N$ converges weakly to $\pi_{(1+c')\rho}$.

Proof. By the recurrence relation

$$\begin{aligned}(1-q)xP_N = &-\sqrt{q}\,\sqrt{\gamma+N}\,\sqrt{N+1}\,P_{N+1} + \big[N + q(\gamma+N)\big]P_N \\ &- \sqrt{q}\,\sqrt{\gamma+N-1}\,\sqrt{N}\,P_{N-1}\end{aligned}$$

of the Meixner orthogonal polynomials, it immediately follows that

$$\mathbb{E}(X_N) = \int x P_N^2 d\mu_q^\gamma = v_N.$$

According to Theorem 3.1, $\varphi = \varphi_N$ the Laplace transform (14) of $P_N^2 d\mu_q^\gamma$ satisfies the differential equation in $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\,(\lambda) < \log \frac{1}{q}$,

$$a_2(\lambda)\varphi'' + a_1(\lambda)\varphi' + a_0(\lambda)\varphi = 0$$

where

$$\begin{aligned}
a_2(\lambda) &= (1+q)^2 \mathrm{th}\Big(\frac{\lambda}{2}\Big) - 2q\,\mathrm{sh}\,\lambda, \\
a_1(\lambda) &= 2(1+q)\big[q\gamma - (1-q)N\big]\mathrm{th}\Big(\frac{\lambda}{2}\Big) + \frac{(1+q)^2}{2}\,\mathrm{ch}^{-2}\Big(\frac{\lambda}{2}\Big) - 2q\gamma\,\mathrm{sh}\,\lambda - 2q\,\mathrm{ch}\,\lambda, \\
a_0(\lambda) &= \big[q\gamma - (1-q)N\big]^2 \mathrm{th}\Big(\frac{\lambda}{2}\Big) + \frac{(1+q)}{2}\big[q\gamma - (1-q)N\big]\mathrm{ch}^{-2}\Big(\frac{\lambda}{2}\Big) - q\gamma\,\mathrm{e}^{\lambda}.
\end{aligned}$$

Denote by $\psi(\lambda) = \psi_N(\lambda) = \mathrm{e}^{-\lambda v_N}\varphi_N(\lambda)$ the Laplace transform of the centered random variable $X_N - v_N$. Then ψ satisfies the differential equation (16). It is easily checked that, as $qN^2 \to \rho$, $N \to \infty$, for every λ fixed,

$$2v_N a_2(\lambda) + a_1(\lambda) = \frac{1}{2}\,\mathrm{ch}^{-2}\Big(\frac{\lambda}{2}\Big) + o(N)$$

while

$$v_N^2 a_2(\lambda) + v_N a_1(\lambda) + a_0(\lambda) = -2(1+c')\rho\,\mathrm{sh}(\lambda) + o(N).$$

Hence, the limiting differential equation is given, as in the Poisson case, by

$$\mathrm{sh}(\lambda)\Psi'' + \Psi' - 4(1+c')\rho\,\mathrm{sh}(\lambda)\mathrm{ch}^2\Big(\frac{\lambda}{2}\Big)\Psi = 0.$$

The conclusion follows similarly.

When $\gamma N \to \infty$ with q fixed, change λ into $\varepsilon\lambda$, $\varepsilon \to 0$, to get that $\psi^{(\varepsilon)}(\lambda) = \psi(\varepsilon\lambda)$ solves the differential equation

$$\lambda\psi^{(\varepsilon)''} + \psi^{(\varepsilon)'} - \frac{q}{(1-q)^2}\Big[(4N+2) + \frac{q}{(1-q)^2}\,\gamma\varepsilon^2\lambda^2\Big]\gamma\varepsilon^2\lambda\psi^{(\varepsilon)} + o(\varepsilon) = 0$$

where $o(\varepsilon)$ means tending to 0 as $\gamma\varepsilon^2 \sim 1$. We then conclude as for Proposition 4.1. $\square$

The following corollary on the equilibrium measure $\widehat{\mu}^N = E\big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i/N}\big)$ of the Meixner ensemble is established as in the Poisson case.

Corollary 4.4. *Let Q^N be the Coulomb gas distribution (1) associated to the negative binomial law μ_q^γ with parameters $0 < q < 1$ and $\gamma > 0$. As $\gamma = \gamma_N \sim c'N$, $N \to \infty$, $c' \geq 0$, $\widehat{\mu}^N$ converges weakly to*

$$\frac{2}{1-q}\,\sqrt{qU(c'+U)}\,\xi + \frac{1}{1-q}\,\big[U + q(c'+U)\big]$$

where ξ has the arcsine law on $(-1,+1)$ and U is uniform on $[0,1]$ and independent from ξ.

According to the description of the Marchenko-Pastur law in [Le], the law that appears in Corollary 4.4 may be considered as a generalized Marchenko-Pastur distribution. In the limit as $q \to 1$ from the Meixner ensemble to the Laguerre ensemble, we effectively recover a Marchenko-Pastur law. The density of this distribution is explicitely given in [Joha1]. Compare also [K-VA].

c) The Krawtchouk polynomials. Let $\mu = \mu_p^K$ be the binomial distribution on $\{0,1,\ldots,K\}$ with parameter of success $0 < p < 1$ and denote by P_N, $0 \leq N \leq K$, the normalized (in $\mathrm{L}^2(\mu)$) Krawtchouk orthogonal polynomials for μ_p^K. The corresponding result is obtained by the formal change $\gamma = -K$ and $q = -\frac{p}{1-p}$ in the Meixner ensemble.

Of course, this may be justified by carefully reproducing the various steps of the proof of Proposition 4.3.

Proposition 4.5. *Let X_N be a random variable on $\mathbb{N}$ with distribution $P_N^2 d\mu_p^K$. Then $\mathbb{E}(X_N) = v_N$ where*

$$v_N = (1-2p)N + pK.$$

i) If $K = K_N \sim \kappa N$, $N \to \infty$, $\kappa \geq 1$, and p is fixed, then X_N/N converges weakly to $u\xi + v$ where

$$u^2 = 4p(1-p)(\kappa - 1) \quad \text{and} \quad v = p(\kappa - 1) + 1 - p.$$

ii) If $K = K_N \sim \kappa N$, $N \to \infty$, $\kappa \geq 1$, and $pN^2 \to \rho \geq 0$, $N \to \infty$, then $X_N - v_N$ converges weakly to $\pi_{(\kappa-1)\rho}$.

Corollary 4.6. *Let Q^N be the Coulomb gas distribution (1) associated to the binomial law μ_p^K on $\{0, 1, \ldots, K\}$ with parameter of success $0 < p < 1$. As $K = K_N \sim \kappa N$, $N \to \infty$, $\kappa \geq 1$, $\widehat{\mu}^N$ converges weakly to*

$$2\sqrt{p(1-p)U(\kappa - U)}\,\xi + p(\kappa - U) + (1-p)U$$

where ξ has the arcsine law on $(-1, +1)$ and U is uniform on $[0,1]$ and independent from ξ.

d) The Hahn polynomials. Let $\mu = \mu_{\alpha,\beta}^K$ be the distribution (13), and denote by P_N, $0 \leq N \leq K$, the normalized (in $\mathrm{L}^2(\mu)$) Hahn orthogonal polynomials for $\mu_{\alpha,\beta}^K$. Let $\varphi = \varphi_N$ be the Laplace transform (14) of $P_N^2 d\mu_{\alpha,\beta}^K$. According to Theorem 3.1, φ satisfies a differential equation of the fourth order. In the limits however, this equation is turned into a second order equation similar to the ones of the preceding statements. The result is the following. We skip the somewhat tedious details of the proof, and furthermore only examine the first regime.

Proposition 4.7. *Let X_N be a random variable on $\mathbb{N}$ with distribution $P_N^2 d\mu_{\alpha,\beta}^K$. Then $\mathbb{E}(X_N) = v_N$ where*

$$v_N = \frac{(\alpha+\beta+N+1)(\alpha+N+1)(K-N)}{(\alpha+\beta+2N+1)(\alpha+\beta+2N+2)} + \frac{(\alpha+\beta+K+N+1)(\beta+N)N}{(\alpha+\beta+2N)(\alpha+\beta+2N+1)}.$$

If $K = K_N \sim \kappa N$, $N \to \infty$, $\kappa \geq 1$, $\alpha = \alpha_N \sim aN$, $\beta = \beta_N \sim bN$, $N \to \infty$, $a, b \geq 0$ or $a, b \leq -\kappa$, then X_N/N converges weakly to $u\xi + v$ where

$$v = v(a,b,\kappa) = \frac{(a+b+1)(a+1)(\kappa-1) + (a+b+\kappa+1)(b+1)}{(a+b+2)^2},$$

$$u^2 = u^2(a,b,\kappa) = \frac{2(a+b+\kappa+1)(b+1)}{(a+b+1)(a+1)(\kappa-1)(a+b+2)^4} \times \Big[(a+b+1)^2(a+1)^2(\kappa-1)^2 + (a+b+\kappa+1)^2(b+1)^2\Big].$$

Corollary 4.8. *Let Q^N be the Coulomb gas distribution (1) associated to the distribution (13) on $\{0,1,\ldots,K\}$ with parameters α and β, $\alpha,\beta > -1$ or $\alpha,\beta < -K$. As $K = K_N \sim \kappa N$, $\alpha = \alpha_N \sim a$, $\beta = \beta_N \sim b$, $N \to \infty$, $\kappa \geq 1$, $a,b \geq 0$ or $a,b \leq -\kappa$, $\widehat{\mu}^N$ converges weakly to*

$$U\,u\Big(\frac{a}{U},\frac{b}{U},\frac{\kappa}{U}\Big)\xi + U\,v\Big(\frac{a}{U},\frac{b}{U},\frac{\kappa}{U}\Big)$$

where ξ has the arcsine law on $(-1,+1)$ and U is uniform on $[0,1]$ and independent from ξ.

In the presentation of [Le], the law that appears in Corollary 4.8 may be considered as a generalized form of the equilibrium measures for the Jacobi Unitary Ensemble [Fo], [C-C], [Co]. In the limit as $\kappa \to \infty$ from the Hahn ensemble to the Jacobi ensemble, we effectively recover these measures. Compare also [K-VA].

5. Moment identities and small deviation inequalities

In the second part of this work, we develop the integration by parts strategy to examine the deviation inequalities (9) and (10) for the random growth function $W([cN],N)$ and the length L_n of the longest increasing subsequence in a random permutation. We namely derive expressions for the (factorial) moments of the mean spectral measures of the preceding discrete orthogonal polynomial ensembles from which the small deviation bounds on the rightmost charges of the associated Coulomb gas at the appropriate Tracy-Widom asymptotics may easily be deduced. The approach may be seen as an alternate direct argument for results that may be drawn for the earlier investigation by K. Johansson [Joha1] using large deviation principles together with superadditivity of random growth functions.

We only deal here with the Poisson-Charlier and Meixner ensembles. In particular, we have not been able so far to study similarly the Hahn ensemble, due to the lack of explicit generating functions for the corresponding orthogonal polynomials.

Lemma 5.1. *Let $\mu = \mu^\theta$ be the Poisson measure on $\mathbb{N}$ with parameter $\theta > 0$, and denote by P_ℓ, $\ell \in \mathbb{N}$, the normalized (Poisson-Charlier) orthogonal polynomials for μ^θ. Then, for every integer k,*

$$\int x(x-1)\cdots(x-k+1)\frac{1}{N}\sum_{\ell=0}^{N-1} P_\ell^2\, d\mu^\theta = \sum_{i=0}^{k}\theta^{k-i}\binom{k}{i}^2\frac{1}{N}\sum_{\ell=i}^{N-1}\ell(\ell-1)\cdots(\ell-i+1).$$

Proof. By the integration by parts formula (11), for any, say, polynomial function f on $\mathbb{N}$,

$$\int xf\,d\mu^\theta = \theta \int f(x+1)\,d\mu^\theta. \tag{18}$$

For every $x \in \mathbb{N}$ and every $\ell \geq 1$,

$$P_\ell(x+1) = P_\ell(x) + \sqrt{\frac{\ell}{\theta}}\,P_{\ell-1}(x).$$

Iterating, for every $x \in \mathbb{N}$, $k \geq 1$ and $\ell \geq 1$,

$$P_\ell(x+k) = \sum_{i=0}^{k\wedge\ell} \theta^{-i/2} \binom{k}{i} \big[\ell(\ell-1)\cdots(\ell-i+1)\big]^{1/2} P_{\ell-i}(x).$$

Now, by (18), for every $\ell \in \mathbb{N}$ and $k \geq 1$,

$$\int x(x-1)\cdots(x-k+1)P_\ell^2\,d\mu^\theta = \theta^k \int P_\ell^2(x+k)\,d\mu^\theta.$$

Hence, by orthogonality,

$$\int x(x-1)\cdots(x-k+1)P_\ell^2\,d\mu^\theta = \sum_{i=0}^{k\wedge\ell} \theta^{k-i} \binom{k}{i}^2 \ell(\ell-1)\cdots(\ell-i+1).$$

The conclusion follows. □

Note that Lemma 5.1 may be used to recover Corollary 4.2. In particular, one may observe for further purposes that, from the proof of Lemma 5.1, if $X_N = X_N^\theta$ is a random variable with distribution $P_N^2 d\mu^\theta$, and if $\theta = \theta_N \sim hN$, $N \to \infty$, $h \geq 0$, for every fixed k,

$$\lim_{N\to\infty} \frac{1}{N^k}\,\mathbb{E}\Big(\big(X_N\big)^k\Big) = \sum_{i=0}^{k} h^{k-i} \binom{k}{i}^2.$$

On the other hand, we know from Proposition 4.1 that the limiting distribution of X_N/N is $u\xi+v$ where ξ has the arcsine distribution on $(-1,+1)$ and $u^2 = 4h$, $v = h+1$. Therefore, for every $h \geq 0$ and every $k \in \mathbb{N}$,

$$\mathbb{E}\big([u\xi+v]^k\big) = \sum_{i=0}^{k} h^i \binom{k}{i}^2 \tag{19}$$

for $u^2 = 4h$ and $v = h+1$.

The next lemma describes the corresponding moment identities for the Meixner polynomials.

Lemma 5.2. *Let $\mu = \mu_q^\gamma$ be the negative binomial distribution on $\mathbb{N}$ with parameters $0 < q < 1$ and $\gamma > 0$, and denote by $P_\ell = P_\ell^\gamma$, $\ell \in \mathbb{N}$, the normalized (Meixner) orthogonal polynomials for μ_q^γ. Then, for every integer k,*

$$\begin{aligned}\int x(x-1)\cdots(x-k+1)\frac{1}{N}\sum_{\ell=0}^{N-1} P_\ell^2 d\mu_q^\gamma \\ = \Big(\frac{q}{1-q}\Big)^k \sum_{i=0}^{k} q^{-i}\binom{k}{i}^2 \frac{1}{N}\sum_{\ell=i}^{N-1}\frac{(\gamma+\ell)_{k-i}\,\ell\,!}{(\ell-i)!}.\end{aligned}$$

Proof. We make use of the integration by parts formula (11) to obtain that, for any, say, polynomial function f on $\mathbb{N}$,

$$\int xf d\mu_q^\gamma = \frac{\gamma q}{1-q}\int f(x+1)d\mu_q^{\gamma+1}. \tag{20}$$

For every $x \in \mathbb{N}$ and every $\ell \geq 1$,

$$P_\ell^\gamma(x+1) = P_\ell^\gamma(x) + (q-1)\sqrt{\frac{\ell}{q\gamma}}\,P_{\ell-1}^{\gamma+1}(x).$$

Iterating, for every $x \in \mathbb{N}$, $k \geq 1$ and $\ell \geq 1$,

$$P_\ell^\gamma(x+k) = \sum_{i=0}^{k\wedge\ell}\Big(\frac{q-1}{\sqrt{q}}\Big)^i\binom{k}{i}\big[(\gamma)_i\big]^{-1/2}\big[\ell(\ell-1)\cdots(\ell-i+1)\big]^{1/2}P_{\ell-i}^{\gamma+i}(x).$$

Now, by (20), for every $\ell \in \mathbb{N}$ and $k \geq 1$,

$$\int x(x-1)\cdots(x-k+1)\big(P_\ell^\gamma\big)^2 d\mu_q^\gamma = \Big(\frac{q}{1-q}\Big)^k(\gamma)_k\int P_\ell^\gamma(x+k)^2 d\mu_q^{\gamma+k}.$$

Hence, by orthogonality,

$$\begin{aligned}\int x(x-1)\cdots(x-k+1)\big(P_\ell^\gamma\big)^2 d\mu_q^\gamma \\ = \Big(\frac{q}{1-q}\Big)^k(\gamma)_k\sum_{i,j=0}^{k\wedge\ell}\Big(\frac{q-1}{\sqrt{q}}\Big)^{i+j}\binom{k}{i}\binom{k}{j}\big[(\gamma)_i(\gamma)_j\big]^{-1/2} \\ \times\Big[\frac{\ell\,!}{(\ell-i)!}\cdot\frac{\ell\,!}{(\ell-j)!}\Big]^{1/2}\int P_{\ell-i}^{\gamma+i}P_{\ell-j}^{\gamma+j}d\mu_q^{\gamma+k}.\end{aligned}$$

To make use of this identity, we need to evaluate the integrals

$$\int P_{\ell-i}^{\gamma+i}P_{\ell-j}^{\gamma+j}d\mu_q^{\gamma+k}.$$

To this task, we use the generating function of the Meixner polynomials given by

$$\Big(1-\frac{\lambda}{q}\Big)^{x}(1-\lambda)^{-(x+\gamma)}=\sum_{n=0}^{\infty}\frac{\sqrt{(\gamma)_n}}{n!}\,P_n^{\gamma}(x)\Big(\frac{\lambda}{\sqrt{q}}\Big)^{n},\qquad \lambda<q.$$

Using the identity

$$\begin{aligned}\int\Big(1-\frac{\lambda}{q}\Big)^{x}(1-\lambda)^{-(x+\gamma+i)}\Big(1-\frac{\lambda'}{q}\Big)^{x}(1-\lambda')^{-(x+\gamma+j)}d\mu_q^{\gamma+k}&\\ =(1-\lambda)^{k-i}(1-\lambda')^{k-j}\int\bigg[\Big(1-\frac{\sqrt{\lambda\lambda'}}{q}\Big)^{x}\big(1-\sqrt{\lambda\lambda'}\,\big)^{-(x+\gamma+k)}\bigg]^2 d\mu_q^{\gamma+k}&\end{aligned}$$

for $\lambda,\lambda'<q$, it is easily checked, identifying the series coefficients, that for every $n,m\in\mathbb{N}$,

$$\begin{aligned}\bigg(\frac{(\gamma+i)_n(\gamma+j)_m}{n!\,m!}\bigg)^{1/2}\int P_n^{\gamma+i}P_m^{\gamma+j}d\mu_q^{\gamma+k}&\\ =\sum_{(a,b)\in I}\binom{k-i}{a}\binom{k-j}{b}\big(-\sqrt{q}\big)^{a+b}\,\frac{(\gamma+k)_{n-a}}{(n-a)!}&\end{aligned}$$

where

$$I=\big\{(a,b)\in\mathbb{N}^2;0\leq a\leq(k-i)\wedge n,0\leq b\leq(k-j)\wedge m,n-a=m-b\big\}.$$

Hence, for $n=\ell-i$, $m=\ell-j$,

$$\begin{aligned}\bigg(\frac{(\gamma+i)_{\ell-i}(\gamma+j)_{\ell-j}}{(\ell-i)!(\ell-j)!}\bigg)^{1/2}\int P_{\ell-i}^{\gamma+i}P_{\ell-j}^{\gamma+j}d\mu_q^{\gamma+k}&\\ =\sum_{r=i\vee j}^{k\wedge\ell}\binom{k-i}{r-i}\binom{k-j}{r-j}\big(-\sqrt{q}\big)^{2r-(i+j)}\,\frac{(\gamma+k)_{\ell-r}}{(\ell-r)!}\,.&\end{aligned}$$

After some work, it follows that

$$\int x(x-1)\cdots(x-k+1)\big(P_\ell^{\gamma}\big)^2d\mu_q^{\gamma}=\Big(\frac{q}{1-q}\Big)^{k}\sum_{r=0}^{k\wedge\ell}q^{-r}\binom{k}{r}^2\frac{(\gamma+\ell)_{k-r}\,\ell\,!}{(\ell-r)!}\,.$$

The conclusion follows. □

As in the Poisson case, Lemma 5.2 may be used to recover Corollary 4.4.

As announced, we now draw from the moment identities of Lemmas 5.1 and 5.2 non-asymptotic bounds on the tail distributions of the measures $\frac{1}{N}\sum_{\ell=0}^{N-1}P_\ell^2d\mu$. By (3), if $Q=Q^N$ has the Coulomb gas distribution (1), for every t,

$$Q\Big(\big\{\max_{1\leq i\leq N}x_i\geq t\big\}\Big)\leq\int\sum_{i=1}^{N}\mathbf{1}_{\{x_i\geq t\}}dQ(x)=\int_t^{\infty}\sum_{\ell=0}^{N-1}P_\ell^2d\mu.\tag{21}$$

The rightmost charge under the Coulomb gas Q may thus be estimated by bounds on the tail of the distribution $\frac{1}{N}\sum_{\ell=0}^{N-1} P_\ell^2 d\mu$. To this task, the moment identities put forward will prove tight enough to obtain sharp deviation inequalities. These are used to produce sharp estimates, for fixed N, on the random growth functions W described in the introduction, as well as on the distribution of the longest increasing subsequence in a random partition, at the rate given by the Tracy-Widom asymptotics. Below, we carefully examine the various regimes on the Poisson-Charlier ensemble. The Meixner (and Krawtchouk) ensemble is discussed next, with however fewer details.

a) The Poisson-Charlier ensemble. Let $Y = Y_N$ be a random variable with distribution with density $\frac{1}{N}\sum_{\ell=0}^{N-1} P_\ell^2$ with respect to the Poisson measure $\mu = \mu^\theta$ with parameter $\theta > 0$, As announced, we bound above, for fixed N, the probability $\mathbb{P}(\{Y \geq t\})$ according to the various regimes of the parameter θ.

By Lemma 5.1, for any integer $t \geq k \geq 1$ and any $N > k \geq 1$,

$$\begin{aligned}\mathbb{P}\big(\{Y \geq t\}\big) &\leq \frac{1}{t(t-1)\cdots(t-k+1)}\int x(x-1)\cdots(x-k+1)\frac{1}{N}\sum_{\ell=0}^{N-1} P_\ell^2 d\mu^\theta \\ &\leq \frac{(t-k)!}{t!}\sum_{i=0}^{k}\theta^{k-i}\binom{k}{i}^2\frac{1}{N}\sum_{\ell=i}^{N-1}\frac{\ell\,!}{(\ell-i)!}\,.\end{aligned}$$

By Stirling's asymptotics, there is a numerical constant $C \geq 1$ such that for integers w, z with $1 \leq w \leq z/C$,

$$\frac{1}{Cz^w}\,\mathrm{e}^{B(w,z)} \leq \frac{(z-w)!}{z!} \leq \frac{C}{z^w}\,\mathrm{e}^{B(w,z)} \tag{22}$$

where

$$B(w,z) = \sum_{n\geq 1}\frac{1}{n(n+1)}\,\frac{w^{n+1}}{z^n}\,.$$

Hence, for $1 \leq k \leq N/C$,

$$\mathbb{P}\big(\{Y \geq t\}\big) \leq a_0 + \frac{C}{t^k}\cdot\frac{1}{N}\sum_{\ell=1}^{N-1}\ell^k\sum_{i=0}^{k}\left(\frac{\theta}{\ell}\right)^{k-i}\binom{k}{i}^2\mathrm{e}^{B(k,t)-B(i,N)}$$

where $a_0 = \frac{C\theta^k}{Nt^k}\,\mathrm{e}^{B(k,t)}$ and where $C > 0$ is a large enough numerical constant possibly varying from line to line below.

Set now

$$\omega_N' = \left(1+\sqrt{\frac{\theta}{N}}\right)^2 \quad (\geq 1)$$

and assume that $t \geq \omega'_N N$. Then, for every $i \leq k \leq N/C$,

$$B(k,t) - B(i,N) \leq \frac{1}{2N}\Big[\frac{k^2}{\omega'_N} - i^2\Big] + \sum_{n\geq 2}\frac{1}{n(n+1)}\cdot\frac{1}{{\omega'_N}^n N^n}\,[k^{n+1} - i^{n+1}]$$

$$\leq \frac{(k-i)k}{\sqrt{\omega'_N}\,N} - \frac{\sqrt{\omega'_N}-1}{\omega'_N}\cdot\frac{k^2}{N} + (k-i)\sum_{n\geq 2}\frac{1}{n}\cdot\frac{k^n}{{\omega'_N}^n N^n}$$

$$\leq (k-i)\bigg(\frac{k}{\sqrt{\omega'_N}\,N} + \frac{k^2}{\omega'_N N^2}\bigg) - \frac{\sqrt{\omega'_N}-1}{\omega'_N}\cdot\frac{k^2}{N}\,.$$

Therefore, for $\ell \geq 1$,

$$\sum_{i=0}^{k}\Big(\frac{\theta}{\ell}\Big)^{k-i}\binom{k}{i}^2 \mathrm{e}^{B(k,t)-B(i,N)} \leq \exp\bigg(-\frac{\sqrt{\omega'_N}-1}{\omega'_N}\cdot\frac{k^2}{N}\bigg)\sum_{i=0}^{k} d^{k-i}\binom{k}{i}^2$$

where

$$d = d_\ell = \frac{\theta}{\ell}\,\mathrm{e}^{\frac{k}{\sqrt{\omega'_N}\,N}+\frac{k^2}{\omega'_N N^2}}\,.$$

Clearly

$$\sum_{i=0}^{k} d^{k-i}\binom{k}{i}^2 \leq \big(1+\sqrt{d}\,\big)^{2k}$$

that takes care of the main exponential estimate. However, in order to take into account some polynomial factors, a somewhat improved bound is helpful. To this task it is convenient to observe that, by (19),

$$\sum_{i=0}^{k} d^{k-i}\binom{k}{i}^2 = \mathbb{E}\Big(\big[2\sqrt{d}\,\xi + 1 + d\big]^k\Big)$$

where ξ has the arcsine distribution on $(-1,+1)$. Now, for every $j \geq 0$,

$$\mathbb{E}(\xi^{2j+1}) = 0, \quad \mathbb{E}(\xi^{2j}) = \frac{(2j)!}{2^{2j}(j!)^2} \leq \frac{C}{\sqrt{j+1}}\,,$$

so that

$$\mathbb{E}\Big(\big[2\sqrt{d}\,\xi + 1 + d\big]^k\Big) = \sum_{j=0}^{k}\binom{k}{j}\big(2\sqrt{d}\,\big)^j\mathbb{E}(\xi^j)(1+d)^{k-j}$$

$$\leq 2C\sum_{j=0}^{k}\frac{1}{\sqrt{j+1}}\binom{k}{j}\big(2\sqrt{d}\,\big)^j(1+d)^{k-j}$$

$$\leq 2C\big(1+\sqrt{d}\,\big)^{2k}\sum_{j=0}^{k}\frac{1}{\sqrt{j+1}}\binom{k}{j}p^j(1-p)^{k-j}$$

where

$$p = p_\ell = \frac{2\sqrt{d}}{(1+\sqrt{d}\,)^2}\,.$$

A simple binomial estimate shows that

$$\sum_{i=0}^{k} \frac{1}{\sqrt{j+1}} \binom{k}{j} p^j (1-p)^{k-j} \leq \frac{C}{\sqrt{pk}}$$

so that finally

$$\sum_{i=0}^{k} d^{k-i} \binom{k}{i}^2 \leq \frac{C}{\sqrt{pk}} \left(1+\sqrt{d}\,\right)^{2k}.$$

Summarizing,

$$\mathbb{P}\big(\{Y \geq t\}\big) \leq a_0 + \frac{C}{t^k} \exp\left(- \frac{\sqrt{\omega'_N}-1}{\omega'_N} \cdot \frac{k^2}{N}\right) \frac{1}{N} \sum_{\ell=1}^{N-1} \frac{\ell^k}{\sqrt{p_\ell k}} \left(1+\sqrt{d_\ell}\,\right)^{2k}. \tag{23}$$

Now,

$$p_\ell \geq \frac{1}{C} \min\left(\sqrt{\frac{\ell}{\theta}}, \sqrt{\frac{\theta}{\ell}}\right).$$

On the other hand,

$$\begin{aligned}
\frac{1}{N} \sum_{\ell=1}^{N-1} \left(\frac{\ell}{N}\right)^{k\pm\frac{1}{4}} (1+\sqrt{d_\ell}\,)^{2k} &= \frac{1}{N} \sum_{\ell=1}^{N-1} \left(\frac{\ell}{N}\right)^{\pm\frac{1}{4}} \left(\sqrt{\frac{\ell}{N}} + \sqrt{\frac{\theta}{N}}\, \mathrm{e}^{\frac{k}{2\sqrt{\omega'_N}\,N} + \frac{k^2}{2\omega'_N N^2}}\right)^{2k} \\
&\leq \frac{C}{k} \left(1 + \sqrt{\frac{\theta}{N}}\, \mathrm{e}^{\frac{k}{2\sqrt{\omega'_N}\,N} + \frac{k^2}{2\omega'_N N^2}}\right)^{2k},
\end{aligned}$$

while

$$\begin{aligned}
\frac{1}{{\omega'_N}^k} \left(1 + \sqrt{\frac{\theta}{N}}\, \mathrm{e}^{\frac{k}{2\sqrt{\omega'_N}\,N} + \frac{k^2}{2\omega'_N N^2}}\right)^{2k} &= \left(1 + \frac{\sqrt{\omega'_N}-1}{\sqrt{\omega'_N}} \left(\mathrm{e}^{\frac{k}{2\sqrt{\omega'_N}\,N} + \frac{k^2}{2\omega'_N N^2}} - 1\right)\right)^{2k} \\
&\leq \exp\left(\frac{\sqrt{\omega'_N}-1}{\sqrt{\omega'_N}} \left(\frac{k^2}{\sqrt{\omega'_N} N} + \frac{Ck^3}{\omega'_N N^2}\right)\right).
\end{aligned}$$

As a consequence of the preceding and of (23), for $k \leq N/C$,

$$\mathbb{P}\big(\{Y \geq t\}\big) \leq a_0 + \frac{C}{k^{3/2}} \max\left(\frac{\theta}{N}, \frac{N}{\theta}\right)^{1/4} \left(\frac{\omega'_N N}{t}\right)^k \exp\left(\frac{\sqrt{\omega'_N}-1}{\sqrt{\omega'_N}} \cdot \frac{Ck^3}{\omega'_N N^2}\right).$$

Choose now $t = [\omega'_N N + r]$, $r \geq 1$. Together with (21), we finally get that for every $\theta > 0$, $N \geq 1$, $r \geq 1$ and $1 \leq k \leq N/C$, where $C > 0$ is numerical,

$$\begin{aligned}
&Q^N\Big(\big\{\max_{1\leq i\leq N} x_i \geq \omega'_N N + r\big\}\Big) \\
&\quad \leq N a_0 + \frac{CN}{k^{3/2}} \max\left(\frac{\theta}{N}, \frac{N}{\theta}\right)^{1/4} \left(\frac{N}{[\omega'_N N + r]}\right)^k \exp\left(\frac{\sqrt{\omega'_N}-1}{\sqrt{\omega'_N}} \cdot \frac{Ck^3}{\omega'_N N^2}\right).
\end{aligned} \tag{24}$$

We now distinguish between the various regimes of θ.

When $\theta N \to \rho > 0$ as $N \to \infty$, it has been shown by K. Johansson [Joha2] that the Poisson-Charlier ensemble converges to the Poissonization of the Plancherel measure on partitions. In particular,

$$\lim_{N\to\infty} Q^N\Big(\big\{ \max_{1\leq i\leq N} x_i \geq N-1+t\big\}\Big) = \mathbb{P}\big(\{L_{\mathcal{N}} \geq t\}\big)$$

where $L_n(\sigma)$ is the length of the longest increasing subsequence in a random permutation σ of size n, and where $\mathcal{N}$ is an independent Poisson random variable with parameter $\rho > 0$. Setting $k = \delta N$, $\delta > 0$ small enough, and $r = 2\sqrt{\rho}\,\varepsilon \geq 1$, in the limit as $N \to \infty$ we conclude from (24) that

$$\mathbb{P}\big(\big\{L_{\mathcal{N}} \geq 2\sqrt{\rho}\,(1+\varepsilon)\big\}\big) \leq \frac{C}{(\sqrt{\rho}\,\delta^3)^{1/2}}\,\mathrm{e}^{-2\sqrt{\rho}\,\varepsilon\delta + C\sqrt{\rho}\,\delta^3}.$$

Optimizing in $\delta > 0$ shows that, for any $\rho \geq 1$, any $0 < \varepsilon \leq 1$ and some numerical constant $C > 0$,

$$\mathbb{P}\big(\big\{L_{\mathcal{N}} \geq 2\sqrt{\rho}\,(1+\varepsilon)\big\}\big) \leq C\,\mathrm{e}^{-\sqrt{\rho}\,\varepsilon^{3/2}/C}.$$

Now, for $\rho \geq 1$ integer, since $\mathbb{P}(\{L_n \geq t\})$ is increasing in n,

$$\begin{aligned}
\mathbb{P}\big(\big\{L_{\mathcal{N}} \geq 2\sqrt{\rho}\,(1+\varepsilon)\big\}\big) &= \sum_{n=0}^{\infty} \mathrm{e}^{-\rho}\frac{\rho^n}{n!}\,\mathbb{P}\big(\big\{L_n \geq 2\sqrt{\rho}\,(1+\varepsilon)\big\}\big) \\
&\geq \bigg(\sum_{n=\rho}^{\infty} \mathrm{e}^{-\rho}\frac{\rho^n}{n!}\bigg)\mathbb{P}\big(\big\{L_\rho \geq 2\sqrt{\rho}\,(1+\varepsilon)\big\}\big) \\
&\geq \frac{1}{C}\,\mathbb{P}\big(\big\{L_\rho \geq 2\sqrt{\rho}\,(1+\varepsilon)\big\}\big).
\end{aligned}$$

We thus conclude to the following small deviation version of (10).

Proposition 5.3. *Let $L_n(\sigma)$ be the length of the longest increasing subsequence in a permutation σ of size n taken at random uniformly on the symmetric group S_n. For any $n \geq 1$ and $0 < \varepsilon \leq 1$,*

$$\mathbb{P}\big(\big\{L_n \geq 2\sqrt{n}\,(1+\varepsilon)\big\}\big) \leq C\,\mathrm{e}^{-\sqrt{n}\,\varepsilon^{3/2}/C}$$

where $C > 0$ is numerical.

The second case of interest concerns the regime $\theta = \theta_N \sim hN$, $N \to \infty$, $h \geq 0$, for which we described in the preceding section the asymptotic behavior of the mean spectral measure. As we have seen there, whenever $\theta = \theta_N \sim hN$, $N \to \infty$, $h \geq 0$, we know from Corollary 4.2 that Y_N/N converges weakly to $2\sqrt{hU}\,\xi + h + U$. Together with (24) we bound above the probability that Y_N/N exceeds the right endpoint $\omega' = 2\sqrt{h} + h + 1 = (1 + \sqrt{h})^2$ of the support of $2\sqrt{hU}\,\xi + h + U$. This result may also be seen as a consequence of (9) in the limit from the Meixner ensemble to the Poisson-Charlier ensemble.

Proposition 5.4. *Let $Q = Q^N$ be the Coulomb gas with respect to the Poisson measure with parameter $\theta > 0$. Assume that $\theta = hN$, $h > 0$, and set*

$$\omega' = \big(1 + \sqrt{h}\,\big)^2.$$

Then, for every $N \geq 1$ and $0 < \varepsilon \leq 1$,

$$Q\Big(\big\{\max_{1 \leq i \leq N} x_i \geq \omega' N(1+\varepsilon)\big\}\Big) \leq C\,\mathrm{e}^{-N\varepsilon^{3/2}/C}$$

where $C > 0$ only depends on $h > 0$.

Recall that under the distribution Q^N, the coordinates $x_i = x_i^N$, $1 \leq i \leq N$, depend on N. Let us consider then $\max_{1 \leq i \leq N} \frac{x_i^N}{N}$, $N \geq 1$, as a sequence of random variables on some probability space with respective distributions Q^N, $N \geq 1$. When $\theta = \theta_N \sim hN$, $N \to \infty$, $h \geq 0$, we deduce from the preceding and the Borel-Cantelli lemma that

$$\limsup_{N \to \infty} \max_{1 \leq i \leq N} \frac{x_i^N}{N} \leq \omega' = \big(1 + \sqrt{h}\,\big)^2$$

almost surely. The following corollary indicates that it is a true limit. The result is of course in analogy with the (almost sure) convergence of the largest eigenvalue of random matrices to the right endpoint of the support of the equilibrium measure.

Corollary 5.5. *Under the preceding notation,*

$$\lim_{N \to \infty} \max_{1 \leq i \leq N} \frac{x_i^N}{N} = \big(1 + \sqrt{h}\,\big)^2$$

almost surely.

Proof. For simplicity, we write x_i instead of x_i^N. By the preceding and the Borel-Cantelli lemma, it is enough to show that for every $\varepsilon > 0$ small enough,

$$\sum_N Q^N\Big(\big\{\max_{1 \leq i \leq N} x_i \leq N(\omega' - \varepsilon)\big\}\Big) < \infty. \tag{25}$$

Now,

$$Q^N\Big(\big\{\max_{1 \leq i \leq N} x_i \leq N(\omega' - \varepsilon)\big\}\Big) = P\bigg(\bigg\{\frac{1}{N}\sum_{i=1}^N \mathbf{1}_{x_i \leq N(\omega'-\varepsilon)} \geq 1\bigg\}\bigg).$$

Let f be the Lipschitz function which is equal to 1 on $(-\infty, \omega' - \varepsilon]$, to 0 on $[\omega', +\infty)$, and linear in between. Then, since the law of $2\sqrt{hU}\,\xi + h + U$ is absolutely continuous on $[0, \omega']$,

$$\mathbb{E}\Big(f\big(2\sqrt{hU}\,\xi + h + U\big)\Big) < 1.$$

Hence, by Proposition 4.1, for some N_0 large enough and some $0 < \eta < 1$,

$$E\biggl(\frac{1}{N}\sum_{i=1}^{N} f\Bigl(\frac{x_i}{N}\Bigr)\biggr) \leq 1-\eta$$

for all $N \geq N_0$. Therefore,

$$Q^N\Bigl(\bigl\{\max_{1\leq i\leq N} x_i \leq N(\omega'-\varepsilon)\bigr\}\Bigr) \leq Q^N\biggl(\biggl\{\frac{1}{N}\sum_{i=1}^{N} f\Bigl(\frac{x_i}{N}\Bigr) - E\biggl(\frac{1}{N}\sum_{i=1}^{N} f\Bigl(\frac{x_i}{N}\Bigr)\biggr) \geq \eta\biggr\}\biggr).$$

We now make use of the following elementary lemma.

Lemma 5.6. *Let $F : \mathbb{R}^N \to \mathbb{R}$ be a Lipschitz function such that $\int F dQ^N = 0$. Then*

$$\sup_N \frac{1}{N^6}\int F^4 dQ^N \leq 8\,\|F\|_{\mathrm{Lip}}^4 \sup_N \int \frac{1}{N}\sum_{i=1}^{N}\Bigl(\frac{x_i}{N}\Bigr)^4 dQ^N.$$

Proof. By Jensen's inequality and the mean zero hypothesis,

$$\int F^4 dQ^N \leq \iint \bigl|F(x)-F(y)\bigr|^4 dQ^N(x)dQ^N(y).$$

Hence, for each N,

$$\int F^4 dQ^N \leq \|F\|_{\mathrm{Lip}}^4 \iint |x-y|^4 dQ^N(x)dQ^N(y) \leq 8\,\|F\|_{\mathrm{Lip}}^4 \iint |x|^4 dQ^N.$$

By Cauchy-Schwarz,

$$|x|^4 = \biggl(\sum_{i=1}^{N} x_i^2\biggr)^2 \leq N\sum_{i=1}^{N} x_i^4$$

from which the conclusion follows. □

Apply now this lemma to the Lipschitz function

$$F(x) = \frac{1}{N}\sum_{i=1}^{N} f\Bigl(\frac{x_i}{N}\Bigr) - \int \frac{1}{N}\sum_{i=1}^{N} f\Bigl(\frac{x_i}{N}\Bigr)dQ^N, \quad x=(x_1,\ldots,x_N)\in\mathbb{R}^N,$$

for which $\|F\|_{\mathrm{Lip}} \leq N^{-2}\|f\|_{\mathrm{Lip}}$. Together with the preceding, we get, for every $N \geq 1$,

$$Q^N\Bigl(\bigl\{\max_{1\leq i\leq N} x_i \leq N(\omega'-\varepsilon)\bigr\}\Bigr) \leq \frac{8\|f\|_{\mathrm{Lip}}^4}{\eta^4 N^2}\sup_N \int \frac{1}{N}\sum_{i=1}^{N}\Bigl(\frac{x_i}{N}\Bigr)^4 dQ^N.$$

Following Section 4,

$$\lim_{N\to\infty}\int \frac{1}{N}\sum_{i=1}^{N}\Bigl(\frac{x_i}{N}\Bigr)^4 dQ^N = \mathbb{E}\Bigl(\bigl[2\sqrt{hU}\,\xi + h + U\bigr]^4\Bigr) < \infty.$$

Hence (25) holds, and the proof of Corollary 5.5 is easily completed. $\square$

b) The Meixner ensemble. We only discuss in details the second regime, q fixed, $\gamma = \gamma_N \sim c'N$, $N \to \infty$, $c' \geq 0$, but the first one, γ fixed, $q = \frac{\rho}{N^2}$, $N \to \infty$, $\rho > 0$, may be analyzed in complete analogy with the Poisson-Charlier ensemble (although at a somewhat heavier technical level).

Let now $Y = Y_N$ be random variables with distribution with density $\frac{1}{N}\sum_{\ell=0}^{N-1} P_\ell^2$ with respect to μ_q^γ. Whenever $\gamma = \gamma_N \sim c'N$, $N \to \infty$, $c' \geq 0$, we know from Corollary 4.4 that Y_N/N converges weakly to

$$\frac{2}{1-q}\sqrt{qU(c'+U)}\,\xi + \frac{1}{1-q}\big[U + q(c'+U)\big]. \tag{26}$$

In the following, we bound the probability that Y_N/N exceeds the right endpoint

$$\omega' = \frac{2}{1-q}\sqrt{q(c'+1)} + \frac{1}{1-q}\big[1 + q(c'+1)\big] = \frac{\big(1+\sqrt{q(c'+1)}\,\big)^2}{1-q}$$

of the support of (26).

By Lemma 5.2, for any integer $t \geq k \geq 1$ and any $N > k \geq 1$,

$$\begin{aligned}
\mathbb{P}\big(\{Y \geq t\}\big) &\leq \frac{1}{t(t-1)\cdots(t-k+1)}\int x(x-1)\cdots(x-k+1)\frac{1}{N}\sum_{\ell=0}^{N-1} P_\ell^2\, d\mu_q^\gamma \\
&\leq \frac{(t-k)!}{t!}\,\frac{1}{(1-q)^k}\sum_{i=0}^{k} q^{k-i}\binom{k}{i}^2 \frac{1}{N}\sum_{\ell=i}^{N-1}\frac{(\gamma+\ell)_{k-i}\,\ell\,!}{(\ell-i)!}\,.
\end{aligned}$$

Assume that for some $c', c_0' \geq 0$, $\gamma \leq c'N + c_0'$, and set $c = c'+1 \geq 1$. By (22), for some large enough $C > 0$ depending on q, c' and c_0' (and possibly changing from line to line below), for every $0 \leq i \leq k \leq N/C$ and $\ell \leq N-1$,

$$(\gamma+\ell)_{k-i} \leq (cN + c_0')_{k-i} \leq C(cN)^{k-i}\mathrm{e}^{\widetilde{B}}$$

where

$$\widetilde{B} = \frac{(k-i)^2}{cN} - \frac{(k-i)^2}{2(cN + c_0' + k - i)} \leq \frac{(k-i)^2}{2cN} + \frac{Ck^3}{N^2}\,.$$

By (22) again,

$$\mathbb{P}\big(\{Y \geq t\}\big) \leq a_0 + \frac{C}{[(1-q)t]^k}\,\frac{1}{N}\sum_{\ell=1}^{N-1}\ell^k\sum_{i=0}^{k}\Big(\frac{qcN}{\ell}\Big)^{k-i}\binom{k}{i}^2 \mathrm{e}^{B(k,t)-B(i,N)+\widetilde{B}}$$

where $a_0 = \frac{C(qcN)^k}{N[(1-q)t]^k}\,\mathrm{e}^{B(k,t)+\widetilde{B}}$. Assume that $t \geq \omega' N$. As for the Poisson-Charlier ensemble, for every $i \leq k \leq N/C$,

$$\begin{aligned}
B(k,t) - B(i,N) + \widetilde{B} &\leq \frac{(k-i)^2}{2cN} + \frac{k^2}{2\omega' N} - \frac{i^2}{2N} + \frac{Ck^3}{N^2} \\
&\leq \lambda_1(k-i)\,\frac{k}{N} - \lambda_2\,\frac{k^2}{2N} + \frac{Ck^3}{N^2}
\end{aligned}$$

with

$$\lambda_1 = \frac{1}{c}\left(1 + \frac{c-1}{1+\sqrt{qc}}\right) \quad \text{and} \quad \lambda_2 = \frac{2\sqrt{qc}}{1+\sqrt{qc}}\,\lambda_1.$$

It follows that

$$\sum_{i=0}^{k}\left(\frac{qcN}{\ell}\right)^{k-i}\binom{k}{i}^2 \mathrm{e}^{B(k,t)-B(i,N)+\widetilde{B}} \leq \exp\left(-\frac{\lambda_2 k^2}{2N} + \frac{Ck^3}{N^2}\right)\sum_{i=0}^{k} d^{k-i}\binom{k}{i}^2$$

where

$$d = d_\ell = \frac{qcN}{\ell}\,\mathrm{e}^{\lambda_1 \frac{k}{N}}.$$

We then argue as in the proof of Proposition 5.4 to conclude to following result.

Proposition 5.7. *Let $Q = Q^N$ be a Coulomb gas with respect to the negative binomial distribution with parameters $0 < q < 1$ and $\gamma \geq 0$. Assume that $\gamma \leq c'N + c_0'$ for some $c', c_0' \geq 0$, and set*

$$\omega' = \frac{\left(1+\sqrt{q(c'+1)}\,\right)^2}{1-q}.$$

Then, for every $N \geq 1$ and every $0 < \varepsilon \leq 1$,

$$Q\Big(\Big\{\max_{1\leq i\leq N} x_i \geq \omega' N(1+\varepsilon)\Big\}\Big) \leq C\,\mathrm{e}^{-N\varepsilon^{3/2}/C}$$

where $C > 0$ only depends on q, c', c_0'.

As for the Poisson case, we also have the following corollary.

Corollary 5.8. *Under the preceding notation,*

$$\lim_{N\to\infty}\max_{1\leq i\leq N}\frac{x_i^N}{N} = \omega' = \frac{\left(1+\sqrt{q(c'+1)}\,\right)^2}{1-q}$$

almost surely.

Recall the shape function W of (5). By (6), and the change $\omega = \omega' - 1$, we recover from this corollary (cf. [Joha1]) that for every $c \geq 1$,

$$\lim_{N\to\infty}\frac{1}{N}\,W\big([cN], N\big) = \omega = \frac{\left(1+\sqrt{qc}\,\right)^2}{1-q} - 1$$

almost surely (since $\gamma = [cN] - N \sim c'N$, $c' = c-1$). With respect to the large deviation bound (9) of K. Johansson [Joha1], Proposition 5.7 yields the sharp non-asymptotic inequality at the appropriate small deviation rate

$$\mathbb{P}\Big(\big\{W\big([cN], N\big) \geq \omega N(1+\varepsilon)\big\}\Big) \leq C\,\mathrm{e}^{-N\varepsilon^{3/2}/C}$$

for every $N \geq 1$, $0 < \varepsilon \leq 1$, and some constant $C > 0$ only depending on q and c.

c) The Krawtchouk ensemble. We state the corresponding results for the Krawtchouk ensemble.

Proposition 5.9. *Let $Q = Q^N$ be a Coulomb gas with respect to the binomial distribution on $\{0, 1, \ldots, K\}$ with parameter of success $0 < p < 1$. Assume that $K \geq \kappa N$, $\kappa \geq 1$, and set*

$$\omega' = \sqrt{p(\kappa - 1)} + \sqrt{1 - p}.$$

Then, for every $N \geq 1$ and every $0 < \varepsilon \leq 1$,

$$Q\Big(\big\{ \max_{1 \leq i \leq N} x_i \geq \omega' N(1 + \varepsilon)\big\}\Big) \leq C\, \mathrm{e}^{-N\varepsilon^{3/2}/C}$$

where $C > 0$ only depends on p and κ.

Again, Proposition 5.9 may be interpreted as a tail inequality on some shape function in combinatorial probability, namely Seppäläinen's simplified model of directed first passage percolation [Se2]. Indeed, let $M, N \geq 1$ and consider $w(i, j)$, $i, j \in \mathbb{N}$, independent Bernoulli random variables with probability of success p, $0 < p < 1$. Set

$$W = W(M, N) = \max \sum_{(i,j) \in \pi} w(i, j)$$

where now the maximum is running over all sequences $\pi = (k, j_k)_{1 \leq k \leq M}$ such that $1 \leq j_1 \leq \cdots \leq j_M \leq N$, i.e., with respect to (5), up/right paths from $(1, 1)$ to (M, N) with exactly one element in each column. It is shown in [Joha2] that whenever $Q = Q^N$ denotes the Coulomb gas associated to the binomial distribution with size $K = M + N - 1$ and parameter p, then, for every $t \geq 0$,

$$\mathbb{P}\big(\{W \geq t\}\big) = Q^N\Big(\big\{ \max_{1 \leq i \leq N} x_i \geq t + N - 1\big\}\Big).$$

Therefore, as a consequence of Proposition 5.9, for every $N \geq 1$ and $0 < \varepsilon \leq 1$,

$$\mathbb{P}\Big(\big\{W\big([cN], N\big) \geq \omega N(1 + \varepsilon)\big\}\Big) \leq C\, \mathrm{e}^{-N\varepsilon^{3/2}/C}$$

where

$$\omega = \omega' - 1 = \sqrt{pc} + \sqrt{1 - p} - 1$$

and $C > 0$ only depends on p and $c \geq 1$. Presumably, such a small deviation inequality may also be shown to follow from some large deviation principle together with superadditivity. As discussed in [Joha2], [Joha3], the Krawtchouk ensemble is also related to zig-zag paths in random domino tilings of the Aztec diamond, and Proposition 5.9 above may be used to produce bounds for this model.

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