Gaussian Riesz transforms

A famous result in harmonic analysis (cf. e.g. [11]) expresses that, for any $1 < p < \infty$, there is a constant $C_p > 0$ such that for any function $f : \mathbb{R}^n \to \mathbb{R}$ in $C_c^\infty(\mathbb{R}^n)$,

$$\frac{1}{C_p} \int_{\mathbb{R}^n} |\nabla f|^p d\lambda_n \,\leq\, \int_{\mathbb{R}^n} \big|\sqrt{-\Delta} f\big|^p d\lambda_n \,\leq\, C_p \int_{\mathbb{R}^n} |\nabla f|^p d\lambda_n. \tag{1}$$

Here $\sqrt{-\Delta}$ is the fractional Laplacian, which may be defined by multiplication by $|x|$ of the Fourier transform (the Fourier transform of $\sqrt{-\Delta} f$ is the Fourier transform of f multiplied by $|x|$).

This result is also a multi-dimensional extension of the boundedness of the classical Hilbert transform

$$\mathcal{H}_{\mathbb{R}} f(x) \,=\, \frac{1}{\pi}\,\text{p.v.} \int_{\mathbb{R}} \frac{f(x-y)}{y}\, d\lambda_1(y) \,=\, \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{\{|y| > \varepsilon\}} \frac{f(x-y)}{y}\, d\lambda_1(y). \tag{2}$$

The Hilbert transform may indeed be viewed via multiplication by $i\,\mathrm{sgn}(x)$ of the Fourier transform of f. Its multi-dimensional extension in $\mathbb{R}^n$ amounts to the multiplication by $\frac{ix_k}{|x|}$, $k = 1, \ldots, n$, of the Fourier transform, yielding the family R_k of Riesz transforms, for which, symbolically,

$$R_k \,=\, \partial_k (-\Delta)^{-1/2}, \quad k = 1, \ldots, n.$$

The family of inequalities (1) is therefore formulated equivalently as the boundedness in $\mathrm{L}^p(\lambda_n)$, $1 < p < \infty$, of the Riesz transform operator $\nabla(-\Delta)^{-1/2}$. This result is typically established by complex analysis techniques, cf. [11].

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The Gaussian Riesz transform inequalities due to P.-A. Meyer [8], are the analogue of (1) when the Laplacian Δ is replaced by the Ornstein-Uhlenbeck operator $\mathrm{L} = \Delta - x \cdot \nabla$, with invariant measure the standard Gaussian measure $d\gamma_n(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}|x|^2} d\lambda_n(x)$ on $\mathbb{R}^n$.

Theorem 1 (The Meyer inequalities)**.** *For any $1 < p < \infty$, there is a constant $C_p > 0$, independent of n, such that*

$$\frac{1}{C_p} \int_{\mathbb{R}^n} |\nabla f|^p d\gamma_n \le \int_{\mathbb{R}^n} \big|\sqrt{-\mathrm{L}} f\big|^p d\gamma_n \le C_p \int_{\mathbb{R}^n} |\nabla f|^p d\gamma_n \tag{3}$$

for every smooth $f : \mathbb{R}^n \to \mathbb{R}$.

The class of functions for which the inequalities hold true extend to suitably domains for which the integrals are well-defined. This aspect will be mostly ignored throughout the note (under the simplified language of "smooth functions"). An important feature of the result is the independence of the constant C_p upon the dimension n of the underlying state space, allowing for extensions to infinite-dimensional Gaussian measures, as considered in [8, 7]. (It should be pointed out that the constants in the Euclidean version (1) may also be chosen independent of n, cf. [10] and the references therein.) These inequalities are actually motivated by, and pivotal in, the study of the Malliavin stochastic calculus of variations cf. [7, 9].

The purpose of this post is to sketch the proof of the Meyer inequalities emphasized by G. Pisier in [10], relying on a transference argument to the boundedness of the Hilbert transform on the torus. Another proof is also proposed in [6].

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1 Gaussian Riesz transforms

The Ornstein-Uhlenbeck operator is defined by

$$\mathrm{L}f \,=\, \Delta f \,-\, x \cdot \nabla f \tag{4}$$

acting on smooth functions $f : \mathbb{R}^n \to \mathbb{R}$ (cf. [8, 3, 1]...). The differential operator L, as a drifted Laplacian, fulfills the basic integration by parts formula with respect to the Gaussian measure γ_n,

$$\int_{\mathbb{R}^n} f(-\mathrm{L}g) d\gamma_n \,=\, \int_{\mathbb{R}^n} \nabla f \cdot \nabla g \, d\gamma_n \tag{5}$$

for suitably smooth functions $f, g : \mathbb{R}^n \to \mathbb{R}$.

In this Gaussian context, the square root operator $\sqrt{-\mathrm{L}}$ may be given a simple spectral description via Fourier-Hermite expansions. To illustrate the picture, consider the one-dimensional case, and recall that the spectrum the Ornstein-Uhlenbeck operator $-\mathrm{L}$ is the set $\mathbb{N}$ of the integers, with Hermite polynomials as eigenfunctions. A mean-zero function f in $\mathrm{L}^2(\gamma_1)$ (or even a polynomial if necessary) is thus represented in the family $(h_k)_{k\in\mathbb{N}}$ of Hermite polynomials (cf. [2]) in the form

$$f \,=\, \sum_{k\in\mathbb{N}} a_k h_k.$$

Since $-\mathrm{L}h_k = kh_k$, $k \in \mathbb{N}$, formally

$$\sqrt{-\mathrm{L}} f \,=\, \sum_{k\in\mathbb{N}} \sqrt{k}\, a_k h_k.$$

A similar meaning can be given to the fractional powers of $-\mathrm{L}$, even negative ones provided the operators are applied to mean-zero functions.

According to (5), and as for the standard Laplacian, for every function f in some appropriate domain,

$$\int_{\mathbb{R}^n} \big[\sqrt{-\mathrm{L}} f\big]^2 d\gamma_n \,=\, \int_{\mathbb{R}^n} f(-\mathrm{L}f) d\gamma_n \,=\, \int_{\mathbb{R}^n} |\nabla f|^2 d\gamma_n.$$

The Gaussian Riesz transform issue is therefore to extend these comparisons to $\mathrm{L}^p(\gamma_n)$, $1 < p < \infty$, and to reach the analogue of (1) for the Ornstein-Uhlenbeck operator L.

2 Ornstein-Uhlenbeck semigroup and gradient formulas

To briefly recall some basic facts [8, 3, 1], the Ornstein-Uhlenbeck operator $\mathrm{L} = \Delta \,-\, x \cdot \nabla$ generates a Markov semigroup $(P_t)_{t\geq 0}$, which may be represented by the integral Mehler formula as

$$P_t f(x) \;=\; \int_{\mathbb{R}^n} f\big(e^{-t}x + \sqrt{1-e^{-2t}}\,y\big)\, d\gamma_n(y) \tag{6}$$

for all $t > 0$, $x \in \mathbb{R}^n$, and suitable functions $f : \mathbb{R}^n \to \mathbb{R}$.

To illustrate, in dimension one, the action of P_t, letting $f = \sum_{k\in\mathbb{N}} a_k h_k$, then

$$P_t f \;=\; \sum_{k\in\mathbb{N}} e^{-kt} a_k h_k.$$

Accordingly, the Gaussian Riesz transform $\nabla(-\mathrm{L})^{-1/2}$ may be constructed from the Ornstein-Uhlenbeck semigroup by the formulas

$$(-\mathrm{L})^{-1/2} \;=\; \int_0^\infty P_t \,\frac{dt}{\sqrt{\pi t}} \tag{7}$$

(since $\int_0^\infty e^{-kt}\frac{dt}{\sqrt{\pi t}} = \frac{1}{\sqrt{k}}$, $k \geq 1$), and thus

$$\nabla(-\mathrm{L})^{-1/2} \;=\; \int_0^\infty \nabla P_t \,\frac{dt}{\sqrt{\pi t}}\,. \tag{8}$$

To this representation, the gradient formulas for P_t may be added ($t > 0$, $x \in \mathbb{R}^n$),

$$\begin{aligned} \nabla P_t f(x) \;&=\; e^{-t}\int_{\mathbb{R}^n} \nabla f\big(e^{-t}x + \sqrt{1-e^{-2t}}\,y\big) d\gamma_n(y) \quad \big(\;=\; e^{-t} P_t(\nabla f)(x)\,\big) \\ &=\; \frac{e^{-t}}{\sqrt{1-e^{-2t}}}\int_{\mathbb{R}^n} y\, f\big(e^{-t}x + \sqrt{1-e^{-2t}}\,y\big) d\gamma_n(y) \end{aligned} \tag{9}$$

as vector integrals (the second identity being the result of integration by parts).

3 A representation formula for the Gaussian Riesz transform

The proof of Theorem 1 presented here is due to G. Pisier [10], and is based on a transference principle to the Riesz transform on the torus via the Gaussian rotational invariance.

It is enough to prove the existence, for any $1 < p < \infty$, of a constant $C_p > 0$ (independent of n) such that

$$\big\|\nabla(-\mathrm{L})^{-1/2} f\big\|_{\mathrm{L}^p(\gamma_n)} \;\leq\; C_p \,\|f\|_{\mathrm{L}^p(\gamma_n)} \tag{10}$$

for any smooth (mean-zero) function $f : \mathbb{R}^n \to \mathbb{R}$ (even just a polynomial, linear combination of Hermite polynomials). Namely, changing f into $(-\mathrm{L})^{1/2} f$ yields the left-hand side in (3).

To reach the right-hand side, by duality with $\frac{1}{p} + \frac{1}{q} = 1$, and integration by parts,

$$\begin{aligned}
\big\|(-\mathrm{L})^{1/2} f\big\|_{\mathrm{L}^p(\gamma_n)} &= \sup_{\|g\|_{\mathrm{L}^q(\gamma_n)} \leq 1} \int_{\mathbb{R}^n} g(-\mathrm{L})^{1/2} f \, d\gamma_n \\
&= \sup_{\|g\|_{\mathrm{L}^q(\gamma_n)} \leq 1} \int_{\mathbb{R}^n} (-\mathrm{L})^{-1/2} g \, (-\mathrm{L} f) d\gamma_n \\
&= \sup_{\|g\|_{\mathrm{L}^q(\gamma_n)} \leq 1} \int_{\mathbb{R}^n} \nabla(-\mathrm{L})^{-1/2} g \cdot \nabla f \, d\gamma_n,
\end{aligned}$$

from which the claim follows by (10) applied to the function g, and to q instead of p.

This section is devoted to the first steps of the proof, with in particular a suitable representation formula for the Gaussian Riesz transform.

According therefore to (7) and (9), for a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ and $x \in \mathbb{R}^n$,

$$\nabla(-\mathrm{L})^{-1/2} f(x) = \int_0^\infty \bigg(\int_{\mathbb{R}^n} y \, f\big(e^{-t} x + \sqrt{1 - e^{-2t}} \, y\big) d\gamma_n(y) \bigg) \frac{e^{-t} dt}{\sqrt{\pi \, t(1 - e^{-2t})}} \, .$$

Changing y into $-y$, it also holds true that

$$\nabla(-\mathrm{L})^{-1/2} f(x) = -\int_0^\infty \bigg(\int_{\mathbb{R}^n} y \, f\big(e^{-t} x - \sqrt{1 - e^{-2t}} \, y\big) d\gamma_n(y) \bigg) \frac{e^{-t} dt}{\sqrt{\pi \, t(1 - e^{-2t})}} \, .$$

At this stage, it is useful to perform the change of variables $e^{-t} \mapsto \cos\theta$, $\theta \in [0, \frac{\pi}{2})$ so that the first formula for $\nabla(-\mathrm{L})^{-1/2} f(x)$ above reads

$$\begin{aligned}
\nabla(-\mathrm{L})^{-1/2} f(x) &= \int_0^{\frac{\pi}{2}} \bigg(\int_{\mathbb{R}^n} y \, f\big(x\cos(\theta) + y \sin(\theta)\big) d\gamma_n(y) \bigg) \frac{d\theta}{\sqrt{\pi |\log(\cos(\theta))|}} \\
&= \lim_{\varepsilon \to 0} \int_\varepsilon^{\frac{\pi}{2}} \bigg(\int_{\mathbb{R}^n} y \, f\big(x\cos(\theta) + y \sin(\theta)\big) d\gamma_n(y) \bigg) \frac{d\theta}{\sqrt{\pi |\log(\cos(\theta))|}}
\end{aligned}$$

while the second one yields

$$\begin{aligned}
\nabla(-\mathrm{L})^{-1/2} f(x) &= -\int_0^{\frac{\pi}{2}} \bigg(\int_{\mathbb{R}^n} y \, f\big(x\cos(\theta) - y \sin(\theta)\big) d\gamma_n(y) \bigg) \frac{d\theta}{\sqrt{\pi |\log(\cos(\theta))|}} \\
&= -\int_{-\frac{\pi}{2}}^{0} \bigg(\int_{\mathbb{R}^n} y \, f\big(x\cos(\theta) + y \sin(\theta)\big) d\gamma_n(y) \bigg) \frac{d\theta}{\sqrt{\pi |\log(\cos(\theta))|}} \\
&= -\lim_{\varepsilon \to 0} \int_{-\frac{\pi}{2}}^{-\varepsilon} \bigg(\int_{\mathbb{R}^n} y \, f\big(x\cos(\theta) + y \sin(\theta)\big) d\gamma_n(y) \bigg) \frac{d\theta}{\sqrt{\pi |\log(\cos(\theta))|}} \, .
\end{aligned}$$

Adding the two expressions, and using Fubini's theorem, it appears that

$$\nabla(-\mathrm{L})^{-1/2} f(x) = \int_{\mathbb{R}^n} y \, J(x,y) \, d\gamma_n(y)$$

where, for every $x, y \in \mathbb{R}^n$,

$$J(x,y) \;=\; \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\big(x\cos(\theta) + y\sin(\theta)\big)\varphi(\theta)d\theta \tag{11}$$

and

$$\varphi(\theta) = \frac{\mathrm{sgn}(\theta)}{2\sqrt{\pi\,|\log(\cos(\theta))|}}\,, \quad \theta \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}). \tag{12}$$

The function J is therefore understood as a principal-value integral in order to avoid the singularity at the origin, that is

$$\begin{aligned} J(x,y) \;&=\; \text{p.v.} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\big(x\cos(\theta) + y\sin(\theta)\big)\varphi(\theta)d\theta \\ &=\; \lim_{\varepsilon\to 0} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\big(x\cos(\theta) + y\sin(\theta)\big)\varphi(\theta)\mathbb{1}_{\{|\theta|>\varepsilon\}}d\theta. \end{aligned}$$

For every $x \in \mathbb{R}^n$, by duality and Hölder's inequality,

$$\begin{aligned} \bigg|\int_{\mathbb{R}^n} yJ(x,y)\,d\gamma_n(y)\bigg| \;&=\; \sup_{|c|\le 1} \int_{\mathbb{R}^n} \langle c, y\rangle J(x,y)\,d\gamma_n(y) \\ &\le\; \sup_{|c|\le 1} \bigg(\int_{\mathbb{R}^n} \big|\langle c, y\rangle\big|^q d\gamma_n(y)\bigg)^{\frac{1}{q}} \bigg(\int_{\mathbb{R}^n} \big|J(x,y)\big|^p d\gamma_n(y)\bigg)^{\frac{1}{p}} \end{aligned}$$

where q is the conjugate of $1 < p < \infty$. Hence

$$\bigg|\int_{\mathbb{R}^n} yJ(x,y)\,d\gamma_n(y)\bigg| \;\le\; K_p\bigg(\int_{\mathbb{R}^n} \big|J(x,y)\big|^p d\gamma_n(y)\bigg)^{\frac{1}{p}}$$

where $K_p = \big(\int_{\mathbb{R}} |x|^q d\gamma_1(x)\big)^{\frac{1}{q}}$ only depends on p. As a consequence,

$$\big\|\nabla(-\mathrm{L})^{-1/2}f\big\|_p^p \;\le\; K_p^p \int_{\mathbb{R}^n\times\mathbb{R}^n} \big|J(x,y)\big|^p\,d\gamma_n(x)d\gamma_n(y).$$

4 The transference argument

It therefore remains to show that, for $f : \mathbb{R}^n \to \mathbb{R}$,

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \big|J(x,y)\big|^p\,d\gamma_n(x) \;\le\; C_p\,\|f\|^p_{\mathrm{L}^p(\gamma_n)} \tag{13}$$

where $C_p > 0$ is another constant only depending on p, possibly changing from line to line below. To this task, the argument will develop the method of rotations to reduce the result to the boundedness of the Riesz transform on the torus. The method of rotations is classically used in the study of singular integrals on $\mathbb{R}^n$, see [4, 11], and transference methods are formalized in [5].

To make full use of the transference principle, it is useful to recall that the Hilbert transform on the unit circle $\mathbb{T} = (-\pi, \pi]$,

$$\begin{aligned}\mathcal{H}_{\mathbb{T}} g(\omega) &= \text{p.v.} \int_{\mathbb{T}} g(\omega+\theta) \cot\left(\tfrac{\theta}{2}\right) \tfrac{d\theta}{2\pi} \\ &= \lim_{\varepsilon\to 0} \int_{\{|\theta|>\varepsilon\}} g(\omega+\theta) \cot\left(\tfrac{\theta}{2}\right) \tfrac{d\theta}{2\pi}, \quad \omega \in \mathbb{T},\ g \in C_c^\infty(\mathbb{T}),\end{aligned}$$

is bounded in $\mathrm{L}^p(\mathbb{T})$ for any $1 < p < \infty$ (see [11] or [7, Chapter II Appendix]).

For each $\omega \in \mathbb{T}$, denote by R_ω the rotation operator on $\mathbb{R}^n \times \mathbb{R}^n$ defined by

$$R_\omega(x, y) = \big(x\cos(\omega) + y\sin(\omega), -x\sin(\omega) + y\cos(\omega)\big), \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Given $F \in \mathrm{L}^p(\gamma_n \otimes \gamma_n)$, for every $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, define

$$\omega \in \mathbb{T} \mapsto u_{x,y}(\omega) = F\big(R_\omega(x, y)\big).$$

Note that by the invariance of the Gaussian measure $\gamma_n \otimes \gamma_n$ under the rotations R_ω, for every $\omega \in \mathbb{T}$,

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \big|u_{x,y}(\omega)\big|^p d\gamma_n(x) d\gamma_n(y) = \|F\|^p_{\mathrm{L}^p(\gamma_n\otimes\gamma_n)}. \tag{14}$$

In particular, by Fubini's theorem, $u_{x,y} \in \mathrm{L}^p(\mathbb{T})$ for $\gamma_n \otimes \gamma_n$-almost every $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$.

Notice next that φ of (12) has the form

$$\varphi(\theta) = \frac{\operatorname{sgn}(\theta)}{2\sqrt{\pi|\log(\cos(\theta))|}} = \frac{1}{2\sqrt{2\pi}} \cot\left(\tfrac{\theta}{2}\right) + r(\theta)$$

where $r \in \mathrm{L}^\infty(\mathbb{T})$[1]. If Φ is then the operator

$$\Phi(g)(\omega) = \text{p.v.} \int_{\mathbb{T}} g(\omega+\theta)\varphi(\theta) \tfrac{d\theta}{2\pi}, \quad \omega \in \mathbb{T},\ g \in \mathrm{L}^p(\mathbb{T}),$$

by the L^p-boundedness of the Hilbert transform on the torus, Φ is a bounded operator on $\mathrm{L}^p(\mathbb{T})$. Hence, for almost every $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\big\|\Phi(u_{x,y})\big\|^p_{\mathrm{L}^p(\mathbb{T})} \le C_p \, \|u_{x,y}\|^p_{\mathrm{L}^p(\mathbb{T})}. \tag{15}$$

[1]Here the function φ is initially defined on $(-\frac{\pi}{2}, \frac{\pi}{2})$, and needs to be extended to $(-\pi, \pi]$ by letting $\varphi \equiv 0$ outside $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Using that $R_{\omega+\theta} = R_\omega \circ R_\theta$,

$$\Phi(u_{x,y})(\omega) \; = \; \text{p.v.} \int_{\mathbb{T}} u_{R_\omega(x,y)}(\theta)\varphi(\theta)d\theta. \tag{16}$$

Combining then (15) and (16), and integrating over (x,y),

$$\begin{aligned} \int_{\mathbb{R}^n \times \mathbb{R}^n} \bigg(\int_{\mathbb{T}} \bigg| \text{p.v.} \int_{\mathbb{T}} u_{R_\omega(x,y)}(\theta)\varphi(\theta)d\theta \bigg|^p d\omega \bigg) d\gamma_n(x)d\gamma_n(y) & \\ \leq \; C_p \int_{\mathbb{R}^n \times \mathbb{R}^n} \|u_{x,y}\|^p_{\mathrm{L}^p(\mathbb{T})} d\gamma_n(x)d\gamma_n(y) & \\ = \; C_p \, \|F\|^p_{\mathrm{L}^p(\gamma_n \otimes \gamma_n)} & \end{aligned} \tag{17}$$

where the invariance property (14) is used in the last step.

Using again the rotational invariance of Gaussian measures, the integral

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \bigg| \text{p.v.} \int_{\mathbb{T}} u_{R_\omega(x,y)}(\theta)\varphi(\theta)d\theta \bigg|^p d\gamma_n(x)d\gamma_n(y)$$

is independent of the value of ω. By (17) and Fubini's theorem, it finally follows that

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \bigg| \text{p.v.} \int_{\mathbb{T}} u_{x,y}(\theta)\varphi(\theta)d\theta \bigg|^p d\gamma_n(x)d\gamma_n(y) \; \leq \; C_p \, \|F\|^p_{\mathrm{L}^p(\gamma_n \otimes \gamma_n)}.$$

Recalling the definition (11) of J, the proof of (13) is complete after setting $F(x,y) = f(x)$, $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$.

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