Logarithmic Sobolev and transportation inequalities

If μ and ν are probability measures on the Borel sets of $\mathbb{R}^n$, denote by

$$\mathrm{H}\big(\mu\,|\nu\big) \;=\; \int_{\mathbb{R}^n} f \log f \, d\nu \tag{1}$$

if μ is absolutely continuous with respect to ν with Radon-Nikodym derivative f, $+\infty$ if not, the *relative entropy* of μ with respect to ν.

For ν the standard Gaussian measure $d\gamma_n(x) = \frac{1}{(2\pi)^{\frac{n}{2}}}\, e^{-\frac{1}{2}|x|^2} d\lambda_n(x)$, the relative entropy $\mathrm{H}(\cdot\,|\gamma_n)$ enjoys two remarkable inequalities.

The first one, called the *logarithmic Sobolev inequality*, expresses that, for every probability measure μ,

$$\mathrm{H}\big(\mu\,|\gamma_n\big) \;\leq\; \frac{1}{2}\,\mathrm{I}\big(\mu\,|\gamma_n\big) \tag{2}$$

where

$$\mathrm{I}\big(\mu\,|\gamma_n\big) \;=\; \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f}\, d\gamma_n \tag{3}$$

is the Fisher information of (the density f of) μ with respect to γ_n (whenever well-defined).

The second one is the *(quadratic) transportation cost inequality* expressing that, for every probability measure μ with a finite second moment,

$$\mathrm{W}_2(\mu, \gamma_n)^2 \;\leq\; 2\,\mathrm{H}\big(\mu\,|\gamma_n\big) \tag{4}$$

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where

$$\mathrm{W}_2(\mu, \nu) \, = \, \inf_{\pi} \bigg(\int_{\mathbb{R}^n \times \mathbb{R}^n} |x-y|^2 d\pi(x,y) \bigg)^{\frac{1}{2}},$$

is the quadratic Kantorovich (Wasserstein) distance between two probability measures μ and ν on the Borel sets of $\mathbb{R}^n$ with a finite second moment, the infimum being running over all couplings π on $\mathbb{R}^n \times \mathbb{R}^n$ with respective marginals μ and ν.

This post (a bit dense) is devoted to a brief description of these two inequalities, and their applications and relationships with related functional inequalities. As main features, illustrations to Gaussian concentration, and an hierarchy from isoperimetry to the Poincaré inequality via the logarithmic Sobolev and the transportation inequalities are emphasized (in particular the former is formally stronger than the latter).

The inequalities are detailed for the standard Gaussian probability measure (and may be extended to arbitrary, even infinite-dimensional, Gaussian distributions [8]), but they actually concern large families of probability measures. In particular the hierarchy outlined in Section 7 may be developed in general (and makes full sense within this setting). Standard references on the subject are [16, 6]...

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References

1 Logarithmic Sobolev inequality

The logarithmic Sobolev inequality has been emphasized by L. Gross in [10].

Theorem 1 (The logarithmic Sobolev inequality). *For every smooth function $f : \mathbb{R}^n \to \mathbb{R}$ with $\int_{\mathbb{R}^n} f^2 d\gamma_n = 1$,*

$$\int_{\mathbb{R}^n} f^2 \log f^2 \, d\gamma_n \leq 2 \int_{\mathbb{R}^n} |\nabla f|^2 d\gamma_n. \tag{5}$$

By homogeneity, for every smooth, square integrable, function $f : \mathbb{R}^n \to \mathbb{R}$,

$$\mathrm{Ent}_{\gamma_n}(f^2) = \int_{\mathbb{R}^n} f^2 \log f^2 \, d\gamma_n - \int_{\mathbb{R}^n} f^2 \, d\gamma_n \log \left(\int_{\mathbb{R}^n} f^2 \, d\gamma_n \right) \leq 2 \int_{\mathbb{R}^n} |\nabla f|^2 d\gamma_n. \tag{6}$$

It is a simple matter to check that the inequality is sharp on the exponential functions $f(x) = e^{\langle a,x\rangle - |a|^2}$, $x \in \mathbb{R}^n$, where $a \in \mathbb{R}^n$.

This inequality can take various equivalent forms. For example, changing f^2 into $f > 0$, for any (smooth strictly positive) probability density f with respect to γ_n (i.e. $\int_{\mathbb{R}^n} f \, d\gamma_n = 1$),

$$\int_{\mathbb{R}^n} f \log f \, d\gamma_n \leq \frac{1}{2} \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} \, d\gamma_n \tag{7}$$

which amounts to (2) from the introduction.

It is an important feature that the inequality and the constants do not depend on the dimension of the underlying state space (in contrast in particular with the classical Sobolev inequalities). Actually, by a simple tensorization property, and the product structure of γ_n (similar to the one emphasized for the Poincaré inequality in [1]), it is enough to establish the logarithmic Sobolev inequality (5) in dimension one. By affine transformations, the logarithmic Sobolev inequality may be formulated for arbitrary Gaussian measures. Due to its dimension-free character, infinite-dimensional Gaussian measures may also be considered.

The Gaussian logarithmic Sobolev inequality is sometimes considered, and proved, in its Euclidean (dimensional) form, with respect to the Lebesgue measure λ_n, obtained after a simple, equivalent, change of function,

$$\int_{\mathbb{R}^n} f^2 \log f^2 d\lambda_n \leq \frac{n}{2} \log \left(\frac{2}{n\pi e} \int_{\mathbb{R}^n} |\nabla f|^2 d\lambda_n \right) \tag{8}$$

for every smooth function $f : \mathbb{R}^n \to \mathbb{R}$ such that $\int_{\mathbb{R}^n} f^2 d\lambda_n = 1$ (cf. [16, 6]).

2 Hypercontractivity

Another main aspect of the pioneering work [10] by L. Gross is the equivalence (in a general Dirichlet form framework) of logarithmic Sobolev inequalities with hypercontractivity properties.

For the Gaussian measure γ_n, let

$$P_t f(x) \,=\, \int_{\mathbb{R}^n} f\big(e^{-t}x + \sqrt{1-e^{-2t}}\,y\big)d\gamma_n(y), \quad t \geq 0,\ x \in \mathbb{R}^n, \tag{9}$$

be the Ornstein-Uhlenbeck semigroup (acting on suitable functions $f : \mathbb{R}^n \to \mathbb{R}$). Recall (cf. e.g. [6, 2]) that the Markov semigroup $(P_t)_{t\geq 0}$ is invariant and symmetric with respect to γ_n, and that its infinitesimal generator $\mathrm{L} = \Delta - x \cdot \nabla$ fulfills the integration by parts formula

$$\int_{\mathbb{R}^n} f(-\mathrm{L}g)d\gamma_n \,=\, \int_{\mathbb{R}^n} \nabla f \cdot \nabla g \, d\gamma_n \tag{10}$$

for every smooth functions $f, g : \mathbb{R}^n \to \mathbb{R}$.

By Jensen's inequality on the integral representation (9), the operators P_t are contractions in all $\mathrm{L}^p(\gamma_n)$, $p \geq 1$, spaces. As a result, the logarithmic Sobolev inequality (5), holding for all smooth functions f, is equivalent to the following hypercontractivity property, emphasized first by E. Nelson [13].

Theorem 2 (Hypercontractivity)**.** *For every (measurable) function $f : \mathbb{R}^n \to \mathbb{R}$, every $1 < p < q < \infty$, and every $t > 0$ such that $e^{2t} \geq \frac{q-1}{p-1}$,*

$$\|P_t f\|_q \,\leq\, \|f\|_p \tag{11}$$

(where $\|\cdot\|_p$ is the $\mathrm{L}^p(\gamma_n)$ norm).

The proof of this equivalence amounts to take the time derivative of $R(t) = \|P_t f\|_{q(t)}$ where $q(t) = 1 + e^{2t}(p-1)$, $t \geq 0$, which yields the logarithmic Sobolev inequality up to a power-type change of function. Namely, given a non-negative smooth function f on $\mathbb{R}^n$, the chain rule formula yields, on the one hand, an entropy term as derivative of L^p-norm, and on the other hand, the Ornstein-Uhlenbeck operator from the heat equation $\frac{\partial}{\partial t} P_t f = \mathrm{L} P_t f$, so to obtain

$$q(t)^2 R(t)^{q(t)-1} R'(t) \,=\, q'(t)\,\mathrm{Ent}_{\gamma_n}\big((P_t f)^{q(t)}\big) + q(t)^2 \int_{\mathbb{R}^n} (P_t f)^{q(t)-1} \mathrm{L} P_t f d\gamma_n.$$

Next, by the integration by parts formula (10),

$$\begin{aligned} &q(t)^2\, R(t)^{q(t)-1} R'(t) \\ &\quad= q'(t)\,\mathrm{Ent}_{\gamma_n}\big((P_t f)^{q(t)}\big) - 2\big(q(t)-1\big) \int_{\mathbb{R}^n} \frac{q(t)^2}{2}\, |\nabla P_t f|^2 (P_t f)^{q(t)-2} d\gamma_n. \end{aligned} \tag{12}$$

Since $q'(t) = 2(q(t) - 1)$, the logarithmic Sobolev inequality (6) applied to $(P_t f)^{q(t)/2}$ yields that $R'(t) \leq 0$. Hence $R(t) \leq R(0)$ which is the claim. The actual equivalence with hypercontractivity is proved similarly.

3 Proof of the logarithmic Sobolev inequality

In the note [12], "more than 15 proofs of the logarithmic Sobolev inequality" (and hypercontractivity) are put forward. The simplest appears to be the one by D. Bakry and M. Émery [5] based on heat flow arguments. While the classical heat equation is used in [12], this section presents the same idea with the Ornstein-Uhlenbeck semigroup $(P_t)_{t\geq 0}$.

Let thus $f : \mathbb{R}^n \to \mathbb{R}$ be smooth, and such that $0 < c \leq f \leq C < \infty$, these constraints being easily lifted at the end of the proof. Then, since $P_0 g = g$ and $P_\infty g = \int_{\mathbb{R}^n} g \, d\gamma_n$ (for any suitable $g : \mathbb{R}^n \to \mathbb{R}$),

$$\begin{aligned} \mathrm{Ent}_{\gamma_n}(f) &= \int_{\mathbb{R}^n} f \log f \, d\gamma_n - \int_{\mathbb{R}^n} f \, d\gamma_n \log \bigg(\int_{\mathbb{R}^n} f \, d\gamma_n \bigg) \\ &= - \int_0^\infty \frac{d}{dt} \int_{\mathbb{R}^n} P_t f \log P_t f \, d\gamma_n \, dt. \end{aligned}$$

Now

$$\frac{d}{dt} \int_{\mathbb{R}^n} P_t f \log P_t f \, d\gamma_n = \int_{\mathbb{R}^n} \mathrm{L} P_t f \log P_t f \, d\gamma_n + \int_{\mathbb{R}^n} \mathrm{L} P_t f \, d\gamma_n = - \int_{\mathbb{R}^n} \frac{|\nabla P_t f|^2}{P_t f} \, d\gamma_n$$

by the integration by parts formula (10) (and invariance $\int_{\mathbb{R}^n} \mathrm{L} g \, d\gamma_n = 0$). Now, from the integral representation of P_t, $\nabla P_t f = e^{-t} P_t(\nabla f)$ (as vectors). Next, by the Cauchy-Schwarz inequality with respect to P_t (as an integral),

$$\big| P_t(\nabla f) \big|^2 \leq \big[P_t\big(|\nabla f|\big) \big]^2 \leq P_t f \, P_t \Big(\frac{|\nabla f|^2}{f} \Big).$$

Therefore,

$$\int_{\mathbb{R}^n} \frac{|\nabla P_t f|^2}{P_t f} \, d\gamma_n \leq e^{-2t} \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} \, d\gamma_n$$

by invariance of P_t, and the conclusion follows after integration in t.

4 Infimum-convolution inequalities

The Gross link between the logarithmic Sobolev inequality and hypercontractivity may be developed in parallel at the level of infimum-convolution inequalities.

Consider the basic Hamilton-Jacobi initial value problem

$$\begin{aligned} \frac{\partial v}{\partial t} + \frac{1}{2} |\nabla v|^2 \ &= \ 0 \quad \text{in} \ \ \mathbb{R}^n \times (0, \infty), \\ v \ &= \ f \quad \text{on } \mathbb{R}^n \times \{t = 0\}. \end{aligned} \tag{13}$$

Solutions of (13) are described by the Hopf-Lax representation formula as infimum-convolutions. Namely, given a (Lipschitz continuous) function f on $\mathbb{R}^n$, define the one-parameter family of infimum-convolutions of f with the quadratic cost as

$$Q_t f(x) \ = \ \inf_{y \in \mathbb{R}^n} \Big[f(y) + \frac{1}{2t} |x - y|^2 \Big], \quad t > 0, \, x \in \mathbb{R}^n. \tag{14}$$

The family $(Q_t)_{t \geq 0}$ (with the convention $Q_0 f = f$) defines a semigroup with infinitesimal (non-linear) generator $-\frac{1}{2} |\nabla f|^2$. That is, $v = v(x, t) = Q_t f(x)$ is a solution of the Hamilton-Jacobi initial value problem (13) (at least almost everywhere). Actually, if in addition f is bounded, the Hopf-Lax formula $Q_t f$ is the relevant mathematical solution of (13), that is its unique viscosity solution (cf. [9]). It is also fruitful to think of the Hamilton-Jacobi equation as the limiting point of a family of heat equations obtained after the addition of a small noise (and an exponential change of function). With this in mind, the following development is of no surprise ([7]).

Once this has been recognized, it is not difficult indeed to follow Gross's idea for the Hamilton-Jacobi equation. Namely, letting now $S(t) = \|e^{Q_t f}\|_{\lambda(t)}$, $t \geq 0$, for some function $\lambda(t)$ with $\lambda(0) = a$, $a \in \mathbb{R}$, the analogue of (12) reads as

$$\lambda(t)^2 \, S(t)^{\lambda(t) - 1} S'(t) \ = \ \lambda'(t) \, \mathrm{Ent}_{\gamma_n} \big(e^{\lambda(t) Q_t f} \big) - \int_{\mathbb{R}^n} \frac{\lambda(t)^2}{2} \big| \nabla Q_t f \big|^2 e^{\lambda(t) Q_t f} d\gamma_n. \tag{15}$$

By the logarithmic Sobolev inequality (6) applied to $e^{\lambda(t) Q_t f}$, $S'(t) \leq 0$ as soon as $\lambda'(t) = 1$, $t \geq 0$. As a result, the logarithmic Sobolev inequality shows that, for every $t \geq 0$, every $a \in \mathbb{R}$ and every (say bounded) function f,

$$\big\| e^{Q_t f} \big\|_{a + t} \ \leq \ \big\| e^f \big\|_a, \tag{16}$$

a family of hypercontractive inequalities for the infimum-convolution semigroup $(Q_t)_{t \geq 0}$. Conversely, if (16) holds for every $t \geq 0$ and some $a \neq 0$, then the logarithmic Sobolev inequality holds true.

5 Transportation cost inequality

If $t = 1$ and $a \to 0$, (16) reads

$$\int_{\mathbb{R}^n} e^{Q_1 f} d\gamma_n \ \leq \ e^{\int_{\mathbb{R}^n} f \, d\gamma_n} \tag{17}$$

for any (bounded measurable, or suitably integrable) function $f : \mathbb{R}^n \to \mathbb{R}$. The point is that this inequality, holding for every bounded measurable function $f : \mathbb{R}^n \to \mathbb{R}$, is an infimum-convolution reformulation of the transportation cost inequality (4). To this purpose, the Monge-Kantorovitch theorem (cf. [16]) expresses that, for every probability measures μ and ν on $\mathbb{R}^n$ (with a second moment),

$$\mathrm{W}_2(\mu, \nu)^2 \;=\; \sup \left[\int_{\mathbb{R}^n} Q_1 f \, d\mu - \int_{\mathbb{R}^n} f \, d\nu \right] \tag{18}$$

where the supremum is running over bounded measurable functions $f : \mathbb{R}^n \to \mathbb{R}$. On the other hand, given a probability μ with a second moment, if g is the Radon-Nikodym derivative of μ with respect to γ_n,

$$\mathrm{H}\big(\mu \,|\, \gamma_n\big) \;=\; \int_{\mathbb{R}^n} g \log g \, d\gamma_n \;=\; \sup \int_{\mathbb{R}^n} g\psi \, d\gamma_n$$

where the supremum is taken over all functions ψ such that $\int_{\mathbb{R}^n} e^{\psi} d\gamma_n \leq 1$. For any bounded measurable function f, $\psi = Q_1 f - \int_{\mathbb{R}^n} f \, d\gamma_n$ belongs to this family by (17), so that

$$\int_{\mathbb{R}^n} Q_1 f \, d\mu - \int_{\mathbb{R}^n} f \, d\gamma_n \;=\; \int_{\mathbb{R}^n} g\psi \, d\gamma_n \;\leq\; \int_{\mathbb{R}^n} g \log g \, d\gamma_n .$$

Taking the supremum over all f's, the inequality (4) holds true. The full equivalence between (17) and (4) is deduced in the same way.

As a result, the following transportation cost inequality, first discovered by M. Talagrand [15], holds true.

Theorem 3 (The transportation cost inequality)**.** *For every Borel probability measure μ on $\mathbb{R}^n$ with a finite second moment,*

$$\mathrm{W}_2(\mu, \gamma_n)^2 \;\leq\; 2\, \mathrm{H}\big(\mu \,|\gamma_n\big). \tag{19}$$

The preceding investigation actually reveals that the (Gross) logarithmic Sobolev inequality (2) formally implies the (Talagrand) transportation cost inequality (4) (via the infimum-convolution inequality (17)). This relationship, discussed below in Section 7, is fully relevant for arbitrary measures satisfying one of these inequalities and was emphasized by F. Otto and C. Villani [14].

Theorem 4 (The Otto-Villani theorem)**.** *The logarithmic Sobolev inequality* (5) *implies the transportation cost inequality* (19).

6 Concentration inequalities and the Herbst argument

Whenever $F : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz, with Lipschitz semi-norm $\|F\|_{\mathrm{Lip}}$,

$$Q_1 F \;\geq\; F - \frac{1}{2}\,\|F\|_{\mathrm{Lip}}^2.$$

Therefore (17) for such a Lipschitz function F yields

$$\int_{\mathbb{R}^n} e^F d\gamma_n \;\leq\; e^{\int_{\mathbb{R}^n} F d\gamma_n + \frac{1}{2}\,\|F\|_{\mathrm{Lip}}^2}, \tag{20}$$

which is exactly the Laplace concentration bounds as developed in the post [3], which entail all the relevant information towards the (sharp) Gaussian concentration inequalities.

Now (20) may be also be reached (more) directly from the logarithmic Sobolev inequality by the so-called Herbst argument (cf. [11]).

Let $F : \mathbb{R}^n \to$ be bounded and smooth, such that $\|F\|_{\mathrm{Lip}} \leq 1$. In particular, since F is assumed to be regular enough, it holds true that $|\nabla F| \leq 1$ at every point. Apply next the logarithmic Sobolev inequality (6) to $f^2 = e^{\lambda F}$ for every $\lambda \in \mathbb{R}$. Note first that

$$\int_{\mathbb{R}^n} |\nabla F|^2 d\gamma_n \;=\; \frac{\lambda^2}{4} \int_{\mathbb{R}^n} |\nabla F|^2 e^{\lambda F} d\gamma_n \;\leq\; \frac{\lambda^2}{4} \int_{\mathbb{R}^n} e^{\lambda F} d\gamma_n.$$

Next, setting $H(\lambda) = \int_{\mathbb{R}^n} e^{\lambda F} d\gamma_n$, $\lambda \in \mathbb{R}$, (6) yields

$$\lambda H'(\lambda) - H(\lambda) \log H(\lambda) \;\leq\; \frac{\lambda^2}{2}\, H(\lambda).$$

In other words, if $K(\lambda) = \frac{1}{\lambda} \log H(\lambda)$ (with $K(0) = \frac{H'(0)}{H(0)} = \int_{\mathbb{R}^n} F d\gamma_n$), $K'(\lambda) \leq \frac{1}{2}$ for every λ. Therefore,

$$K(\lambda) \;=\; K(0) + \int_0^\lambda K'(u) du \;\leq\; \int_{\mathbb{R}^n} F d\gamma_n + \frac{\lambda}{2}\,,$$

and hence, for every $\lambda \in \mathbb{R}$,

$$\int_{\mathbb{R}^n} e^{\lambda F} d\gamma_n \;=\; H(\lambda) \;\leq\; e^{\lambda \int_{\mathbb{R}^n} F d\gamma_n + \frac{\lambda^2}{2}},$$

which amounts to (20) for this class of functions. The boundedness and smoothness assumptions are easily removed. Replace F by $F_N = \max(-N, \min(F, N)$, $N \geq 1$, and apply (20) to $P_\varepsilon F_N$, $\varepsilon > 0$. As $\varepsilon \to 0$ and $N \to \infty$, the family of inequalities (20) extends to all Lipschitz functions F on $\mathbb{R}^n$.

7 A hierarchy of inequalities

There is a hierarchy between the most important (geometric) and functional inequalities satisfied the Gaussian measure, which can be properly formalized, and made relevant, for families of probability measures (cf. e.g. [16, 6]).

Namely,

$$\begin{array}{c}\text{isoperimetric inequality}\\ \Downarrow \\ \text{logarithmic Sobolev inequality}\\ \Downarrow \\ \text{transportation cost inequality}\\ \Downarrow \\ \text{Poincaré inequality}\end{array}$$

The following hints toward this picture apply for arbitrary probability measures satisfying one or more of these functional inequalities. A possible path along this series of implication starts with the functional form of the Gaussian isoperimetric inequality

$$\mathcal{I}\bigg(\int_{\mathbb{R}^n} f\, d\gamma_n\bigg) \leq \int_{\mathbb{R}^n} \sqrt{\mathcal{I}(f)^2 + |\nabla f|^2}\, d\gamma_n, \tag{21}$$

for any smooth $f : \mathbb{R}^n \to [0,1]$, where $\mathcal{I}$ is the isoperimetric profile of γ_n (cf. [4]). For a given smooth, bounded, function f on $\mathbb{R}^n$, the inequality (21) applied to εf^2 as $\varepsilon \to 0$ yields the logarithmic Sobolev inequality (5) (use in particular the asymptotics $\mathcal{I}(s) \sim s\sqrt{2\log\left(\frac{1}{s}\right)}$ as $s \to 0$).

That the logarithmic Sobolev inequality implies the quadratic transportation cost inequality has been illustrated in Section 5. Finally, starting form the dual formulation (17) of the latter applied to εf, a Taylor expansion as $\varepsilon \to 0$ yields the Poincaré inequality

$$\mathrm{Var}_{\gamma_n}(f) \leq \int_{\mathbb{R}^n} |\nabla f|^2 d\gamma_n$$

presented in [1].

References

[1] The Gaussian Poincaré inequality. *The Gaussian Blog.*

[2] Some basic properties and characterizations of Gaussian measures and variables. *The Gaussian Blog.*

[3] Gaussian concentration inequalities. *The Gaussian Blog.*

[4] The Gaussian isoperimetric inequality. *The Gaussian Blog.*

[5] D. Bakry, M. Émery. Diffusions hypercontractives. *Séminaire de Probabilités XIX, Lecture Notes in Math.* 1123, 177–206. Springer (1985).

[6] D. Bakry, I. Gentil, M. Ledoux. *Analysis and geometry of Markov diffusion operators.* Grundlehren der mathematischen Wissenschaften 348. Springer (2014).

[7] S. Bobkov, I. Gentil, M. Ledoux. Hypercontractivity of Hamilton-Jacobi equations. *J. Math. Pures Appl.* 80, 669–696 (2001).

[8] V. Bogachev. *Gaussian measures.* Math. Surveys Monogr. 62. American Mathematical Society (1998).

[9] L. Evans. *Partial differential equations.* American Mathematical Society (1997).

[10] L. Gross. Logarithmic Sobolev inequalities. *Amer. J. Math.* 97, 1061–1083 (1975).

[11] M. Ledoux. *The concentration of measure phenomenon.* Mathematical Surveys and Monographs 89. American Mathematical Society (2001).

[12] M. Ledoux. "More than fifteen proofs of the logarithmic Sobolev inequality" (2015).

[13] E. Nelson. A quartic interaction in two dimensions. *Mathematical Theory of Elementary Particles* 69–73. M.I.T. Press, Cambridge (1966); E. Nelson. The free Markov field. *J. Funct. Anal.* 12, 211–227 (1973).

[14] F. Otto, C. Villani. Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality. *J. Funct. Anal.* 173, 361–400 (2000).

[15] M. Talagrand. Transportation cost for Gaussian and other product measures. *Geom. Funct. Anal.* 6, 587–600 (1996).

[16] C. Villani. *Topics in optimal transportation.* Graduate Studies in Mathematics 58. American Mathematical Society (2003).