Hermite polynomials

Let γ_n be the standard Gaussian measure

$$d\gamma_n(x) \,=\, \frac{1}{(2\pi)^{\frac{n}{2}}}\, e^{-\frac{1}{2}|x|^2} d\lambda_n(x)$$

on the Borel sets of $\mathbb{R}^n$. There is a natural orthonormal basis of the Hilbert space $\mathrm{L}^2(\gamma_n)$ of the square-integrable functions with respect to γ_n given by the so-called Hermite, or Hermite-Weber, polynomials. In dimension $n = 1$, they may be introduced in various ways, for example via the Rodrigues formula

$$\widetilde{h}_k(x) \,=\, (-1)^k\, e^{\frac{1}{2}x^2}\, \frac{d^k}{dx^k}\big(e^{-\frac{1}{2}x^2}\big), \quad x \in \mathbb{R},\ k \in \mathbb{N}. \tag{1}$$

The Hermite polynomials share a number of basic properties related to the Gaussian measure. The post briefly exposes some basic facts and results, which are available in the standard textbooks on the subject.

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1 Generating function

For every $\lambda \in \mathbb{R}$, the expansion in $x \in \mathbb{R}$,

$$e^{\lambda x - \frac{1}{2}\lambda^2} = \sum_{k=0}^{\infty} \frac{\lambda^k}{\sqrt{k!}} h_k(x) \tag{2}$$

defines polynomials h_k, $k \in \mathbb{R}$, called the *Hermite polynomials*. The function $e^{\lambda x - \frac{1}{2}\lambda^2}$ is called the generating function of the Hermite polynomials. For every $k \in \mathbb{N}$, h_k is a polynomial of degree k. For example

$$h_0(x) = 1, \quad h_1(x) = x, \quad h_2(x) = \frac{1}{\sqrt{2}}(x^2 - 1).$$

There is the explicit expression, $k \in \mathbb{N}$, $x \in \mathbb{R}$,

$$h_k(x) = \sum_{\ell=0}^{\lfloor k/2 \rfloor} (-1)^\ell \frac{\sqrt{k!}}{2^\ell \ell! (k-2\ell)!} x^{k-2\ell}$$

although not always very tractable. A more useful formula is

$$h_k(x) = \frac{1}{\sqrt{k!}} \mathbb{E}\big((x+iG)^k\big)$$

where G is a standard normal random variable (consequence of the fact that $\mathbb{E}(e^{\lambda(x+iG)}) = e^{\lambda x - \frac{1}{2}\lambda^2}$).

The normalization by $\sqrt{k!}$ in the series expansion (2) ensures that the polynomials h_k are normalized in $\mathrm{L}^2(\gamma_1)$, that is

$$\int_{\mathbb{R}} h_k^2 \, d\gamma_1 = 1 \quad \text{for every } k \in \mathbb{N},$$

but other normalizations are possible, and used in the literature, as in the Rodrigues formula from (1) (for which $\widetilde{h}_k = \sqrt{k!}\, h_k$, $k \in \mathbb{N}$).

The Hermite polynomials are moreover orthogonal in $\mathrm{L}^2(\gamma_1)$,

$$\int_{\mathbb{R}} h_k h_\ell \, d\gamma_1 \, = \, \delta_{k\ell}, \quad k, \ell \in \mathbb{N},$$

as a consequence of

$$\sum_{j=0}^{\infty} \frac{\lambda^{2j}}{j!} \, = \, e^{\lambda^2} \, = \, \int_{\mathbb{R}} \big(e^{\lambda x - \frac{1}{2}\lambda^2} \big)^2 d\gamma_1 \, = \, \sum_{k,\ell=0}^{\infty} \frac{\lambda^{k+\ell}}{\sqrt{k!}\sqrt{\ell!}} \int_{\mathbb{R}} h_k h_\ell \, d\gamma_1.$$

The complete system h_k, $k \in \mathbb{N}$, therefore defines an orthonormal basis of $\mathrm{L}^2(\gamma_1)$.

2 Recurrence formula

In the present normalization, it may be checked, by differentiation in x of the series expansion (2), that

$$h_k' \, = \, \sqrt{k} \, h_{k-1}, \quad k \geq 1. \tag{3}$$

Taking the derivative in λ of the generating function

$$(x - \lambda) \, e^{\lambda x - \frac{1}{2}\lambda^2} \, = \, \sum_{k=1}^{\infty} \frac{k \lambda^{k-1}}{\sqrt{k!}} \, h_k(x) \, = \, \sum_{k=0}^{\infty} \sqrt{k+1} \, \frac{\lambda^k}{\sqrt{k!}} \, h_{k+1}(x)$$

yields the three term recurrence formula

$$x h_k \, = \, \sqrt{k+1} \, h_{k+1} + \sqrt{k} \, h_{k-1}, \quad k \geq 1. \tag{4}$$

These two relationships actually connects (1) and (2) since, from (1) with the normalization by $\sqrt{k!}$,

$$\begin{aligned} \sqrt{k} \, h_{k-1} \, = \, h_k' \, = \, \frac{1}{\sqrt{k!}} \, \widetilde{h}_k' \, &= \, \frac{1}{\sqrt{k!}} \Big(x \widetilde{h}_k + (-1)^k \, e^{\frac{1}{2}x^2} \, \frac{d^{k+1}}{dx^{k+1}} \big(e^{-\frac{1}{2}x^2} \big) \Big) \\ &= \, x h_k - \frac{1}{\sqrt{k!}} \, \widetilde{h}_{k+1} \\ &= \, x h_k - \sqrt{k+1} \, h_{k+1}. \end{aligned}$$

3 Multi-dimensional Hermite polynomials

Multi-dimensional Hermite polynomials on $\mathbb{R}^n$ are defined as products of one-dimensional polynomials with multi-index. Namely, for $\underline{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$H_{\underline{k}}(x) \, = \, h_{k_1}(x_1) \cdots h_{k_n}(x_n).$$

Equivalently, an expansion of the multi-dimensional generating function, for $x \in \mathbb{R}^n$, $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$, yields

$$e^{\lambda \cdot x - \frac{1}{2}|\lambda|^2} = \sum_{\underline{k} \in \mathbb{N}^n} \frac{\lambda_1^{k_1} \cdots \lambda_n^{k_n}}{\sqrt{k_1!} \cdots \sqrt{k_n!}} \, H_{\underline{k}}(x).$$

In the same way as in the real case, the family $H_{\underline{k}}$, $\underline{k} \in \mathbb{N}^n$, defines an orthonormal basis of the Hilbert space $\mathrm{L}^2(\gamma_n)$. The so-called chaos decomposition (Fourier-Hermite expansion) of a function in $\mathrm{L}^2(\gamma_n)$ takes the form

$$f = \sum_{k \in \mathbb{N}^n} f_{\underline{k}} \, H_{\underline{k}} = \sum_{k=0}^{\infty} \bigg(\sum_{|\underline{k}|=k} f_{\underline{k}} \, H_{\underline{k}} \bigg)$$

where the $f_{\underline{k}}$'s are real numbers, and $|\underline{k}| = k_1 + \cdots + k_n$. The sums under parentheses actually represents the so-called homogeneous Wiener chaos of order k, which may be considered more generally.

Such Hilbert space decomposition may indeed be achieved similarly in infinite-dimension in the context of abstract Wiener spaces. Namely, in the notation of [1], let $(E, \mathcal{H}, \mu)$ be an abstract Wiener space, and let $(\xi_k)_{k \in \mathbb{N}} \subset E^*$ be any fixed orthonormal basis of E_2^*, the closure of E^* in $\mathrm{L}^2(\mu)$ (take any weak-star dense sequence of the unit ball of E^* and orthonormalize it with respect to μ using the Gram-Schmidt procedure). If $\alpha = (\alpha_0, \alpha_1, \dots) \in \mathbb{N}^{(\mathbb{N})}$, i.e. $|\alpha| = \alpha_0 + \alpha_1 + \cdots < \infty$, set

$$H_\alpha = \sqrt{\alpha!} \prod_i h_{\alpha_i} \circ \xi_i$$

(where $\alpha! = \alpha_0! \alpha_1! \cdots$). Then the family (H_α) constitutes an orthonormal basis of $\mathrm{L}^2(\mu)$.

In addtion, for each integer $k \geq 1$, set

$$\mathcal{W}^{(k)}(\mu) = \big\{ F \in \mathrm{L}^2(\mu); \langle F, H_\alpha \rangle = \textstyle\int_E F H_\alpha d\mu = 0 \text{ for all } \alpha \text{ such that } |\alpha| \neq k \big\}.$$

Then, any function F in $\mathrm{L}^2(\mu)$ (i.e. $\int_E \|F\|^2 d\mu < \infty$) may be developed as $F = \sum_{k=0}^{\infty} \Psi_k$ where

$$\Psi_k = \sum_{|\alpha|=k} \langle \Psi, H_\alpha \rangle \, H_\alpha$$

is an element of $\mathcal{W}^{(k)}(\mu)$, $k \geq 1$ ($\Psi_0 = \int_E F d\mu$).

4 Eigenvectors of the Ornstein-Uhlenbeck operator

Recall ([2]) the Ornstein-Uhlenbeck operator L acting on smooth functions $f : \mathbb{R}^n \to \mathbb{R}$ as

$$\mathrm{L} f = \Delta f - x \cdot \nabla f.$$

The Ornstein-Uhlenbeck operator L is the infinitesimal generator of the Ornstein-Uhlenbeck operator semigroup $(P_t)_{t\geq 0}$ which admits the Mehler representation

$$P_t f(x) \,=\, \int_{\mathbb{R}^n} f\big(e^{-t}x + \sqrt{1-e^{-2t}}\, y\big) d\gamma_n(y), \quad t \geq 0,\ x \in \mathbb{R}^n.$$

The spectrum of the operator L is $\mathbb{N}$, with eigenfunctions given by the Hermite polynomials

$$\mathrm{L} H_{\underline{k}} \,=\, -k\, H_{\underline{k}} \tag{5}$$

with $k = k_1 + \cdots + k_n$. This may be checked in various ways, for example by the action of the operators P_t, $t \geq 0$, on the generating function

$$P_t\big(e^{\lambda\cdot x - \frac{1}{2}|\lambda|^2}\big) \,=\, e^{\lambda e^{-t}\cdot x - \frac{1}{2}|\lambda|^2 e^{-2t}}.$$

Hence $P_t H_{\underline{k}} = e^{-kt} H_{\underline{k}}$, from which (5) follows.

As a consequence of (5), for any (smooth) function $f : \mathbb{R}^n \to \mathbb{R}$, and any $\underline{k} \in \mathbb{N}^n$,

$$k \int_{\mathbb{R}^n} f\, H_{\underline{k}}\, d\gamma_n \,=\, -\int_{\mathbb{R}^n} f\, \mathrm{L} H_{\underline{k}}\, d\gamma_n \,=\, \int_{\mathbb{R}^n} \nabla f \cdot \nabla H_{\underline{k}}\, d\gamma_n,$$

which is a generalized form of the basic integration by parts formula

$$\int_{\mathbb{R}^n} x f \, d\gamma_n \,=\, \int_{\mathbb{R}^n} \nabla f \, d\gamma_n$$

(as vector integrals).

5 Brownian martingales

The definition of the Hermite polynomials via the generating series (2) may be extended to include a time parameter $t \geq 0$ in the form

$$e^{\lambda x - \frac{1}{2}\lambda^2 t} \,=\, \sum_{k=0}^{\infty} \frac{\lambda^k}{\sqrt{k!}}\, h_k(x,t), \quad \lambda \in \mathbb{R},\ x \in \mathbb{R}. \tag{6}$$

A remarkable property expresses that, for every $k \geq 1$, $(h_k(B_t,t))_{t\geq 0}$ is a martingale, with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ of a Brownian motion $(B_t)_{t\geq 0}$ (defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$). When $k = 1$, this is the martingale property of Brownian motion itself, for $k = 2$, it expresses that $(B_t^2 - t)_{t\geq 0}$ is a martingale.

For a quick proof, it may be noticed that, for every $\lambda \in \mathbb{R}$,

$$e^{\lambda B_t - \frac{1}{2}\lambda^2 t}, \quad t \geq 0,$$

is a martingale since, for $s < t$,

$$\mathbb{E}\big(e^{\lambda B_t} \,|\, \mathcal{F}_s\big) \;=\; e^{\lambda B_s}\, \mathbb{E}\big(e^{\lambda (B_t - B_s)} \,|\, \mathcal{F}_s\big) \;=\; e^{\lambda B_s + \frac{1}{2}\lambda^2 (t-s)}$$

as $B_t - B_s$ is independent from $\mathcal{F}_s$ with law $\mathcal{N}(0, t-s)$. But then, for every $\lambda \in \mathbb{R}$,

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{\sqrt{k!}}\, \mathbb{E}\big(h_k(B_t, t) \,|\, \mathcal{F}_s\big) \;=\; \sum_{k=0}^{\infty} \frac{\lambda^k}{\sqrt{k!}}\, h_k(B_s, s)$$

from which the claim follows.

6 A proof of the Gaussian Poincaré inequality

Fourier-Hermite expansions provide a simple argument towards Poincaré inequalities, in the spirit of the Wirtinger inequality for the uniform measure on the sphere. This section briefly exposes the argument in the Gaussian case. The Gaussian Poincaré inequality is developed in the post [3].

Theorem 1 (The Gaussian Poincaré inequality). *For any locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ in $\mathrm{L}^2(\gamma_n)$,*

$$\mathrm{Var}_{\gamma_n}(f) \;\leq\; \int_{\mathbb{R}^n} |\nabla f|^2 d\gamma_n. \tag{7}$$

Theorem 1 may be given a simple proof by a series expansion in Hermite polynomials. Namely, in dimension one to start with,

$$f - \int_{\mathbb{R}} f \, d\gamma_1 \;=\; \sum_{k \geq 1}^{\infty} f_k h_k$$

where f_k are real coefficient. Starting if necessary from a finite sum, by (3)

$$f' \;=\; \sum_{k \geq 1}^{\infty} f_k \sqrt{k}\, h_{k-1}.$$

Since the Hermite polynomials form an orthonormal basis of $\mathrm{L}^2(\gamma_1)$,

$$\int_{\mathbb{R}} \big| f - \textstyle\int_{\mathbb{R}} f d\gamma_1 \big|^2 d\gamma_1 \;=\; \textstyle\sum_{k \geq 1} f_k^2$$

while

$$\int_{\mathbb{R}} f'^2 d\gamma_1 \;= =\; \sum_{k \geq 1} k\, f_k^2$$

from which the inequality immediately follows. A density argument may then be used to complete the picture.

The same proof applies in $\mathbb{R}^n$. It may actually also be observed, by the product property of the Gaussian measure $\gamma_n = \gamma_1 \otimes \cdots \otimes \gamma_1$, that the Poincaré inequality easily tensorizes so that it is enough to establish it in dimension one (cf. [3]).

References

[1] Admissible shift, reproducing kernel Hilbert space, and abstract Wiener space. *The Gaussian Blog*.

[2] Mehler kernel, and the Ornstein-Uhlenbeck operator. *The Gaussian Blog*.

[3] The Gaussian Poincaré inequality. *The Gaussian Blog*.