Some geometric inequalities for Gaussian measures

Gaussian measures share some surprising geometric inequalities. The isoperimetric inequality, already discussed in [1], is one of them, and some others are presented here. Among them, the Gaussian correlation inequality has aroused great interest over the last 60 years.

Let, as usual, γ_n be the standard Gaussian measure on the Borel sets of $\mathbb{R}^n$, with density $\frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}|x|^2}$, $x \in \mathbb{R}^n$, with respect to the Lebesgue measure. The *Gaussian correlation inequality* states that for any symmetric convex sets A, B in $\mathbb{R}^n$,

$$\gamma_n(A \cap B) \geq \gamma_n(A)\, \gamma_n(B). \tag{1}$$

The same result holds true for any centered Gaussian measure on a Banach space E, and symmetric convex sets in E.

A detailed history of the problem can be found in [5]. In dimension 2, the result goes back to L. Pitt [18]. When one of the sets A or B is a symmetric strip, the inequality was proved independently by C. Khatri [12] and Z. Šidák [20]. It was extended to the case when one of the sets is a symmetric ellipsoid by G. Hargé [11]. The final step was achieved in a striking short contribution by T. Royen in 2014 [19].

The note emphasizes a number of related inequalities on the Gaussian measure of geometric flavour. Its pattern is modeled on the 2002 review article [15] by R. Latała, with the remarkable feature that all the conjectures exposed therein have now been solved.

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1 The Gaussian isoperimetric inequality

The Gaussian isoperimetric inequality is extensively discussed in the corresponding chapter of this blog [1], with a number of various proofs.

Recall the distribution function

$$\Phi(t) \,=\, \frac{1}{\sqrt{2\pi}} \int_{-t}^{\infty} e^{-\frac{1}{2}x^2} dx, \quad t \in \mathbb{R},$$

of the standard normal law on the real line (with the convention $\Phi(-\infty) = 0$, $\Phi(+\infty) = 1$).

The Gaussian isoperimetric profile is defined by

$$\mathcal{I}(s) \,=\, \varphi_1 \circ \Phi^{-1}(s), \quad s \in [0, 1]. \tag{2}$$

The function $\mathcal{I}$ is symmetric along the vertical line $s = \frac{1}{2}$, and such that $\mathcal{I}(0) = \mathcal{I}(1) = 0$.

Given $r > 0$, $A_r = \{x \in \mathbb{R}^n; \inf_{a \in A} |x - a| \leq r\}$ is the (closed) r-neighborhood of a set A in $\mathbb{R}^n$. The (Gaussian) outer Minkowski content of Borel set A is defined as

$$\gamma^+(A) \,=\, \liminf_{r \to 0} \frac{1}{r} \big[\gamma(A_r) - \gamma(A)\big].$$

Theorem 1 (The Gaussian isoperimetric inequality)**.** *For any Borel set A in $\mathbb{R}^n$,*

$$\gamma^+(A) \,\geq\, \mathcal{I}\big(\gamma(A)\big). \tag{3}$$

Equality is achieved on the half-spaces $H = \{x \in \mathbb{R}^n; \langle x, u\rangle \leq h\}$ where u is a unit vector and $h \in \mathbb{R}$.

The measure of a half-space is computed in dimension one as $\gamma(H) = \Phi(h)$, and its boundary measure is

$$\gamma^+(H) \,=\, \liminf_{r\to 0} \frac{1}{r}\left[\Phi(h+r) - \Phi(h)\right] \,=\, \varphi_1(h).$$

The Gaussian isoperimetric inequality thus expresses equivalently that, if H is a half-space such that $\Phi(h) = \gamma(H) = \gamma(A)$, then

$$\gamma^+(A) \,\geq\, \gamma^+(H), \tag{4}$$

and half-spaces are the extremal sets of the Gaussian isoperimetric problem.

Integrating along the neighborhoods, (4) is equivalently formulated as

$$\gamma(A_r) \,\geq\, \gamma(H_r), \quad r > 0, \tag{5}$$

provided that $\gamma(A) = (\geq)\ \gamma(H)$, or

$$\Phi^{-1}\big(\gamma(A_r)\big) \,\geq\, \Phi^{-1}\big(\gamma(A)\big) + r, \quad r > 0 \tag{6}$$

(since $\gamma(H_r) = \Phi(h+r)$).

Linear (affine) transformations yield the isoperimetric statement for any Gaussian measure. The dimension-free character allows furthermore for an infinite-dimensional formulation on an abstract Wiener space $(E, \mathcal{H}, \mu)$ as, for example,

$$\Phi^{-1}\big(\mu(A + r\mathcal{K})\big) \,\geq\, \Phi^{-1}\big(\mu(A)\big) + r, \quad r \geq 0,$$

where $\mathcal{K}$ is the unit ball of the reproducing kernel Hilbert space $\mathcal{H}$ (cf. [2]), and

$$A + r\mathcal{K} \,=\, \{a + rh\,; a \in A, h \in \mathcal{K}\}.$$

(Due to the linear structure, on the Euclidean space $\mathbb{R}^n$, $A_r = A + rB(0,1)$ where $B(0,1)$ is the (closed) Euclidean unit ball.)

2 The Ehrhard inequality

The classical Brunn-Minkowski inequality in Euclidean space states that for any Borel sets A and B in $\mathbb{R}^n$,

$$\mathrm{vol}_n\big(\theta A + (1-\theta)B\big) \,\geq\, \theta\,\mathrm{vol}_n(A) + (1-\theta)\,\mathrm{vol}_n(B), \quad \theta \in [0,1]. \tag{7}$$

(If A and B are subsets of $\mathbb{R}^n$, $A + B = \{a + b\,; a \in A, b \in B\}$.) This remarkable and powerful geometric inequality, with numerous consequences and applications, may be used in particular to recover the standard isoperimetric inequality in $\mathbb{R}^n$. The task is to show that, for fixed volume, balls are the extremal sets of the isoperimetric problem. That is, in the integrated form, whenever $\mathrm{vol}_n(A) = (\geq)\ \mathrm{vol}_n(B)$ where B is some ball,

$$\mathrm{vol}_n\big(A + B(0,r)\big) \geq \mathrm{vol}_n\big(B + B(0,r)\big)$$

for every $r > 0$. If $B = B(0, r_0)$ for some r_0, the choice in (7) of $B = B(0, \frac{\theta r}{1-\theta})$ such that $\theta = \frac{r_0}{r_0+r} \in (0,1)$, yields on the left-hand side $\theta^n \,\mathrm{vol}_n(A + B(0,r))$ while, by the choice of θ, the right-hand side is equal to

$$\begin{aligned}\theta \ \mathrm{vol}_n\big(B(0,r_0)\big) + (1-\theta)\,\mathrm{vol}_n\bigg(B\Big(0, \frac{\theta r}{1-\theta}\Big)\bigg) \qquad&\\ = \theta\, r_0^n \,\mathrm{vol}_n\big(B(0,1)\big) + (1-\theta)\,\frac{\theta^n r^n}{(1-\theta)^n}\,\mathrm{vol}_n\big(B(0,1)\big)&\\ = \theta^n (r_0 + r)^n \,\mathrm{vol}_n\big(B(0,1)\big)&\\ = \theta^n \,\mathrm{vol}_n\big(B(0, r_0 + r)\big)&\\ = \theta^n \,\mathrm{vol}_n\big(B(0, r_0) + B(0,r)\big),&\end{aligned}$$

which is therefore the result.

Gaussian measures satisfy a similar property, in the form of the log-concavity inequality

$$\log \gamma_n\big(\theta A + (1-\theta)B\big) \geq \theta \log \gamma_n(A) + (1-\theta)\log \gamma_n(B), \quad \theta \in [0,1]. \tag{8}$$

This inequality extends to any Gaussian measure μ on a separable Banach space E, and any Borel sets A and B in E (cf. [5]). However, the log-concavity of the measure does not imply the Gaussian isoperimetry.

In 1983, A. Ehrhard [10] emphasized an improved form of log-concavity of Gaussian measures through the inverse Φ^{-1} of the distribution function Φ the standard normal distribution.

Theorem 2 (The Ehrhard inequality)**.** *For any Borel sets A, B in $\mathbb{R}^n$, and any $\theta \in [0,1]$,*

$$\Phi^{-1}\big(\gamma_n(\theta A + (1-\theta)B)\big) \geq \theta\, \Phi^{-1}\big(\gamma_n(A)\big) + (1-\theta)\,\Phi^{-1}\big(\gamma_n(B)\big).$$

Theorem 2 extends to any Gaussian measure on a separable Banach space.

It is not difficult to see how Ehrhard's inequality includes isoperimetry. Indeed, applying it to $\frac{1}{\theta}A$ and to $B = \frac{r}{1-\theta}B(0,1)$, $r > 0$, $\theta \in (0,1)$, where $B(0,1)$ is the (closed) Euclidean unit ball, yields

$$\begin{aligned}&\Phi^{-1}\big(\gamma_n(A + (1-\theta)^{-1} rB(0,1))\big)\\ &\qquad \geq \theta\,\Phi^{-1}\big(\gamma_n(\theta^{-1}A)\big) + (1-\theta)\,\Phi^{-1}\big(\gamma_n\big((1-\theta)^{-1} rB(0,1)\big)\big).\end{aligned}$$

As $\theta \to 1$,

$$\Phi^{-1}\big(\gamma_n(A + rB(0,1))\big) \;\geq\; \Phi^{-1}\big(\gamma_n(A)\big) + r,$$

which is one form of Gaussian isoperimetry (6).

Theorem 2 was established for convex sets by A. Ehrhard [10] using Gaussian symmetrization techniques. It was extended to the case of only one of the sets A, B to be convex (good enough to recover isoperimetry) in [13]. C. Borell [8] finally proved the full result using pde tools on the functional version, in the form of the following Prékopa-Leindler-type inequality. If $f, g, h : \mathbb{R}^n \to [0,1]$ are measurable, and $\theta \in [0,1]$, are such that

$$\Phi^{-1}\big(h(\theta x + (1-\theta)y)\big) \;\geq\; \theta\,\Phi^{-1}\big(f(x)\big) + (1-\theta)\,\Phi^{-1}\big(g(y)\big),$$

for all $x, y \in \mathbb{R}^n$, then

$$\Phi^{-1}\bigg(\int_{\mathbb{R}^n} h d\gamma_n\bigg) \geq \theta\,\Phi^{-1}\bigg(\int_{\mathbb{R}^n} f d\gamma_n\bigg) + (1-\theta)\Phi^{-1}\bigg(\int_{\mathbb{R}^n} g d\gamma_n\bigg).$$

Applied to $f = \mathbb{1}_A$ and $g = \mathbb{1}_B$ yields the statement in Theorem 2 (and this functional form is actually equivalent to it when considering the level sets of functions defined on $\mathbb{R}^{n+1}$).

The proof in [8] is based on a parabolic maximum principle applied to the second order differential operator on $\mathbb{R}^n \times \mathbb{R}^n$,

$$\mathcal{E} = \Delta_x + \Delta_y + 2\sum_{i=1}^{n} \partial_{x_i}\partial_{y_i}$$

and the functional

$$C(t,x,y) \;=\; U_h\big(t, \theta x + (1-\theta)y\big) - \theta\,U_f(t,x) - (1-\theta)\,U_g(t,y),$$

$t \geq 0$, $x, y \in \mathbb{R}^n$, where, for $q = h, f, g$, $U_q = \Phi^{-1}(u_q)$ and

$$u_q(t,x) \;=\; \int_{\mathbb{R}^n} q\big(x + \sqrt{t}\,z\big)d\gamma_n(z).$$

Alternate proofs have been presented in [21] or [17].

3 The S-inequality

The S inequality is a type of isoperimetric inequality with respect to homotheties, with strips as extremal sets.

Theorem 3 (The S-inequality). *Let A be a symmetric closed convex set in $\mathbb{R}^n$, and let $S = \{x \in \mathbb{R}^n; |x_1| \leq s\}$, $s \geq 0$, be a strip such that $\gamma_n(A) = \gamma_n(S)$. Then*

$$\gamma_n(tA) \;\geq\; \gamma_n(tS) \quad \text{for } t \geq 1,$$

and

$$\gamma_n(tA) \;\leq\; \gamma_n(tS) \quad \text{for } 0 \leq t \leq 1.$$

This theorem has been established by R. Latała and K. Oleszkiewicz [14], relying on technical arguments and some clever real-line inequalities. It was observed from the S-inequality by S. Szarek (cf. [14], that the moment comparison of Gaussian random vectors (cf. [2]) are the same as in the real case. That is, if X is a centered Gaussian random vector on a separable Banach space E with norm $\|\cdot\|$, then

$$\frac{\big(\mathbb{E}(\|X\|^q)\big)^{1/q}}{\big(\mathbb{E}(|g|^q)\big)^{1/q}} \;\leq\; \frac{\big(\mathbb{E}(\|X\|^p)\big)^{1/p}}{\big(\mathbb{E}(|g|^p)\big)^{1/p}}$$

for any $0 \leq p \leq q$, where g has distribution $\mathcal{N}(0,1)$ on $\mathbb{R}$.

4 The B-inequality

The B-inequality for Gaussian measure is another statement about convex sets.

Theorem 4 (The B-inequality). *Let A be a symmetric closed convex set in $\mathbb{R}^n$. For every $\alpha, \beta > 0$,*

$$\gamma_n\big(\sqrt{\alpha\beta}A\big) \;\geq\; \sqrt{\gamma_n(\alpha A)\,\gamma_n(\beta A)}. \tag{9}$$

In an equivalent formulation, the map $t \mapsto \gamma_n(e^t A)$ is log-concave on $\mathbb{R}$.

The B-inequality has been established by D. Cordero-Erausquin, M. Fradelizi and B. Maurey in [9]. A interesting feature of the proof is that it is connected to (but lies much deeper than) the Gaussian Poincaré inequality for functions f which are orthogonal to constants and linear functions, for which the constant is improved as

$$\mathrm{Var}_{\gamma_n}(f) \;\leq\; \frac{1}{2}\int_{\mathbb{R}^n} |\nabla f|^2 d\gamma_n.$$

This is in particular clear on the Hermite expansion proof of the Gaussian Poincaré inequality [3].

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