

# Brownian motion and Wiener measure

Brownian motion, or Wiener process, is a fundamental Gaussian process, both describing the random motion of a particule in a fluid and introducing a basic continuous-time stochastic process whose law, the Wiener measure, defines a Gaussian measure or vector on the infinite-dimensional space of continuous functions.

The wealth of properties, results, studies on Brownian motion and the Wiener measure, and related processes, would require a specific blog. The purpose here is only to emphasize a few specific features related to their Gaussian structure. Basic textbooks on Brownian motion, such as the renowned [7, 4, 6], or the more recent [5], easily cover the brief material exposed here.

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# 1 Brownian motion

A random process  $X = (X_t)_{t \geq 0}$ , indexed by  $[0, \infty)$  defined on some probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , is said to be Gaussian if, for every  $t_1, \dots, t_n \geq 0$ , the vector  $(X_{t_1}, \dots, X_{t_n})$  is Gaussian in  $\mathbb{R}^n$ . It is assumed below that  $X = (X_t)_{t \geq 0}$  is centered in the sense that  $\mathbb{E}(X_t) = 0$  for every  $t \geq 0$ . Following the characterization of Gaussian vectors, the finite-dimensional distributions of the process  $X = (X_t)_{t \geq 0}$  are fully determined by the covariance function  $\Sigma(s, t) = \mathbb{E}(X_s X_t)$ ,  $s, t \geq 0$ .

There are many examples of Gaussian covariance functions in this context. One of them is of most interest and given by

$$\Sigma(s, t) = \mathbb{E}(X_s X_t) = \min(s, t), \quad s, t \geq 0. \quad (1)$$

It is easily seen that it is indeed a covariance function. A few properties of this covariance function are worthwhile emphasizing. First, for each  $s < t$ , the law of  $X_t - X_s$  is Gaussian with law  $\mathcal{N}(0, t-s)$  ( $X_0 = 0$  almost surely). Then, the increments of the process  $X = (X_t)_{t \geq 0}$  are independent: for  $0 \leq t_0 < t_1 < \dots < t_n$ , the random variables

$$X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}$$

are independent (with respective laws  $\mathcal{N}(0, t_1 - t_0), \mathcal{N}(0, t_2 - t_1), \dots, \mathcal{N}(0, t_n - t_{n-1})$ ). Equivalently, the vector  $\left(\frac{X_k - X_{k-1}}{\sqrt{t_k - t_{k-1}}}\right)_{1 \leq k \leq n}$  is of law  $\mathcal{N}(0, \text{Id})$  in  $\mathbb{R}^n$ .

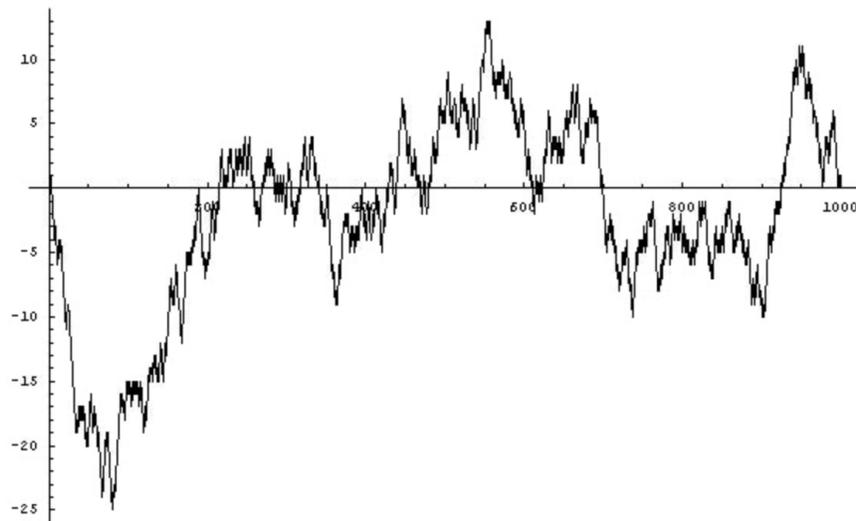
These properties actually characterize the (finite-dimensional) distribution of the so-called *Brownian motion*, a process describing the stochastic and without memory motion of a

particle in a fluid, typically denoted by  $B = (B_t)_{t \geq 0}$ , a notation adopted below. A Brownian motion  $B = (B_t)_{t \in [0, T]}$  indexed on some interval  $[0, T]$ ,  $T > 0$ , is of course constructed similarly.

From a functional point of view, since

$$\mathbb{E}(|B_s - B_t|^2) = |s - t|, \quad s, t \geq 0,$$

Kolmogorov's continuity theorem (cf. [1]) ensures that there is a version of the process  $B$  (denoted in the same way) with almost surely continuous trajectories  $t \in [0, \infty) \mapsto B_t$ . They may be even shown almost surely  $\alpha$ -Hölder continuous for every  $\alpha < \frac{1}{2}$ . However, the paths are almost surely nowhere differentiable.



## 2 Series expansions

An alternate viewpoint on the construction and continuity of the Brownian paths is provided by uniform convergence of series. This aspect is developed in more generality in the study of abstract Wiener spaces [2], but the specific model of Brownian motion allows for an explicit study.

Let  $(e_k)_{k \in \mathbb{N}}$  be an orthonormal basis of the Hilbert space  $L^2([0, 1])$  with respect to the Lebesgue measure  $\lambda_1$ . Introduce the so-called Schauder functions

$$h_k(t) = \int_{[0, t]} e_k d\lambda_1 = \langle \mathbb{1}_{[0, t]}, e_k \rangle_{L^2([0, 1])}, \quad k \in \mathbb{N},$$

which define an orthonormal basis of a Hilbert space known as the Cameron-Martin reproducing kernel Hilbert space  $\mathcal{H}$  associated to the covariance structure  $\Sigma$ . The Hilbertian structure of  $\mathcal{H}$  is indeed induced by  $L^2([0, 1])$  via the scalar product

$$\langle h_k, h_\ell \rangle_{\mathcal{H}} = \int_{[0,1]} h'_k h'_\ell d\lambda_1 = \int_{[0,1]} e_k e_\ell d\lambda_1$$

for  $k, \ell \in \mathbb{N}$ .

On some probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , consider a sequence  $(g_k)_{k \in \mathbb{N}}$  of independent standard normal variables, and set, for each  $n \geq 0$ ,

$$B_t^n = \sum_{k=0}^n g_k h_k(t), \quad t \in [0, 1].$$

It is immediate to check that, for every  $t \in [0, 1]$ ,  $B_t^n$  converges as  $n \rightarrow \infty$ , almost surely and in  $L^2(\mathbb{P})$ , to a random variable  $B_t$  with law  $\mathcal{N}(0, t)$ , and that moreover  $\mathbb{E}(B_s B_t) = \min(s, t)$  for every  $s, t \in [0, 1]$ . Indeed,

$$\mathbb{E}(B_s^n B_t^n) = \sum_{k=0}^n h_k(s) h_k(t) = \sum_{k=0}^n \langle \mathbb{1}_{[0,s]}, e_k \rangle_{L^2([0,1])} \langle \mathbb{1}_{[0,t]}, e_k \rangle_{L^2([0,1])}$$

which converges, by Parseval's identity, to

$$\langle \mathbb{1}_{[0,s]}, \mathbb{1}_{[0,t]} \rangle_{L^2([0,1])} = \int_{[0,1]} \mathbb{1}_{[0,s]} \mathbb{1}_{[0,t]} d\lambda_1 = \min(s, t).$$

The finite-dimensional distributions thus coincide with the definition of a Brownian motion  $B = (B_t)_{t \in [0,1]}$  on  $[0, 1]$ .

It may actually be shown that the sequence  $(B^n)_{n \in \mathbb{N}}$  converges, almost surely and in  $L^2(\mathbb{P})$ , in  $C([0, 1])$  (for the uniform norm), ensuring therefore the continuity of the Brownian paths  $B = (B_t)_{t \in [0,1]}$ , and representing  $B$  as the series  $B = \sum_{k \in \mathbb{N}} g_k h_k$ . This series expansion holds true for any orthonormal basis  $(e_k)_{k \in \mathbb{N}}$  of  $L^2([0, 1])$  (cf. [2]), but specific choices are of more interest, and provide a simplified treatment of the uniform convergence.

For example, the Haar basis, labeled as  $e_\ell^{(n)}$ ,  $n \in \mathbb{N}$ ,  $\ell \in I(n)$ ,  $I(n)$  being the set of odd integers between 0 and  $2^n$ , is defined by

$$e_\ell^{(n)} = 2^{(n-1)/2} (\mathbb{1}_{[\frac{\ell-1}{2^n}, \frac{\ell}{2^n})} - \mathbb{1}_{[\frac{\ell}{2^n}, \frac{\ell+1}{2^n})})$$

(with  $e_1^{(0)} = \mathbb{1}_{[0,1]}$ ). For  $h_\ell^{(n)}$ ,  $n \in \mathbb{N}$ ,  $\ell \in I(n)$ , the associated Schauder sequence, and  $g_\ell^{(n)}$ ,  $n \in \mathbb{N}$ ,  $\ell \in I(n)$ , a sequence of independent standard normal random variables, the sequence of functions

$$B_t^n = \sum_{m=0}^n \sum_{\ell \in I(m)} g_\ell^{(m)} h_\ell^{(m)}(t), \quad t \in [0, 1],$$

may easily be shown to converge uniformly almost surely. Namely, for every  $m \in \mathbb{N}$ ,

$$\mathbb{P}\left(\max_{\ell \in I(m)} |g_\ell^{(m)}| \geq m\right) \leq \sum_{\ell \in I(m)} \mathbb{P}(|g_\ell^{(m)}| \geq m) \leq 2^m e^{-\frac{1}{2}m^2}.$$

Hence, by the Borel-Cantelli lemma, there exists a measurable set  $\Omega_0$  with  $\mathbb{P}(\Omega_0) = 1$  such that, for every  $\omega \in \Omega_0$ , there exists  $m_0(\omega)$  with  $\max_{\ell \in I(m)} |g_\ell^{(m)}(\omega)| \leq m$  for every  $m \geq m_0(\omega)$ . Now, at the level  $m$ , the Schauder functions  $h_\ell^{(m)}$ ,  $\ell \in I(m)$ , have disjoint supports and maximal height  $2^{-(m+1)/2}$ , so that, uniformly in  $t \in [0, 1]$ ,

$$\sum_{m \geq m_0(\omega)} \sum_{\ell \in I(m)} |g_\ell^{(m)}(\omega)| |h_\ell^{(m)}(t)| \leq \sum_{m \geq m_0(\omega)} m 2^{-(m+1)/2} < \infty.$$

Therefore  $B^n$ ,  $n \in \mathbb{N}$ , converges almost surely in  $C([0, 1])$ , and its limit  $B = (B_t)_{t \in [0, 1]}$  is a Brownian motion (on  $[0, 1]$ ), represented by the expansion

$$B_t = \sum_{n \in \mathbb{N}} \sum_{\ell \in I(n)} g_\ell h_\ell^{(n)}, \quad t \in [0, 1]. \quad (2)$$

Another classical series representation stems for the trigonometric basis of  $L^2([0, 1])$  leading, for example, to the expansion

$$B_t = g_0 t + \sqrt{2} \sum_{k=1}^{\infty} g_k \frac{\sin(\pi k t)}{\pi k}, \quad t \in [0, 1]. \quad (3)$$

While the preceding series representations define a Brownian motion on  $[0, 1]$ , it is easily extended to a process  $(B_t)_{t \geq 0}$  indexed on  $[0, \infty)$ . For example, consider a sequence  $(b_t^n)_{t \in [0, 1]}$ ,  $n \geq 1$ , of independent Brownian motions on  $[0, 1]$ , and set

$$B_t = b_1^1 + \dots + b_1^n + b_{t-n}^{n+1}, \quad t \in [n, n+1), \quad n \geq 0$$

(with the convention  $b_1^0 = 0$ ). It is clear that the process  $(B_t)_{t \geq 0}$  thus defined fulfills all the axioms of the Brownian motion as described by its finite-dimensional distributions in the first section, and is almost surely continuous.

It is also possible to consider a standard Brownian motion with values in  $\mathbb{R}^n$  as

$$B_t = (b_t^1, \dots, b_t^n), \quad t \geq 0,$$

where  $b^1, \dots, b^n$  are independent one-dimensional Brownian motions.

### 3 Invariance principle

Another view on the construction of Brownian motion is provided by a limit of random walks, describing thus Brownian motion as a kind of infinitesimal random walk.

Given a sequence  $(Y_n)_{n \geq 1}$  of independent identically distributed real random variables on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , with mean zero and variance one, the classical Central Limit Theorem (cf. [3]) expresses that the sequence

$$\frac{1}{\sqrt{n}} \sum_{k=1}^n Y_k, \quad n \geq 1,$$

converges in distribution to a random variable with law  $\mathcal{N}(0, 1)$ .

Consider now the sequence of continuous functions on  $[0, 1]$ ,

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} (Y_k + (nt - \lfloor nt \rfloor) Y_{\lfloor nt \rfloor + 1}), \quad t \in [0, 1], \quad n \geq 1,$$

where  $\lfloor \cdot \rfloor$  is the integer part function, and with the convention  $\sum_{k=1}^0 = 0$ , interpolating linearly between the values  $\frac{1}{\sqrt{n}} \sum_{\ell=1}^k Y_\ell$  at points  $\frac{k}{n}$ ,  $k = 0, 1, \dots, n$ . It is a main achievement, known as Donsker's invariance principle, that this sequence converges almost surely, in the uniform topology, to a Brownian motion on  $[0, 1]$ .

While providing at the same time another approach to the Brownian paths, this result also consecrates the latter as the universal limiting Gaussian process as limit of random walks.

### 4 Wiener measure

The process  $B = (B_t)_{t \in [0, 1]}$  with covariance (1) constructed in the preceding sections, in particular explicitly as a uniformly convergent series, may therefore also be considered as a Gaussian random vector with values in the space  $E = C([0, 1])$  of continuous functions on  $[0, 1]$  equipped with the uniform norm. The law of this Gaussian vector  $B$  defines a probability distribution  $\mu$  on the Borel sets of  $C([0, 1])$ , called the *Wiener measure*. Under the Wiener measure on  $C([0, 1])$ , the coordinate maps  $x \in C([0, 1]) \mapsto x(t)$ ,  $t \in [0, 1]$ , such that  $x(0) = 0$ , follow the Brownian motion distribution, also called Wiener process, with the traditional notation  $W : [0, 1] \mapsto W(t) = W_t$  (used in this section).

In the language of [2], the Wiener measure thus induces an abstract Wiener space structure  $(E, \mathcal{H}, \mu)$  where the reproducing kernel Hilbert space  $\mathcal{H}$  is identified with the Cameron-Martin Hilbert space of absolutely continuous function  $h$  with (almost everywhere) derivative

$h'$  in  $L^2([0, 1])$  (with respect to the Lebesgue measure). The Wiener measure may be constructed in the more general framework of abstract Wiener spaces, but the combined definition of classical Brownian motion and Wiener measure on the space of continuous functions is a privileged example.

In particular, it might be useful to recall (from e.g. [2]) the Cameron-Martin translation formula in this context, expressing that for  $h \in \mathcal{H}$ , the shifted measure  $\mu(\cdot + h)$  has density

$$\exp\left(-\frac{1}{2}\int_0^1 h'(t)^2 dt - \int_0^1 h'(t)dW(t)\right) \quad (4)$$

with respect to  $\mu$ .

In this formula, the Wiener integral  $\int_{[0,1]} f(t) dW(t) = \int_{[0,1]} f_t dB_t$  of a function  $f = f_t = f(t)$ ,  $t \in [0, 1]$ , in  $L^2([0, 1])$  is a simplified (prior) version of the Itô integral. Whenever  $f = \sum_{i=1}^n c_i \mathbb{1}_{[t_{i-1}, t_i]}$ ,  $c_1, \dots, c_n \in \mathbb{R}$ ,  $0 \leq t_0 < t_1 < \dots < t_n \leq 1$ , is a step function,

$$\int_{[0,1]} f_t dW_t = \sum_{i=1}^n c_i (W_{t_i} - W_{t_{i-1}}).$$

By independence and normal distributions of the increments  $W_{t_i} - W_{t_{i-1}}$ ,  $i = 1, \dots, n$ ,

$$\mathbb{E}\left(\left|\int_{[0,1]} f_t dW_t\right|^2\right) = \sum_{i=1}^n c_i^2 (t_i - t_{i-1}) = \int_{[0,1]} f_t^2 d\lambda_1(t).$$

On the basis of this identity, a density argument in the Hilbert space  $L^2([0, 1])$  allows for the definition of  $\int_{[0,1]} f_t dW_t$ , for any measurable function  $f = f_t$ ,  $t \in [0, 1]$ , such that  $\int_{[0,1]} f_t^2 d\lambda_1(t) < \infty$ , as a random variable with law  $\mathcal{N}(0, \sigma^2)$ ,  $\sigma^2 = \int_{[0,1]} f_t^2 d\lambda_1(t)$ .

## 5 Scaling, time reversal, and reflection principle

Some remarkable properties of the Brownian process  $B = (B_t)_{t \geq 0}$  may be recorded.

For any  $\alpha > 0$ , the process

$$\alpha B_{\frac{t}{\alpha^2}}, \quad t \geq 0,$$

is again a Brownian motion (check the covariances). Brownian motion is said to be stable of index 2.

Next, the process

$$t B_{\frac{1}{t}}, \quad t > 0$$

(vanishing at  $t = 0$ ) is also a Brownian motion. Brownian motion is said to be invariant by time reversal.

Finally, the famous reflection principle expresses that for any  $t \geq 0$  and  $a \geq 0$ ,

$$\mathbb{P}\left(\sup_{0 \leq s \leq t} B_s \geq a\right) = 2\mathbb{P}(B_t \geq a) = \mathbb{P}(|B_t| \geq a). \quad (5)$$

## 6 Martingale and strong Markov properties

Given a Brownian motion  $B = (B_t)_{t \geq 0}$  on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , consider, for every  $t \geq 0$ ,  $\mathcal{F}_t$  the  $\sigma$ -field generated by  $B_s$ ,  $s \leq t$ . Clearly  $\mathcal{F}_0$  is the trivial field, and  $(\mathcal{F}_t)_{t \geq 0}$  forms an increasing sequence of  $\sigma$ -fields, called a filtration.

The adapted family  $(B_t, \mathcal{F}_t)_{t \geq 0}$  is a martingale, that is, for every  $s \leq t$ ,

$$\mathbb{E}(B_t | \mathcal{F}_s) = B_s. \quad (6)$$

This is rather immediate since  $B_t - B_s$  is independent from  $\mathcal{F}_s$  by the independence of the increments of Brownian motion (and the fact that  $\mathcal{F}_s$  is also generated by  $B_u - B_v$ ,  $v < u \leq s$ ).

As a second result, the adapted family  $(B_t^2 - t, \mathcal{F}_t)_{t \geq 0}$  is also a martingale, which may be expressed, for every  $s \leq t$ , by

$$\mathbb{E}(B_t^2 - B_s^2 | \mathcal{F}_s) = t - s \quad (7)$$

For the proof, since  $\mathbb{E}(B_s B_t | \mathcal{F}_s) = B_s \mathbb{E}(B_t | \mathcal{F}_s) = B_s^2$  by (6),

$$\mathbb{E}(B_t^2 - B_s^2 | \mathcal{F}_s) = \mathbb{E}((B_t - B_s)^2 | \mathcal{F}_s) = \mathbb{E}((B_t - B_s)^2) = t - s$$

using again that  $B_t - B_s$  is independent from  $\mathcal{F}_s$ .

It is a famous result by P. Lévy that Brownian motion is the only continuous martingale  $(X_t, \mathcal{F}_t)_{t \geq 0}$  such that  $(X_t^2 - t, \mathcal{F}_t)_{t \geq 0}$  is also a martingale.

Brownian motion enjoys the strong Markov property: for any stopping time  $T$  of the filtration  $(\mathcal{F}_t)_{t \geq 0}$ , conditionally on  $\{T < \infty\}$ , the process  $(B_{T+t} - B_T)_{t \geq 0}$  is a Brownian motion independent of the process  $(B_t)_{t \in [0, T]}$ .

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