Tail inequalities for the eigenvalues of the GUE and related models: some results and questions

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Abstract

In a concise exposition, the note reviews some basic tail inequalities reflecting the local asymptotics of the eigenvalues (extreme, bulk, spacing) of the Gaussian Unitary Ensemble, and of related models[1], raising at the same time (sometimes implicitly) questions seemingly not investigated in the literature (at least to the author's knowledge).

Let $X = X^n$ be a $n \times n$ random matrix from the Gaussian Unitary Ensemble (GUE), with thus complex Gaussian entries with mean zero and variance one, independent save for the condition that the matrix is Hermitian, with (real) eigenvalues $\lambda_1^n \leq \cdots \leq \lambda_n^n$. As is classical, the (renormalized) spectral measure $\frac{1}{n}\sum_{j=1}^n \delta_{\frac{1}{\sqrt{n}}\lambda_j^n}$ converges, as $n \to \infty$, to the semi-circle distribution with density $\rho(x) = \frac{1}{2\pi}\sqrt{4-x^2}\,\mathbb{1}_{[-2,+2]}(x)$, $x \in \mathbb{R}$.

For simplicity, the eigenvalues will be denoted $\lambda_1 \leq \cdots \leq \lambda_n$.

1 Asymptotics of the GUE eigenvalues

The following are the three main fluctuations results on the local behavior of the eigenvalues of the GUE. They are presented in general references, such as e.g. [1, 44, 42, 23].

[1] Only minimal references are provided throughout the text, and the considered models are not detailed and discussed.

1.1 Asymptotics of the extreme eigenvalues [47]

The normalized largest eigenvalue $\frac{1}{\sqrt{n}}\,\lambda_n$ converges almost surely to 2. In distribution,

$$n^{\frac{1}{6}}\big(\lambda_n - 2\sqrt{n}\,\big) \;\to\; F_{\mathrm{TW}} \tag{1}$$

as $n \to \infty$, where F_{TW} is the Tracy-Widom distribution, firstly described as the Fredholm determinant

$$F_{\mathrm{TW}}(t) \;=\; \det\Big(\big[\mathrm{Id} - K_{\mathrm{Ai}}\big]_{\mathrm{L}^2([t,\infty))}\Big), \quad t \in \mathbb{R},$$

of the integral operator associated to the Airy kernel

$$K_{\mathrm{Ai}}(x,y) \;=\; \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}, \quad x, y \in \mathbb{R},$$

with Ai the special Airy function solution of $\mathrm{Ai}'' = x\mathrm{Ai}$ with the asymptotics $\mathrm{Ai}(x) \sim \frac{e^{-\frac{2}{3}x^{\frac{3}{2}}}}{2\sqrt{\pi}x^{\frac{1}{4}}}$ as $x \to \infty$.

It is part of the main contribution [47] to provide an alternate analytic description of F_{TW} in terms of some Painlevé equation as

$$F_{\mathrm{TW}}(t) \;=\; \exp\bigg(-\int_t^\infty (x-t)q(x)^2 dx\bigg), \quad t \in \mathbb{R},$$

where $q = q(x)$ is the solution to the Painlevé II ordinary differential equation with infinite boundary conditon given by the Airy function

$$q'' \;=\; xq + 2q^3, \qquad q(x) \;\sim\; \mathrm{Ai}(x) \quad x \to \infty.$$

By symmetry, the normalized smallest eigenvalue $\frac{1}{\sqrt{n}}\,\lambda_1$ converges to -2 with fluctuations around this value given similarly by the Tracy-Widom distribution.

1.2 Asymptotics of the bulk eigenvalues [30]

Let $t_j = t_j^n$ be the theoretical location of the j-th particule, $j = 1, \ldots, n$, defined by $\int_{-\infty}^{t_j} \rho(x)dx = \frac{j}{n}$. In the bulk, for $\frac{j}{n} = \frac{j(n)}{n} \to \theta \in (0,1)$,

$$\sqrt{\frac{2\pi^2}{\log n}}\;\rho(t_j)\,\sqrt{n}\,\big(\lambda_j - t_j\sqrt{n}\,\big) \tag{2}$$

converges weakly to the standard normal law.

1.3 Asymptotics of spacings of the bulk eigenvalues [45]

Whenever $\varepsilon n \leq j \leq (1-\varepsilon)n$ for some $\varepsilon \in (0, \frac{1}{2})$, in distribution,

$$\rho(t_j)\,\sqrt{n}\,(\lambda_{j+1} - \lambda_j) \;\to\; F_{\mathrm{G}} \tag{3}$$

as $n \to \infty$, where F_{G} is the Gaudin distribution, firstly described as the derivative of the Fredholm determinant

$$\det\Big(\big[\mathrm{Id} - K_{\mathrm{Sine}}\big]_{\mathrm{L}^2([0,t])}\Big), \quad t \geq 0,$$

of the integral operator associated to the Sine kernel

$$K_{\mathrm{Sine}}(x,y) \;=\; \frac{\sin(\pi(x-y))}{\pi(x-y)}, \quad x, y \in \mathbb{R}.$$

The contribution [31] (prior to [47]) provides an alternate analytic description of F_{G} in terms of some Painlevé equation as

$$F_{\mathrm{G}}(t) \;=\; \frac{\sigma(\pi t)}{\pi t} \exp\bigg(\int_0^{\pi t} \frac{\sigma(x)}{x}\,dx\bigg), \quad t > 0,$$

where $\sigma = \sigma(x)$ solves the Painlevé V ordinary differential equation

$$(x\sigma'')^2 + 4(x\sigma' - \sigma)\big(x\sigma' - \sigma + (\sigma')^2\big) \;=\; 0$$

with boundary condition $\sigma(x) \sim -\frac{x}{\pi}$ as $x \to 0$.

2 Tail inequalities on the GUE eigenvalues

On the basis of the preceding distributional limits, it is of interest to try to quantify the asymptotics via (sharp) small deviation inequalities for fixed n, reflecting the fluctuation statements. Without further mention, $C > 0$ denotes a numerical constant, and $n \geq 1$ is a fixed integer (possibly bigger than some numerical C).

2.1 Tail inequalities on the extreme eigenvalues

The right and left tail asymptotics of the Tracy-Widom distribution F_{TW} are classical (see e.g. [1])

$$1 - F_{\mathrm{TW}}(t) \;\sim\; \frac{1}{t^{\frac{3}{2}}}\, e^{-\frac{4}{3} t^{\frac{2}{3}}}, \quad t \to \infty\,; \qquad F_{\mathrm{TW}}(t) \;\sim\; \frac{1}{|t|^{\frac{1}{8}}}\, e^{-\frac{1}{12}|t|^3}, \quad t \to -\infty.$$

The following statement from [41], relying on a contour analysis of the trace of the underlying Hermite kernel in the Fredholm determinant representation of the gap probability, provides sharp right-tail estimates on the largest eigenvalue in the fluctuation regime towards the Tracy-Widom law in accordance with the preceding asymptotics. (It is mentioned in [41] that it could also be deduced from classical uniform Plancherel-Rotach asymptotics for Hermite polynomials.)

Proposition 1. *There are constants $C > 0$ and $\delta > 0$ such that if $1 \leq t \leq \delta n^{\frac{1}{6}}$,*

$$\frac{1}{Ct^{\frac{3}{2}}}\, e^{-\frac{4}{3}t^{\frac{3}{2}}} \;\leq\; \mathbb{P}\big(n^{\frac{1}{6}}(\lambda_n - 2\sqrt{n}\,) \geq t\big) \;\leq\; \frac{C}{t^{\frac{3}{2}}}\, e^{-\frac{4}{3}t^{\frac{3}{2}}}. \tag{4}$$

With less precision, it is shown in [2] (upper bound) by the Fredholm determinant representation, and in [37] (both upper and lower bounds) by a tridiagonal matrix representation (see the next section), that

$$\frac{1}{C}\, e^{-C\, t^{\frac{3}{2}}} \;\leq\; \mathbb{P}\big(n^{\frac{1}{6}}(\lambda_n - 2\sqrt{n}\,) \geq t\big) \;\leq\; C\, e^{-\frac{1}{C}\, t^{\frac{3}{2}}}$$

for every $t \leq n^{\frac{2}{3}}$. The large deviation tails are more classically Gaussian in the sense that

$$\mathbb{P}\big(\big|\lambda_n - 2\sqrt{n}\,\big| \geq t\big) \;\leq\; C\, e^{-\frac{1}{C}\, t^2} \tag{5}$$

for every $t \geq \sqrt{n}$.

Concerning the left-tail estimates, below the mean, the following statement is available from [37]. It seems that the precise constant in the exponent, and the polynomial prefactor, have not been worked out so far.

Proposition 2. *For some numerical constant $C > 0$,*

$$\frac{1}{C}\, e^{-C\, t^3} \;\leq\; \mathbb{P}\big(n^{\frac{1}{6}}(\lambda_n - 2\sqrt{n}\,) \leq -t\big) \;\leq\; C\, e^{-\frac{1}{C}t^3} \tag{6}$$

for any $t \leq n^{\frac{2}{3}}$ ($t \leq \frac{1}{C}n^{\frac{2}{3}}$ for the lower bound).

2.2 Tail inequalities on the bulk eigenvalues

Gaussian tails are expected from the asymptotics of eigenvalues in the bulk. For every $t \in \mathbb{R}$, let $\mathcal{N}_t = \sum_{j=1}^n 1_{\{\lambda_j \leq t\}}$ be the eigenvalue counting function. Due to the determinantal structure of the GUE, it is known (see e.g. [9]) that $\mathcal{N}_t$ has the same distribution as a sum of independent Bernoulli random variables. Bernstein's inequality for example (although other, sharper, inequalities may be used, such as Bennett's inequality in [46]) may then be applied to get that for every $u \geq 0$

$$\mathbb{P}\Big(\big| \mathcal{N}_t - \mathbb{E}(\mathcal{N}_t) \big| \geq u \Big) \, \leq \, 2 \exp \bigg(- \frac{u^2}{2\sigma_t^2 + u} \bigg)$$

where σ_t^2 is the variance of $\mathcal{N}_t$. Now, on the one hand [46, 28],

$$\sup_{t \in \mathbb{R}} \bigg| \mathbb{E}(\mathcal{N}_t) - n \int_{-\infty}^{t/\sqrt{n}} \rho(x) dx \bigg| \, \leq \, C$$

for some numerical constant $C > 0$, while, for $\varepsilon \in (0, \frac{1}{2})$ [30],

$$\sup_{t \in ((-2+\varepsilon)\sqrt{n}, (2-\varepsilon)\sqrt{n}\,)} \sigma_t^2 \, \leq \, C_\varepsilon \log n.$$

Using that $\mathcal{N}_t \geq j$ if and only if $\lambda_j \leq t$, these observations may be combined to yield the following proposition [15, 46].

Proposition 3. *For every $\varepsilon \in (0, \frac{1}{2})$, there exist $C, c > 0$ only depending on ε such that for all $\varepsilon n \leq j \leq (1 - \varepsilon)n$ and $c \leq t \leq Cn$,*

$$\mathbb{P}\big(\sqrt{n}\, \big| \lambda_j - t_j \sqrt{n}\, \big| \geq t \big) \, \leq \, 4 \exp \bigg(- \frac{t^2}{C(\log n + t)} \bigg). \tag{7}$$

This inequality captures the order of growth of the variance of eigenvalues in the bulk as it implies that $\mathrm{Var}(\sqrt{n}\, \lambda_j) \, \leq \, C \log n$. Sharper tail inequalities may certainly be developed.

2.3 Tail inequalities on the spacings of bulk eigenvalues

The Gaudin density p and all of its derivatives are smooth, bounded, and rapidly decreasing on $(0,\infty)$. The Wigner surmise predicts that $p(s) \sim \frac{1}{2}\pi s\, e^{-\pi s^2/4}$ as $s \to \infty$. It is therefore expected that the tails of $\sqrt{n}\,(\lambda_{j+1} - \lambda_j)$ reflect this decay. The question does not seem to have been investigated so far. It might even be that the asymptotic result has not been extended to the Gaussian Orthogonal Ensemble.

3 Related models

3.1 Gaussian Beta Ensemble

The beta analogues of the GUE are point processes defined on $\mathbb{R}$ which n-level joint density extends the formula for the joint density of the eigenvalues of the GUE with a parameter $\beta > 0$, $\beta = 2$ corresponding to the GUE, $\beta = 1$ to the real Gaussian Orthogonal Ensemble (GOE). They have been represented by tridiagonal random matrices in [18]. On the basis of this representation, it is shown in [43] that, with λ_n the largest particule, or eigenvalue in the matrix representation, of the GβE,

$$n^{\frac{1}{6}}\big(\lambda_n - 2\sqrt{n}\,\big) \,\to\, F_{\mathrm{TW}_\beta} \tag{8}$$

where F_{TW_β} is a general β-Tracy-Widom distribution, with $F_{\mathrm{TW}_\beta} = F_{\mathrm{TW}}$ when $\beta = 2$, and with similar tails

$$1 - F_{\mathrm{TW}_\beta}(t) \,\sim\, \frac{1}{t^{\frac{3\beta}{4}}}\, e^{-\frac{2\beta}{3}t^{\frac{2}{3}}}, \quad t \to \infty\,; \qquad F_{\mathrm{TW}_\beta}(t) \,\sim\, \frac{1}{|t|^{\frac{\beta}{16}}}\, e^{-\frac{\beta}{24}|t|^3}, \quad t \to -\infty.$$

The upper bound in Proposition 1 has been extended in [24] to the GOE ($\beta = 1$) as

$$\mathbb{P}\big(n^{\frac{1}{6}}(\lambda_n - 2\sqrt{n}\,) \geq t\big) \,\leq\, \frac{C}{t^{\frac{3}{4}}}\, e^{-\frac{2}{3}t^{\frac{3}{2}}} \tag{9}$$

for any $1 \leq t \leq n^{\frac{1}{4}}$, using the uniform Plancherel-Rotach asymptotic estimates for Hermite polynomials.

In general, it is shown in [37], on the basis of the tridiagonal description, that for some numerical constant $C > 0$, and every $\beta \geq 1$,

$$\frac{1}{C}\, e^{-C\beta\, t^{\frac{3}{2}}} \,\leq\, \mathbb{P}\big(n^{\frac{1}{6}}(\lambda_n - 2\sqrt{n}\,) \geq t\big) \,\leq\, C\, e^{-\frac{1}{C}\beta\, t^{\frac{3}{2}}} \tag{10}$$

and

$$\frac{1}{C}\, e^{-C\beta t^3} \;\leq\; \mathbb{P}\big(n^{\frac{1}{6}}(\lambda_n - 2\sqrt{n}\,) \leq -t\big) \;\leq\; C\, e^{-\frac{1}{C}\beta t^3} \tag{11}$$

for any $t \leq n^{\frac{2}{3}}$ ($t \leq \frac{1}{C} n^{\frac{2}{3}}$ for the lower bound of the left-tail inequality).

A version of the asymptotics in the bulk for the GβE is developed in [3]. Beta ensembles with non-quadratic (non-Gaussian) potentials may also be considered in this framework (cf. e.g. [10, 11]).

3.2 Wigner matrices

The local asymptotics of eigenvalues as outlined in Section 1 for the GUE have been extended to large families of Wigner matrices (cf. [44, 23, 45] and subsequent works).

The sharp upper bound in (4) of Proposition 1 has been extended in [24] to families of (both real and complex) Wigner matrices by the local relaxation flow and Green function comparison method introduced in [22] (cf. [23]), although only up to the range $t \leq \delta(\log n)^{\frac{2}{3}}$. The lower bound in the left-tail inequality (6) in Proposition 2 is extended up to $t \leq \delta(\log n)^{\frac{1}{3}}$. A weaker version have been discussed previously in [25].

3.3 Wishart matrices and Laguerre Beta Ensemble

The Laguerre Unitary Ensembles LUE is the ensemble YY^* in which Y is an $n \times m$ matrix comprised of i.i.d. complex Gaussians with mean zero and variance one. The real orthogonal ensemble may be considered as well, and is in particular of interest towards applications in multivariate statistics. There is a general beta version LβE of the associated point process of the eigenvalues (with, as for the GβE, the correspondence $\beta = 2$ for the LUE and $\beta = 1$ for the real Laguerre Orthogonal Ensemble), which may be also built from tridiagonal matrices [18]. The discussion below is presented for this family.

In the asymptotic regime, it is namely proved in [43] that for $m + 1 > n \to \infty$ with $\frac{m}{n} \to \gamma \geq 1$, and $\lambda_{n,m}$ the largest particule, or eigenvalue, of the LβE,

$$\frac{(\sqrt{mn}\,)^{1/3}}{(\sqrt{m} + \sqrt{n}\,)^{4/3}} \Big(\lambda_{n,m} - \big(\sqrt{m} + \sqrt{n}\,\big)^2\Big) \;\to\; F_{\mathrm{TW}_\beta}. \tag{12}$$

For simplicity in the exposition, assume $m = [\gamma n]$, $\gamma \geq 1$, and set $\lambda_n = \lambda_{n,m}$. Then the results in [37], relying again on the tridiagonal representation, show that, for some numerical constant $C > 0$, with $a = a(\gamma) = (1 + \sqrt{\gamma})^2$,

$$\mathbb{P}\big(n^{-\frac{1}{3}}(\lambda_n - an\,) \geq t\big) \;\leq\; C\, e^{-\frac{1}{C}\beta\sqrt{\gamma}\, t^{\frac{3}{2}}} \tag{13}$$

and

$$\mathbb{P}\big(n^{-\frac{1}{3}}(\lambda_n - an\,) \leq -t\big) \;\leq\; C\, e^{-\frac{1}{C}\beta\sqrt{\gamma}\, t^3} \tag{14}$$

for any $t \leq n^{\frac{2}{3}}$.

The analysis of [41] and [24] towards Proposition 1 and (9) supports the idea that a sharp form of, at least (13) and its lower bound, should be available for $\beta = 2$ and $\beta = 1$. The left-tail lower bound

$$\mathbb{P}\big(n^{-\frac{1}{3}}(\lambda_n - an\,) \leq -t\big) \;\geq\; \frac{1}{C}\, e^{-C\beta\sqrt{\gamma}\, t^3} \tag{15}$$

is established in [6] (see also [21]).

When $\gamma > 1$, the lowest eigenvalue exhibits a similar Tracy-Widom asymptotics at the so-called soft-edge, with expected similar tails. When $\gamma = 1$, there is an hard-edge phenomenon. In this case actually, the law of the least eigenvalue is explicit (at least in the real case), as a solution of the differential equation for the Tricomi function [19], from which tail inequalities might potentially be deduced. In the special case $m = n + 1$, the distribution is simply exponential.

The right-tail inequality (13) is extended to families of non-Gaussian covariance matrices with sub-Gaussian entries in [25].

3.4 Directed last passage percolation

Let $(Z_{i,j})_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ be an infinite array of independent exponential random variables with parameter 1. For $m \geq n \geq 1$, let

$$H(m,n) \;=\; \max\bigg\{ \sum_{(i,j)\in\pi} Z_{i,j}\,;\; \pi \in \Pi_{m,n} \bigg\},$$

where $\Pi_{m,n}$ is the set of all up/right paths in $\mathbb{N}\times\mathbb{N}$ joining $(1,1)$ to (m,n), be the directed last passage time on the rectangle $[(1,1),(m,n)]$ in $\mathbb{N}\times\mathbb{N}$, also known as point-to-point last passage time.

It is a result of [32] that, for each $\gamma \geq 1$,

$$\frac{1}{bn^{\frac{1}{3}}}\,\big(H([\gamma n],n) - an\big) \;\to\; F_{\mathrm{TW}} \tag{16}$$

as $n \to \infty$, where $a = a(\gamma) = (1+\sqrt{\gamma})^2$ and $b = b(\gamma) = \gamma^{-\frac{1}{6}}(1+\sqrt{\gamma})^{\frac{4}{3}}$.

The study in [32] also provides large deviation estimates

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\big(H([\gamma n],n) \geq (a+\varepsilon)n\big) \;=\; -J(\varepsilon) \tag{17}$$

for each $\varepsilon > 0$ where J is an explicit rate function such that $J(x) > 0$ if $x > 0$. On the left of the mean,

$$\lim_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}\big(H([\gamma n], n) \leq (a-\varepsilon)n\big) \; = \; -I(\varepsilon) \tag{18}$$

for each $\varepsilon > 0$ where $I(x) > 0$ if $x > 0$. A superadditivity argument allows in (17) for the upper bound

$$\mathbb{P}\big(H([\gamma n], n) \geq (a+\varepsilon)n\big) \; \leq \; e^{-J(\varepsilon)n} \tag{19}$$

for any $n \geq 1$ and $\varepsilon > 0$, the relevant information on J being that $J(\varepsilon) \sim \frac{4}{3}\big(\frac{\varepsilon}{b}\big)^{\frac{3}{2}}$ as $\varepsilon \to 0$. (cf. [32]).

In the following, set $H_n = H([\gamma n], n)$, $\gamma \geq 1$, for simplicity.

It is actually shown in [32] that $H(m,n)$ has the same distribution as the largest eigenvalue of the LUE (with the only minor modification that the matrix Y of the complex Wishart matrix YY^* has entries that are independent complex Gaussian variables with mean zero and variance $\frac{1}{2}$). As such, the bounds (13) and (14) are available for H_n in the form

$$\mathbb{P}\big(n^{-\frac{1}{3}}(H_n - an\,) \geq t\big) \; \leq \; C\,e^{-\frac{1}{C}t^{\frac{3}{2}}} \tag{20}$$

and

$$\mathbb{P}\big(n^{-\frac{1}{3}}(H_n - an\,) \leq -t\big) \; \leq \; C\,e^{-\frac{1}{C}t^{3}} \tag{21}$$

for $t \leq n^{\frac{2}{3}}$, where $C > 0$ only depends on γ.

The correct leading order terms in the upper tail exponent (20) seems to be obtained in [7] by asymptotic analysis on the related Totally Asymmetric Simple Exclusion Process (TASEP) model, at least for the point-to-line (or half-line) model of last passage percolation.

The preceding asymptotic investigation may be considered similarly for random variables $Z_{i,j}$ with a geometric distribution rather than exponential, as in the original contribution [32]. (Since then, a number of further distributions and models exhibiting such behaviours have been investigated.) The fluctuations are actually established initially for geometric distributions in [32] (with suitable values of a, b), the exponential case being seen as the limit of the geometric model with parameter tending to 1. The large deviation tail inequality (19) to the right of the mean holds similarly. Actually, the classical orthogonal polynomial hierarchy, from the Meixner Ensemble (geometric law) to the Laguerre one (exponential and gamma laws), and then the Gaussian Ensemble, suggests that (19) holds true for the GUE as

$$\mathbb{P}\big(\lambda_n \geq (2+\varepsilon)\sqrt{n}\,\big) \; \leq \; e^{-J(\varepsilon)n} \tag{22}$$

with $J(\varepsilon) = \int_0^\varepsilon \sqrt{x(x+4)}dx$, $\varepsilon > 0$ (cf. [8]). Again $J(\varepsilon)$ is of the order of $\varepsilon^{\frac{3}{2}}$ as $\varepsilon \to 0$.

Below the mean, in the context of geometric random variables, a refined Riemann-Hilbert analysis on the determinantal structure of the underlying Meixner Ensemble has been developed in [5] to show that

$$\log \mathbb{P}\big(n^{-\frac{1}{3}}(H_n - a_n) \leq -bt\big) \ = \ -\frac{1}{12}\, t^3 + O(t^4 n^{-\frac{2}{3}}) + O(\log t) \tag{23}$$

uniformly over $M \leq t \leq \delta n^{\frac{2}{3}}$ for some (large) constant $M > 0$ and some (small) constant $\delta > 0$, and every n large enough. Although not written explicitly, it is expected that the same method (even in a simpler form) may be used above the mean to yield

$$\log \mathbb{P}\big(n^{-\frac{1}{3}}(H_n - a_n) \geq bt\big) \ = \ -\frac{4}{3}\, t^{\frac{3}{2}} + O(t^2 n^{-\frac{1}{3}}) + O(\log t) \tag{24}$$

uniformly over $M \leq t \leq \delta n^{\frac{1}{3}}$, again for some (large) constant $M > 0$ and some (small) constant $\delta > 0$, and every n large enough. The same Riemann-Hilbert analysis on the LUE yields (or should yield) (23) in the exponential case, and supposedly also (24) (as well as in the GUE setting). It is not entirely clear however that the preceding expansions can be used towards sharp forms of (20) and (21).

There is a continuous version of the last passage percolation model in terms of families of Brownian motions $B^1, \ldots, B^n$ as

$$\sup_{0=t_0<\cdots<t_{n-1}<t_n=1} \sum_{i=1}^{n} (B^i_{t_i} - B^i_{t_{i-1}}), \tag{25}$$

which has the same distribution as the largest eigenvalue of the GUE [40, 29]. As such, the tail inequalities are described in the GUE section. Tail inequalities at the Tracy-Widom rates for the associated polymer

$$\int_{0<t_1<\cdots<t_{n-1}<1} \exp\bigg(\sum_{i=1}^{n} (B^i_{t_i} - B^i_{t_{i-1}})\bigg) dt_1 \cdots dt_{n-1} \tag{26}$$

have been determined in [36].

3.5 Length of the longest increasing subsequence

If $\ell_n(\sigma)$ denotes the length of the longest increasing subsequence of the permutation σ of the elements $\{1, \ldots, n\}$, it has been shown in [4] that, for σ chosen randomly over all permutations,

$$n^{-\frac{1}{6}}\big(\ell_n(\sigma) - 2\sqrt{n}\,\big) \tag{27}$$

converges weakly, as $n \to \infty$, to the Tracy-Widom distribution F_{TW}.

The Meixner model with geometric parameter q alluded to in the preceding section is also of interest for the Plancherel measure and the length of the longest increasing subsequence in a random permutation. It was namely observed in [33] that, as $q = \theta$, $n \to \infty$, the Meixner orthogonal polynomial Ensemble converges to the θ-Poissonization of the Plancherel measure on partitions. Since the Plancherel measure is the push-forward of the uniform distribution on the symmetric group by the Robinson-Schensted-Knuth correspondence, in this regime, the asymptotic Meixner model yields a Poissonized version of ℓ_n, which could possibly allow for tail inequalities in (27).

Some moderate deviation asymptotics are described in [38, 39] following the original investigation [4].

3.6 Height functions in the KPZ universality class

The directed exponential/geometric last passsage percolation, and the related Totally Asymmetric Simple Exclusion Process (TASEP) model, are most studied members of the one-dimensional Kardar-Parisi-Zhang (KPZ) universality class of stochastic growth models. In both cases, they define a height function $h(x,t)$, where x stands for (one-dimensional) space and t for time. For height functions in the KPZ universality class, at a large time t, under the $\frac{2}{3} - \frac{1}{3}$ scaling in the form

$$\frac{h(ut^{\frac{2}{3}}, t) - t h_{\mathrm{m}}(ut^{-\frac{1}{3}})}{t^{\frac{1}{3}}} \tag{28}$$

with $h_{\mathrm{m}}(x) = \lim_{t\to\infty} \frac{1}{t}\, h(x,t)$ being the (deterministic) macroscopic limit shape, a non trivial limit process should arise, typically related to the GUE Tracy-Widom distribution (Airy processes). It is therefore expected that tail inequalities, similar to the ones presented in the prior sections, may be produced for height functions of various models in the KPZ universality class (including the KPZ equation itself). In this regard, the following non exhaustive recent references (and related works) may be quoted: [34, 13, 14, 12, 16, 17, 20, 21, 26, 27, 35]...

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