Gaussian random matrix ensembles

The standard Gaussian measure $\gamma_n^{\mathbb{C}} = \gamma_n \otimes \gamma_n$ on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ with density

$$\frac{1}{(2\pi)^n}\, e^{-\frac{1}{2}|z|^2} \;=\; \frac{1}{(2\pi)^n}\, e^{-\frac{1}{2}(x^2+y^2)}, \quad z = x + iy \in \mathbb{C}^n,$$

with respect to the Lebesgue measure is invariant under the action (on vectors) of the unitary group. The coordinates of a random vector $(Z_1, \ldots, Z_n)$ with distribution $\gamma_n^{\mathbb{C}}$ are independent complex Gaussian random variables with mean zero and variance 2 (by complex Gaussian random variable with mean zero and variance σ^2, it is meant that the real and imaginary parts are independent centered Gaussian variables with variance $\frac{\sigma^2}{2}$).

The Gaussian measure $\gamma_n^{\mathbb{C}}$ on $\mathbb{C}^n$ admits a natural generalization as a Gaussian probability measure on the space $\mathcal{H}_n \cong \mathbb{R}^{n^2}$ of $n \times n$ Hermitian matrices $M = (M_{ij})_{1\leq i,j\leq n}$ with density

$$\frac{1}{2^{\frac{n}{2}}\pi^{\frac{n^2}{2}}}\, \exp\Big(-\frac{1}{2}\,\mathrm{Tr}(M^2)\Big) \tag{1}$$

with respect to the Lebesgue measure on $\mathcal{H}_n$,

$$\prod_{1\leq i\leq n} dM_{ii} \prod_{1\leq i<j\leq n} d\,\mathrm{Re}\,(M_{ij})\, d\,\mathrm{Im}\,(M_{ij}).$$

The distribution (1) is invariant under the action (on matrices) of the unitary group, and is known as the Gaussian Unitary Ensemble (GUE), a central object in random matrix theory. A random matrix $X = (X_{ij})_{1\leq i,j\leq n}$, defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$, distributed according to the GUE law (1) has entries above the diagonal consisting of independent complex (real on the diagonal) Gaussian random variables with mean zero and variance 1.

Drafted by M. L. **v1** March 2024

The eigenvalues of a GUE random matrix exhibit a number of both striking and universal statistics as the size n of the matrix tends to infinity, both at a global regime (spectral measure) and at a local one (individual behavior of the eigenvalues). The post features some main achievements in this framework. General references on the subject include [14, 2, 18, 8, 16, 7]...

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1 Joint law of the eigenvalues

Given a Hermitian $n \times n$ matrix X, denote by $\lambda_1 \leq \cdots \leq \lambda_n$ the list of its (real) eigenvalues (repeated by multiplicity). As a consequence of the unitary invariance and the Jacobian change of variable formula, the joint law of the sample of the eigenvalues of a GUE matrix is explicitly described as a Coulomb gas distribution.

Proposition 1 (Joint law of the GUE eigenvalues)**.** *If a random matrix X is distributed according to the GUE law* (1)*, the probability distribution of the eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ of X on the Weyl chamber $W = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n; x_1 < \cdots < x_n\}$ has density*

$$\frac{1}{Z_{n,2}} \prod_{1 \leq i < j \leq n} |x_i - x_j|^2 \, e^{-\frac{1}{2}|x|^2} \tag{2}$$

with respect to the Lebesgue measure (where $Z_{n,2} > 0$ is the normalization constant).

By symmetry under permutation of the coordinates, the probability distribution (2) may be, if necessary, extended to the whole of $\mathbb{R}^n$ (as will be implicitly used below in some instances). The Coulomb gas representation is made more explicit as

$$\frac{1}{Z_{n,2}} \exp\left(2 \sum_{1 \leq i < j \leq n} \log\left(|x_i - x_j|\right) - \frac{1}{2} \sum_{i=1}^{n} x_i^2\right), \tag{3}$$

showing a repulsion phenomenon as two "particules" x_i and x_j are getting close (cf. [8]).

2 Gaussian random matrix ensembles

The Gaussian Unitary Ensemble (GUE) admits a real version known as the Gaussian Orthogonal Ensemble (GOE) on the space of $n \times n$ real symmetric matrices M with density

$$\frac{1}{2^{\frac{n}{2}}(2\pi)^{\frac{n(n+1)}{4}}} \exp\Big(-\frac{1}{4} \operatorname{Tr}(M^2)\Big), \tag{4}$$

which is invariant by the orthogonal group. A real symmetric random matrix $X = (X_{ij})_{1 \leq i,j \leq n}$ with the GOE law (4) has entries X_{ij}, $1 \leq i \leq j \leq n$, which are independent centered real-valued Gaussian random variables with variance 1 (2 on the diagonal). By analogy with Proposition 1, the eigenvalues of X have a joint distribution with density

$$\frac{1}{Z_{n,1}} \prod_{1 \leq i < j \leq n} |x_i - x_j| \, e^{-\frac{1}{4}|x|^2}, \quad x \in W. \tag{5}$$

It is possible to consider symplectic ensembles for which the Vandermonde determinant is raised to the power 4, the powers 1, 2, 4 reflecting the underlying geometries of the classical group invariance. A striking observation by I. Dumitriu and A. Edelman [5] is the construction, for any $\beta > 0$, of matrix models such that the distribution of the eigenvalues has density

$$\frac{1}{Z_{n,\beta}} \prod_{1 \leq i < j \leq n} |x_i - x_j|^\beta \, e^{-\frac{\beta}{4}|x|^2}, \quad x \in W, \tag{6}$$

with respect to the Lebesgue measure, known as Gaussian Beta Ensembles (GβE). These are given by tridiagonal matrices of the form

$$M = \frac{1}{\sqrt{\beta}} \begin{pmatrix} g_1 & \chi_{\beta(n-1)} & & & \\ \chi_{\beta(n-1)} & g_2 & \chi_{\beta(n-2)} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & g_{n-1} & \chi_\beta \\ & & & \chi_\beta & g_n \end{pmatrix} \tag{7}$$

where, on the diagonal, $g_1, \ldots, g_n$ are independent Gaussian random variables with mean 0 and variance 2, and on the upper and lower diagonals, $\chi_\beta, \chi_{2\beta}, \ldots, \chi_{(n-1)\beta}$ are independent chi-squared random variables of the indicated parameter, independent from $g_1, \ldots, g_n$.

3 Determinantal correlation functions

The analysis of the GUE is made possible by the determinantal representation of the eigenvalue distribution as a Coulomb gas and the use of orthogonal polynomials, a specific feature related to the fact that the Vandermonde determinant is squared in the joint law of the eigenvalues, allowing for an accessible study and justifying thus the central role of the GUE in random matrix theory.

This determinantal point process representation is the key towards the asymptotics results on eigenvalues of large random matrices, both inside the bulk (spacing between the eigenvalues) and at the edge of the spectrum.

Denote by h_ℓ, $\ell \in \mathbb{N}$, the normalized Hermite polynomials with respect to γ_1, which form an orthonormal basis of $\mathrm{L}^2(\gamma_1)$ [1]. Since, for each ℓ, h_ℓ is a polynomial function of degree ℓ, elementary manipulations on rows or columns show that, up to a constant depending on n,

$$\prod_{1 \le i < j \le n} (x_i - x_j) \,\sim\, \det\big(h_{\ell-1}(x_k)\big)_{1 \le k, \ell \le n}.$$

From the orthogonality properties of the Hermite polynomials, the marginal distributions of dimension $1 \le r \le n$ of the sample $(\lambda_1, \ldots, \lambda_n)$ of the eigenvalues of a GUE matrix may be shown to have densities, with to respect γ_r, proportional to the determinants

$$\det((K_n(x_i, x_j))_{1 \le i,j \le r}$$

of the (Hermite) kernel

$$K_n(x, y) \,=\, \sum_{\ell=0}^{n-1} h_\ell(x) h_\ell(y), \quad x, y \in \mathbb{R}$$

(cf. [3, 2, 8, 18, 16]...).

As illustrations of this description of the GUE correlation functions, for any non-negative measurable function $f : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}\bigg(\frac{1}{n} \sum_{i=1}^n f(\lambda_i)\bigg) = \int_{\mathbb{R}} f \, \frac{1}{n} \, K_n(x, x) \, d\gamma_1(x), \tag{8}$$

or, for any $t \in \mathbb{R}$,

$$\begin{aligned} \mathbb{P}(\lambda_n \le t) \,&=\, \sum_{r=0}^n (-1)^r \sum_{1 \le i_1 < \cdots < i_r \le n} \mathbb{P}\big(\lambda_{i_1} > t, \ldots, \lambda_{i_r} > t\big) \\ &=\, \sum_{r=0}^n \frac{(-1)^r}{r!} \int_{(t,\infty)^r} \det\big(K_n(x_i, x_j)\big)_{1 \le i,j \le r} \, d\gamma_r(x_1, \ldots, x_r). \end{aligned} \tag{9}$$

It is on the basis of such formulas that the asymptotics of the eigenvalues as the size of the matrix tends to infinity may be analyzed.

This correlation analysis is also possible for the eigenvalues of the GOE, but is more delicate and involves Pfaffians instead of determinants. The GβE are investigated by completely different methods.

To emphasize dependence as the size of the matrix tends to infinity, in the rest of the text, X^n is an $n \times n$ random matrix, defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$, following the GUE distribution (1), with eigenvalues $\lambda_1^n \leq \cdots \leq \lambda_n^n$.

4 Asymptotic spectral measure

The first basic asymptotic result is Wigner's theorem [22] about the limiting behavior of the empirical spectral measure on the (scaled) eigenvalues. Let d be a distance (such as the Lévy distance) which metrizes weak convergence of probability measures on the real line.

Theorem 2 (Wigner's theorem)**.** *In probability,*

$$d\bigg(\frac{1}{n}\sum_{i=1}^{n}\delta_{\frac{\lambda_i^n}{\sqrt{n}}}\,,\,\mathrm{sc}\bigg)\;\to\;0$$

where sc *is the semi-circle law with density* $\rho(x) = \frac{1}{2\pi}\sqrt{4-x^2}$ *with respect to the Lebesgue measure on* $(-2, 2)$.

The same statement holds true for random matrices from the GβE [2, 8].

Since the original moment method, numerous proofs of Wigner's theorem have been made available in the literature (cf. [14, 2, 18, 16] and the references therein). An analytic argument may easily be produced on the kernel representation (8), showing, via successive integration by parts with respect to γ_1, that the $2p$-moments of the (normalized) spectral measure,

$$b_n^p \;=\; \mathbb{E}\bigg(\frac{1}{n}\sum_{i=1}^{n}\Big(\frac{\lambda_i^n}{\sqrt{n}}\Big)^{2p}\bigg) \;=\; \frac{1}{n^{p+1}}\int_{\mathbb{R}} x^{2p}K_n(x,x)\,d\gamma_1(x),$$

satisfy the recurrence equation, $p \geq 2$,

$$b_n^p \;=\; 4\,\frac{2p-1}{2p+2}\,b_n^{p-1} + 4\,\frac{2p-1}{2p+2}\cdot\frac{2p-3}{2p}\cdot\frac{p(p-1)}{4n^2}\,b_n^{p-2}$$

($b_n^0 = b_n^1 = 1$), a recursion formula going back to [11]. As $n \to \infty$, the limiting sequence $\chi^p = \lim_{n\to\infty} b_n^p$ satisfies

$$\chi^p \;=\; 4\,\frac{2p-1}{2p+2}\,\chi^{p-1} \;=\; \frac{(2p)!}{p!(p+1)!}$$

which is exactly the recurrence equation of the even moments of the semi-circle law (cf. [10, 13]). This argument shows the convergence of the expected spectral measure $\mathbb{E}\big(\frac{1}{n}\sum_{i=1}^{n}\delta_{\lambda_i^n/\sqrt{n}}\big)$ from which the convergence in probability of the statement is easily deduced by measure concentration arguments [10].

5 Local limits of the GUE eigenvalues

The following are the three main fluctuations results on the local behavior of the eigenvalues of matrices from the GUE. As in the preceding section, X^n is an $n \times n$ random matrix following the GUE distribution (1), with eigenvalues $\lambda_1^n \leq \cdots \leq \lambda_n^n$.

The first statement describes the asymptotic behavior of the suitably scaled largest eigenvalue λ_n^n. It may be drawn from the correlation formula (9) as

$$\lim_{n\to\infty} \mathbb{P}\big(\lambda_n^n \leq 1 + sn^{-2/3}\big) \;=\; \det\Big(\big[\mathrm{Id} - K_{\mathrm{Ai}}\big]_{\mathrm{L}^2(s,\infty)}\Big), \quad s \in \mathbb{R}, \tag{10}$$

where, after suitable asymptotics on the Hermite polynomials known as Plancherel-Rotach asymptotics,

$$K_{\mathrm{Ai}}(x,y) = \frac{\mathrm{Ai}\,(x)\mathrm{Ai}'\,(y) - \mathrm{Ai}'\,(x)\mathrm{Ai}\,(y)}{x-y}\,, \quad x,y \in \mathbb{R},$$

is the Airy kernel associated to the special Airy function solution of $\mathrm{Ai}'' = x\mathrm{Ai}$ with the asymptotics $\mathrm{Ai}(x) \sim \frac{1}{2\sqrt{\pi}x^{1/4}}\, e^{-\frac{2}{3}x^{3/2}}$ as $x \to \infty$.

The Fredholm determinant (10) thus describes the asymptotic behavior of the largest (and by symmetry smallest) eigenvalue of the GUE. A main achievement by C. Tracy and H. Widom [20] was to provide a purely analytic description of (10), known today as the Tracy-Widom distribution.

Theorem 3 (Asymptotics of the extreme eigenvalues)**.** *The normalized largest eigenvalue $\frac{1}{\sqrt{n}}\lambda_n^n$ converges in probability to 2, the right-edge of the support of the semi-circle law sc. In distribution,*

$$n^{\frac{1}{6}}\big(\lambda_n^n - 2\sqrt{n}\big) \;\to\; \mathrm{TW}_2 \tag{11}$$

as $n \to \infty$, where TW_2 follows the Tracy-Widom probability distribution given by

$$F_{\mathrm{TW}_2}(t) \;=\; \exp\bigg(-\int_t^{\infty}(x-t)q(x)^2dx\bigg), \quad t \in \mathbb{R},$$

where $q = q(x)$ is the solution to the Painlevé II ordinary differential equation $q'' = xq + 2q^3$, with asymptotics given by the Airy function $q(x) \sim \mathrm{Ai}(x)$, $x \to \infty$.

This result has been extended to the orthogonal and symplectic ensembles [21], with related limiting distributions TW_1 and TW_4, but also to the GβE [17], for every $\beta > 0$, with however a much different proof based on the explicit tridiagonal matrix representation (7), and a different description of the limiting law TW_β as the random variable

$$\sup_{f \in L} \left[\frac{2}{\sqrt{\beta}} \int_0^\infty f(t)^2 dB(t) - \int_0^\infty \left[f'(t)^2 + t f(t)^2 \right] dt \right.$$

where $B(t)$, $t \geq 0$, is a standard Brownian motion, and L the family of (differentiable) functions $f : [0, \infty) \to \mathbb{R}$ such that $f(0) = 0$, $\int_0^\infty f(t)^2 dt = 1$ and $\int_0^\infty [f'(t)^2 + t f(t)^2] dt < \infty$.

It is worthwhile mentioning, following also [17], that

$$F_{\mathrm{TW}_\beta}(t) \sim e^{\frac{\beta}{24} t^3} \text{ as } t \to -\infty, \quad 1 - F_{\mathrm{TW}_\beta}(t) \sim e^{-\frac{2\beta}{3} t^{\frac{3}{2}}} \text{ as } t \to \infty,$$

showing an asymmetric behavior on the left and right of the largest eigenvalue.

The second local limit concerns eigenvalues in the so-called bulk of the spectrum developed by J. Gustavsson [9]. Recalling the density $\rho(x) = \frac{1}{2\pi} \sqrt{4 - x^2}$ on $(-2, 2)$ of the semi-circle law, let $t_j = t_j^n$ be the theoretical location of the j-th particule, $j = 1, \ldots, n$, defined by $\int_{-\infty}^{t_j} \rho(x) dx = \frac{j}{n}$.

Theorem 4 (Asymptotics of the bulk eigenvalues)**.** *In the bulk, for* $\frac{j}{n} = \frac{j(n)}{n} \to \theta \in (0, 1)$*,*

$$\sqrt{\frac{2\pi^2}{\log n}} \, \rho(t_j) \sqrt{n} \left(\lambda_j^n - t_j \sqrt{n} \right) \tag{12}$$

converges weakly to a standard normal variable.

A key argument of the proof is the calculation of the asymptotic behavior of the variance of the eigenvalue counting function by means of the two-dimensional correlation functions. There is a multivariate central limit theorem for a (finite) sample of eigenvalues in the bulk. The theorem, and its multi-dimensional version, have been extended to the GOE in [15] by an interlacing argument, and to the GβE in [4] by means of optimal local laws on the Stieltjes transform of the spectral measure.

While the preceding statement describes the asymptotic behaviour of the eigenvalues in the bulk of the spectrum, it is of interest to also understand the fluctuations of the spacings reflecting the repulsion property of the underlying Coulomb gas (3). The following statement, achieved by T. Tao [19], provides the suitable answer to this issue, with a proof relying on a delicate approximate independence property between the eigenvalue counting function and the event that there is no spectrum in a small interval, following a careful determinantal process analysis.

Theorem 5 (Asymptotics of spacings of the bulk eigenvalues)**.** *Whenever $\varepsilon n \leq j \leq (1-\varepsilon)n$ for some $\varepsilon \in (0, \frac{1}{2})$, in distribution,*

$$\rho(t_j)\sqrt{n}\,(\lambda_{j+1}^n - \lambda_j^n) \;\to\; \mathrm{G} \tag{13}$$

as $n \to \infty$, where G *has the Gaudin probability distribution $F_{\mathrm{G}} = \int_0^t p(s)ds$, $t \geq 0$, with density p given by*

$$p(s) \;=\; \frac{1}{s^2}\big[\sigma(\pi s)^2 + \pi s\sigma'(\pi s) - \sigma(\pi s)\big]\exp\bigg(\int_0^{\pi s}\frac{\sigma(x)}{x}\,dx\bigg), \quad s > 0,$$

where $\sigma = \sigma(x)$ solves the Painlevé V ordinary differential equation

$$(x\sigma'')^2 + 4(x\sigma' - \sigma)\big(x\sigma' - \sigma + (\sigma')^2\big) \;=\; 0$$

with boundary condition $\sigma(x) \sim -\frac{x}{\pi}$ as $x \to 0$.

The Gaudin distribution is firstly described as the derivative of the Fredholm determinant

$$\det\Big(\big[\mathrm{Id} - K_{\mathrm{Sine}}\big]_{\mathrm{L}^2([0,t])}\Big), \quad t \geq 0,$$

of the integral operator associated to the Sine kernel

$$K_{\mathrm{Sine}}(x, y) \;=\; \frac{\sin(\pi(x-y))}{\pi(x-y)}, \quad x, y \in \mathbb{R}.$$

The contribution [12] (prior to [20]) provides the alternate analytic description of F_{G} in terms a Painlevé equation.

It does not seem that Theorem 5 has been extended so far to the other Gaussian Ensembles, even to the GOE.

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