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# A Law of the Iterated Logarithm for Directed Last Passage Percolation

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**Abstract** Let $\widetilde{H}_N$, $N \geq 1$, be the point-to-point last passage times of directed percolation on rectangles $[(1, 1), ([\gamma N], N)]$ in $\mathbb{N} \times \mathbb{N}$ over exponential or geometric independent random variables, rescaled to converge to the Tracy–Widom distribution. It is proved that for some $\alpha_{\sup} > 0$,

$$\alpha_{\sup} \leq \limsup_{N \to \infty} \frac{\widetilde{H}_N}{(\log \log N)^{2/3}} \leq \left(\frac{3}{4}\right)^{2/3}$$

with probability one, and that $\alpha_{\sup} = \left(\frac{3}{4}\right)^{2/3}$ provided a commonly believed tail bound holds. The result is in contrast with the normalization $(\log N)^{2/3}$ for the largest eigenvalue of a GUE matrix recently put forward by E. Paquette and O. Zeitouni. The proof relies on sharp tail bounds and superadditivity, close to the standard law of the iterated logarithm. A weaker result on the liminf with speed $(\log \log N)^{1/3}$ is also discussed.

**Keywords** Directed last passage percolation · Law of the iterated logarithm · Tracy–Widom distribution · Tail inequalities

**Mathematics Subject Classification** 60F15 · 60B20

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## 1 Introduction and Main Results

Let $(X_{i,j})_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ be an infinite array of independent exponential random variables with parameter 1. For $M \geq N \geq 1$, let

$$H(M, N) = \max\left\{ \sum_{(i,j)\in\pi} X_{i,j}\,;\ \pi \in \Pi_{M,N} \right\},$$

where $\Pi_{M,N}$ is the set of all up/right paths in $\mathbb{N}\times\mathbb{N}$ joining $(1, 1)$ to $(M, N)$, be the directed last passage time on the rectangle $[(1, 1), (M, N)]$ in $\mathbb{N}\times\mathbb{N}$.

It is a result due to Johansson [4] that for each $\gamma \geq 1$,

$$\widetilde{H}_N = \frac{H([\gamma N], N) - aN}{bN^{1/3}},$$

where $a = a(\gamma) = (1+\sqrt{\gamma})^2$ and $b = b(\gamma) = \gamma^{-1/6}(1+\sqrt{\gamma})^{4/3}$, converges as $N \to \infty$ to the Tracy–Widom distribution $F_2$. As is by now classical, the distribution $F_2$ arises as the limit of the rescaled largest eigenvalue

$$\widetilde{\lambda}_N = N^{1/6}\big(\lambda_{\max} - 2\sqrt{N}\big)$$

of the Gaussian Unitary Ensemble (GUE) of size $N$ consisting of an Hermitian matrix with entries that are independent (up to the symmetry condition) complex Gaussian variables with mean zero and variance 1.

In addition to this result, it is also shown in [4] that $H(M, N)$ has the same distribution as the largest eigenvalue of the Laguerre Unitary Ensemble, that is of a complex Wishart matrix $AA^*$ where $A$ is an $N \times M$ matrix with entries that are independent complex Gaussian variables with mean zero and variance $\frac{1}{2}$.

It was recently established by Paquette and Zeitouni [8] that (whenever the GUE is constructed from a given infinite array of Gaussian variables on the same probability space),

$$\limsup_{N\to\infty} \frac{\widetilde{\lambda}_N}{(\log N)^{2/3}} = \left(\frac{1}{4}\right)^{2/3}$$

almost surely. It is reasonable to expect (see [8]) that a similar behaviour, of order $(\log N)^{2/3}$, holds for the largest eigenvalue of a Wishart matrix. One crucial aspect of the investigation [8] is that the subsequence $N = k^3$ carries much of the almost sure behaviour (and determines the limiting value) in contrast with the standard geometric subsequences in the classical block argument of the law of the iterated logarithm (which yields the log log normalization). See e.g. [3] for a survey on the classical law of the iterated logarithm and some relevant references. The work [8] also presents a result on the liminf with rate $(\log N)^{1/3}$, although with non-optimal limits at this point. The different powers $\frac{2}{3}$ and $\frac{1}{3}$ of the normalizations reflect the different right and left tails of the Tracy–Widom distribution (1) (as the square-root of the standard law of the iterated logarithm reflects the symmetric Gaussian tails).

However, in the last passage percolation representation, the almost sure behaviour actually turns out to be much smaller and of more classical log log type.

**Theorem 1** *There exists* $\alpha_{\sup} > 0$ *such that*

$$\alpha_{\sup} \le \limsup_{N\to\infty} \frac{\widetilde{H}_N}{(\log\log N)^{2/3}} \le \left(\frac{3}{4}\right)^{2/3}$$

*with probability one.*

It is expected that $\alpha_{\sup} = \left(\frac{3}{4}\right)^{2/3}$ and we actually provide a proof of it based on the suitable tail estimate which is commonly believed to hold true.

There is a similar, although weaker, result for the liminf.

**Theorem 2** *There exists* $0 < \alpha_{\inf} < \infty$ *such that*

$$-\alpha_{\inf} \le \liminf_{N\to\infty} \frac{\widetilde{H}_N}{(\log\log N)^{1/3}}$$

*with probability one.*

We have not been able to show the existence of $\beta_{\inf} > 0$ such that

$$\liminf_{N\to\infty} \frac{\widetilde{H}_N}{(\log\log N)^{1/3}} \le -\beta_{\inf}$$

with probability one. From the Tracy–Widom asymptotics (see (1) below), it may be conjectured that $\alpha_{\inf} = \beta_{\inf} = (12)^{1/3}$.

The proofs of Theorems 1 and 2 rely on precise tail inequalities on the distribution of $H([\gamma N], N)$ together with blocking arguments on the path representation. Roughly speaking, the powers $\frac{2}{3}$ and $\frac{1}{3}$ reflect the right and left tails of the Tracy–Widom distribution

$$1 - F_2(x) = \mathrm{e}^{-\frac{4}{3}x^{3/2}(1+o(1))}, \qquad F_2(-x) = \mathrm{e}^{-\frac{1}{12}x^3(1+o(1))} \tag{1}$$

as $x \to \infty$ (cf. e.g. [1]), whereas the log log is the result of a block argument along geometric subsequences. One main difference with the random matrix models is that the path representation allows for (point-wise) superadditivity, not available for extremal eigenvalues, which leads to strong decorrelation and the almost sure log log behaviour. Indeed, the identity between the law of $H([\gamma N], N)$ and the law of the largest eigenvalue of a Wishart matrix for fixed $N$ does not extend at the level of the joint distributions of the sequences (in particular correlations between two levels). As a consequence, the proofs here turn out to be simpler than the study developed in [8] which is making use of delicate decorrelation estimates obtained via a hard analysis of the determinantal kernel of the GUE.

The picture on the tail inequalities used in this note is a bit incomplete at this point, impacting the main conclusions, although sharp versions should reasonably hold true.

First, the large deviation estimates developed by Johansson [4] show that

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}\big(H([\gamma N], N) \geq (a+\varepsilon)N\big) = -J(\varepsilon) \tag{2}$$

for each $\varepsilon > 0$ where $J$ is an explicit rate function such that $J(x) > 0$ if $x > 0$. On the left of the mean,

$$\lim_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}\big(H([\gamma N], N) \leq (a-\varepsilon)N\big) = -I(\varepsilon) \tag{3}$$

for each $\varepsilon > 0$ where $I(x) > 0$ if $x > 0$.

A superadditivity argument (see [4] and below) actually allows in (2) for the upper bound

$$\mathbb{P}\big(H([\gamma N], N) \geq (a+\varepsilon)N\big) \leq \mathrm{e}^{-J(\varepsilon)N} \tag{4}$$

for any $N \geq 1$ and $\varepsilon > 0$. The relevant information on $J$ is that (cf. [4])

$$\lim_{\varepsilon\to 0} \frac{J(\varepsilon)}{\varepsilon^{3/2}} = \frac{4}{3b^{3/2}} \,. \tag{5}$$

See also [5].

Below the mean, we can make use of the results of [6] in the random matrix interpretation of $H([\gamma N], N)$ as the largest eigenvalue of a Wishart matrix from which, for some $c, C > 0$ only depending on $\gamma$,

$$\mathbb{P}\big(H([\gamma N], N) \leq (a-\varepsilon)N\big) \leq C\,\mathrm{e}^{-c\varepsilon^3 N^2} \tag{6}$$

for every $\varepsilon > 0$ and $N \geq 1$.

To investigate the lower bound in Theorem 1, we will also need a lower bound on the probability in (4), but the sharp version is not so explicit in the literature. First, in the random matrix description [6], it may be stated that

$$\mathbb{P}\big(H([\gamma N], N) \geq (a+\varepsilon)N\big) \geq c\,\mathrm{e}^{-C\varepsilon^{3/2} N} \tag{7}$$

for every $0 \leq \varepsilon \leq 1$ and $N \geq 1$, where $c, C > 0$ only depend on $\gamma$. This inequality is actually not detailed in [6] but, as explained there, the same arguments may be used.

To further discuss this lower bound in a sharper version, it is of interest to widen the scope. The investigation here may indeed be considered similarly for random variables $X_{i,j}$ with a geometric distribution rather than exponential as in the original contribution [4], and Theorems 1 and 2 extend to this setting. The fluctuations and large deviations are actually established initially for geometric distributions in [4] (with suitable values of $a$, $b$ and a suitable $J$ function), the exponential case being seen as the limit of the geometric model with parameter tending to 1. The tail inequality (4) to the right of the mean holds similarly. Below the mean, in the context of geometric random variables, a refined Riemann–Hilbert analysis on the determinantal structure of the underlying Meixner Ensemble has been developed in [2] to show that

$$\log \mathbb{P}\big(H([\gamma N], N) \le aN - xbN^{1/3}\big) = -\frac{1}{12}x^3 + O(x^4N^{-2/3}) + O(\log x) \quad (8)$$

uniformly over $M \le x \le \delta N^{2/3}$ for some (large) constant $M > 0$ and some (small) constant $\delta > 0$, and every $N$ large enough. Although not written explicitly, it is expected that the same method (even in a simpler form) may be used above the mean to yield

$$\log \mathbb{P}\big(H([\gamma N], N) \ge aN + xbN^{1/3}\big) = -\frac{4}{3}x^{3/2} + O(x^2N^{-1/3}) + O(\log x) \quad (9)$$

uniformly over $M \le x \le \delta N^{1/3}$ for some (large) constant $M > 0$ and some (small) constant $\delta > 0$, and every $N$ large enough. The exact value of the lower bound in Theorem 1 relies on (9). What would actually be needed for this proof is that, for every $\eta > 0$, there exist $M > 0$ (large) and $\kappa > 0$ (small) so that for every $N$ large enough, uniformly over $M \le x \le N^{\kappa}$,

$$\mathbb{P}\big(H([\gamma N], N) \ge aN + xbN^{1/3}\big) \ge \mathrm{e}^{-\frac{4}{3}(1+\eta)x^{3/2}}. \quad (10)$$

The same Riemann–Hilbert analysis on the Laguerre Unitary Ensemble yields (8) in the exponential case, and supposedly also (9) (as well as in the GUE setting). In particular, (8) provides a sharp (two-sided) version of (6), while (9) matches (4) and would provide the sharp version of (7). Another support for (9), or rather (10), is Lemma 7.3 of [8] which yields the sharp lower bound in the framework of the GUE with arguments which should similarly apply to the Laguerre Unitary Ensemble and, with perhaps more work, to the Meixner Ensemble. Taking (10) for granted, we will prove the sharp version of Theorem 1 with $\alpha = \big(\frac{3}{4}\big)^{2/3}$ both in the exponential and geometric cases.[1]

## 2 Proofs

Before addressing the proof of the main results, we emphasize a few useful tools. To start with, to avoid some unessential technicalities, in the definition of $H(M, N)$ (and related quantities of the same type), we will actually consider sums $\sum_{(i,j)\in\pi} X_{i,j} - X_{1,1}$ (that is omitting the common initial point of all paths). It is clear that this change does not alter any of the limits studied here.

[1] As indicated by the referee, (9), or rather (10), should actually be much simpler than (8). The probability to estimate basically amounts to $\mathrm{e}^{\mathrm{Tr}(K)}$ where $K$ is the Meixner kernel restricted to the interval $[aN + xbN^{1/3}, aN + N^{1/3+\delta}]$ which is roughly of order $\mathrm{e}^{-\frac{4}{3}x^{3/2}}$ as $x \to \infty$. The estimates provided in [4] should then potentially yield the conclusion. However, we have not been able to make precise the technical steps towards this goal so that we prefer to state the conclusion conditionally, although indeed the sharp result should hold true.

Next, we recall from [4] the simple but basic superadditivity property. For simplicity, we write below $W_N = H([\gamma N], N)$, $N \geq 1$, $\gamma \geq 1$ being fixed throughout this work. Whenever $1 \leq N \leq L$, let $W_{[N,L]}$ be the maximum of up/right paths joining $([\gamma N], N)$ to $([\gamma L], L)$ in $\mathbb{N} \times \mathbb{N}$ (with therefore the preceding convention, that is omitting $X_{[\gamma N],N}$ in the sums). Then, as is immediate,

$$W_N + W_{[N,L]} \leq W_L \tag{11}$$

and $W_N$ and $W_{[N,L]}$ are independent.

Finally, it will be useful to rely on the following maximal inequality of the type of the classical Ottaviani inequality for sums of independent random variables or vectors (cf. [7]).

**Lemma 3** *For any real numbers $t, s$, and any integers $1 \leq K < L$,*

$$\mathbb{P}\Big(\max_{K \leq N < L}(W_N - aN) \geq t\Big) \leq \frac{\mathbb{P}(W_L - aL \geq t + s)}{\min_{K \leq N < L} \mathbb{P}(W_{L-N} - a(L-N) \geq s)}.$$

*Proof* Let $C_K = \{W_K - aK \geq t\}$ and, for $K < N < L$,

$$C_N = \{W_N - aN \geq t\} \cap \bigcap_{K \leq M < N} \{W_M - aM < t\}.$$

The sets $C_N$, $K \leq N < L$, are disjoint and

$$\bigcup_{K \leq N < L} C_N = \Big\{\max_{K \leq N < L}(W_N - aN) \geq t\Big\}.$$

Then,

$$\begin{aligned}
\mathbb{P}(W_L - aL \geq t + s) &\geq \sum_{K \leq N < L} \mathbb{P}\big(W_L - aL \geq t + s, C_N\big) \\
&\geq \sum_{K \leq N < L} \mathbb{P}\big(W_{[N,L]} - a(L-N) \geq s, C_N\big) \\
&= \sum_{K \leq N < L} \mathbb{P}\big(W_{[N,L]} - a(L-N) \geq s\big)\,\mathbb{P}(C_N)
\end{aligned}$$

where we successively used superadditivity and independence of $W_{[N,L]}$ and $C_N$. Finally, $W_{[N,L]}$ has the same distribution as

$$H\big([\gamma L] - [\gamma N] + 1, L - N + 1\big) \geq W_{L-N}$$

from which the conclusion follows. Note for the further purposes that for $N = L - 1$, $W_{L-N}$ might be 0 so that the result is only of interest for $s \leq -a$. □

We address the proof of the limsup theorem. We argue similarly in the exponential and geometric cases, making clear which tail inequality is used.

*Proof of Theorem 1* Let $\phi : \mathbb{N} \to \mathbb{R}$ be defined by $\phi(n) = (\log\log n)^{2/3}$ if $n \geq \mathrm{e}^{\mathrm{e}}$, and $\phi(n) = 1$ if not, and $n_k = [\rho^k]$, $k \in \mathbb{N}$, for some $\rho > 1$ to be made precise below.

We start with the upper bound. For $\beta > 0$ and $k \geq 1$, let

$$A_k = \left\{ \max_{n_{k-1} \leq N < n_k} \frac{\widetilde{H}_N}{\phi(N)} \geq \beta \right\}.$$

We aim at showing that for every $\beta > \left(\frac{3}{4}\right)^{2/3}$, $\sum_k \mathbb{P}(A_k) < \infty$, so that the conclusion follows by the Borel–Cantelli lemma.

By definition of $\widetilde{H}_N$,

$$\mathbb{P}(A_k) \leq \mathbb{P}\Big( \max_{n_{k-1} \leq N < n_k} (W_N - aN) \geq \beta b n_{k-1}^{1/3} \phi(n_{k-1}) \Big).$$

By the maximal inequality of Lemma 3 (with $s = -a$),

$$\mathbb{P}(A_k) \leq \frac{1}{D} \mathbb{P}\big( W_{n_k} - a n_k \geq \beta b n_{k-1}^{1/3} \phi(n_{k-1}) - a \big)$$

where

$$D = \min_{n_{k-1} \leq N < n_k} \mathbb{P}\big( W_{n_k - N} - a(n_k - N) \geq -a \big)$$

The weak convergence of $(W_N - aN)/bN^{1/3}$ easily ensures that for some $d > 0$, $D \geq d$ for every large $k$. Let then $\beta > \beta' > \left(\frac{3}{4}\right)^{2/3}$. Provided $\rho$ is close enough to 1, for every $k$ large enough,

$$\beta n_{k-1}^{1/3} \phi(n_{k-1}) - a \geq \beta' n_k^{1/3} \phi(n_k).$$

Now, by (4) and (5), for every $0 < \eta < 1$ and every $k$ large enough,

$$\mathbb{P}\big( W_{n_k} \geq a n_k + \beta' b n_k^{1/3} \phi(n_k) \big) \leq \mathrm{e}^{-\frac{4}{3}(1-\eta)\beta'^{3/2}\phi(n_k)^{3/2}}.$$

At this point therefore, for every $k$ large enough,

$$\mathbb{P}(A_k) \leq \frac{1}{d} \mathrm{e}^{-\frac{4}{3}(1-\eta)\beta'^{3/2}\phi(n_k)^{3/2}}.$$

Since $\beta' > \left(\frac{3}{4}\right)^{2/3}$, there is $\eta > 0$ such that the right-hand side of the preceding inequality defines the general term of a convergent series. Hence $\sum_k \mathbb{P}(A_k) < \infty$ which completes the proof of the upper bound.

Next, we turn to the lower bound. Recall that $n_k = [\rho^k]$, $k \in \mathbb{N}$, where $\rho > 1$. Assume first that there exists $\alpha > 0$ such that for any $\rho > 1$,

$$\sum_{k \geq 1} \mathbb{P}\big( W_{[n_{k-1}, n_k]} \geq a(n_k - n_{k-1}) + \alpha b (n_k - n_{k-1})^{1/3} \phi(n_k - n_{k-1}) \big) = \infty. \quad (12)$$

Since the random variables $W_{[n_{k-1},n_k]}$, $k \geq 1$, are independent, by the independent part of the Borel–Cantelli lemma, on a set of probability one, infinitely often in $k \geq 1$,

$$W_{[n_{k-1},n_k]} \;\geq\; a(n_k - n_{k-1}) + \alpha b(n_k - n_{k-1})^{1/3}\phi(n_k - n_{k-1}).$$

On the other hand, according to (6) in the exponential case or (8) in both the exponential and geometric cases, for any $\delta > 0$,

$$\sum_{k\geq 1} \mathbb{P}\big(W_{n_{k-1}} \leq a n_{k-1} - \delta b n_{k-1}^{1/3}\phi(n_{k-1})\big) \;<\; \infty.$$

Hence, almost surely, for every $k$ large enough,

$$W_{n_{k-1}} \;\geq\; a n_{k-1} - \delta b n_{k-1}^{1/3}\phi(n_{k-1}).$$

As a consequence of the superadditivity inequality (11), on a set of probability one, infinitely often in $k$,

$$W_{n_k} \;\geq\; a n_k + \alpha b(n_k - n_{k-1})^{1/3}\phi(n_k - n_{k-1}) - \delta b n_{k-1}^{1/3}\phi(n_{k-1}).$$

For every $\alpha' < \alpha$, if $\rho > 1$ is large enough,

$$\alpha b(n_k - n_{k-1})^{1/3}\phi(n_k - n_{k-1}) - \delta b n_{k-1}^{1/3}\phi(n_{k-1}) \;\geq\; \alpha' b n_k^{1/3}\phi(n_k).$$

Therefore, infinitely often in $k$,

$$\widetilde{H}_{n_k} \;=\; \frac{W_{n_k} - a n_k}{b n_k^{1/3}} \;\geq\; \alpha'\phi(n_k).$$

Hence, since $\alpha' < \alpha$ is arbitrary,

$$\limsup_{N\to\infty} \frac{\widetilde{H}_N}{\phi(N)} \geq \alpha \tag{13}$$

almost surely.

It remains to discuss the choice of $\alpha > 0$ so that (12) holds. Set $m_k = n_k - n_{k-1}$ and recall that $W_{[n_{k-1},n_k]}$ has the same distribution as

$$H\big([\gamma n_k] - [\gamma n_{k-1}] + 1, n_k - n_{k-1} + 1\big) \;\geq\; W_{m_k}.$$

On the basis of (7), for some $c, C > 0$ and every $k \geq 1$ large enough,

$$\mathbb{P}\big(W_{m_k} \geq a m_k + \alpha b m_k^{1/3}\phi(m_k)\big) \;\geq\; c\,\mathrm{e}^{-C\alpha^{3/2}\phi(m_k)^{3/2}}.$$

Provided $\alpha > 0$ is small enough, (12) is satisfied. Now, if we agree that (10) holds true, for every $\eta > 0$ and every $k \geq 1$ large enough,

$$\mathbb{P}\big(W_{m_k} \geq am_k + \alpha bm_k^{1/3}\phi(m_k)\big) \ \geq \ \mathrm{e}^{-\frac{4}{3}(1+\eta)\alpha^{3/2}\phi(m_k)^{3/2}}.$$

In this case, (12) is satisfied for all $\alpha < \big(\frac{3}{4}\big)^{2/3}$, yielding by (13) the conjectured lower bound in Theorem 1. $\square$

Next, we turn to the liminf theorem. Since the superadditivity property is only one-sided, a different (weaker) strategy has to be followed, yielding in particular non-optimal bounds.

*Proof of Theorem 2* Let $\psi(n) = (\log\log n)^{1/3}$ if $n \geq \mathrm{e}^{\mathrm{e}}$, and $\psi(n) = 1$ if not. Let $0 < \tau < 1$ and set here $n_k = [\mathrm{e}^{k^\tau}]$, $k \geq 1$.

By the Borel–Cantelli lemma, it is enough to establish that $\sum_k \mathbb{P}(B_k) < \infty$ where

$$B_k \ = \ \bigg\{ \min_{n_{k-1}<N\leq n_k} \frac{\widetilde{H}_N}{\psi(N)} \leq -2\alpha \bigg\}$$

for some (large enough) $\alpha > 0$. For every $k \geq 1$,

$$\begin{aligned}\mathbb{P}(B_k) \leq & \sum_{N=n_{k-1}+1}^{n_k} \mathbb{P}\bigg(\frac{\widetilde{H}_N}{\psi(N)} \leq -2\alpha, \frac{\widetilde{H}_{n_{k-1}}}{\psi(n_{k-1})} \geq -\alpha\bigg) \\ & + \mathbb{P}\bigg(\frac{\widetilde{H}_{n_{k-1}}}{\psi(n_{k-1})} \leq -\alpha\bigg).\end{aligned} \tag{14}$$

Now

$$\mathbb{P}\bigg(\frac{\widetilde{H}_{n_{k-1}}}{\psi(n_{k-1})} \leq -\alpha\bigg) \ = \ \mathbb{P}\big(W_{n_{k-1}} \leq (a-\varepsilon)n_{k-1}\big)$$

where $\varepsilon n_{k-1} = \alpha bn_{k-1}^{1/3}\psi(n_{k-1})$. By (6) in the exponential case or (8) in both the exponential and geometric cases,

$$\mathbb{P}\big(W_{n_{k-1}} \leq (a-\varepsilon)n_{k-1}\big) \ \leq \ C\,\mathrm{e}^{-c(\alpha b)^3\psi(n_{k-1})^3}.$$

The right-hand side defines the general term of a convergent series whenever $\alpha > 0$ is large enough.

Next, by superadditivity (11),

$$\begin{aligned}&\mathbb{P}\bigg(\frac{\widetilde{H}_N}{\psi(N)} \leq -2\alpha, \frac{\widetilde{H}_{n_{k-1}}}{\psi(n_{k-1})} \geq -\alpha\bigg) \\ &\quad\leq \ \mathbb{P}\big(W_{[n_{k-1},N]} \leq a(N-n_{k-1}) - 2\alpha bN^{1/3}\psi(N) + \alpha bn_{k-1}^{1/3}\,\psi(n_{k-1})\big) \\ &\quad\leq \ \mathbb{P}\big(W_{N-n_{k-1}} \leq (a-\varepsilon)(N-n_{k-1})\big)\end{aligned}$$

where now $\varepsilon > 0$ satisfies

$$\varepsilon(N - n_{k-1}) = \alpha b\big[2N^{1/3}\psi(N) - n_{k-1}^{1/3}\,\psi(n_{k-1})\big].$$

By (6) or (8) again,

$$\mathbb{P}\big(W_{N-n_{k-1}} \leq (a-\varepsilon)(N - n_{k-1})\big) \leq C\mathrm{e}^{-c\,\varepsilon^3(N-n_{k-1})^2}.$$

Now, for every $n_{k-1} < N \leq n_k$,

$$2N^{1/3}\psi(N) - n_{k-1}^{1/3}\,\psi(n_{k-1}) \geq N^{1/3}$$

so that $\varepsilon(N - n_{k-1}) \geq \alpha b N^{1/3}$. In addition, for some $\delta > 0$ and every $k$ large enough,

$$\frac{N}{N - n_{k-1}} \geq \frac{n_{k-1}}{n_k - n_{k-1}} \geq \delta\, k^{1-\tau}\,.$$

Hence,

$$\sum_{N=n_{k-1}+1}^{n_k} \mathrm{e}^{-c\,\varepsilon^3(N-n_{k-1})^2} \leq \sum_{N=n_{k-1}+1}^{n_k} \mathrm{e}^{-c\delta(\alpha b)^3 k^{1-\tau}} \leq \mathrm{e}^{k^\tau}\mathrm{e}^{-c\delta(\alpha b)^3 k^{1-\tau}}.$$

Provided $\tau < \frac{1}{2}$, the right-hand side defines the general term of a convergent series in $k$. Together with the previous step and (14), $\sum_k \mathbb{P}(B_k) < \infty$, and the proof of Theorem 2 is complete. □

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