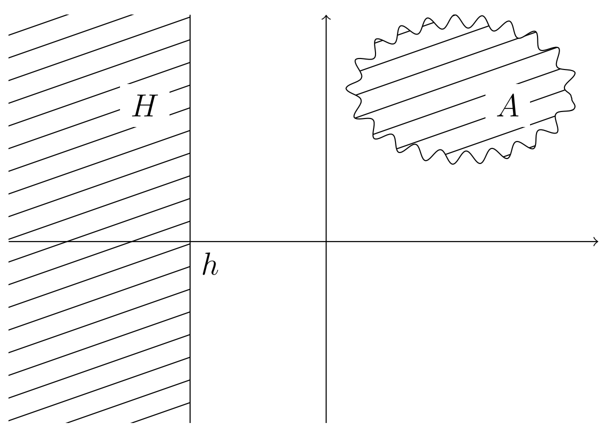
The Gaussian isoperimetric inequality

The classical isoperimetric inequality in Euclidean space expresses that balls are the sets with minimal surface measure given the volume. A similar property holds true on the sphere, on which geodesic balls (caps) are the extremizers of the isoperimetric problem.

Equip now $\mathbb{R}^n$ with the standard Gaussian probability measure γ_n, with density $\frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}|x|^2}$, $x \in \mathbb{R}^n$, with respect to the Lebesgue measure. For a fixed Gaussian measure $\gamma_n(A)$, what are the Borel sets A with the minimal surface measure (in the sense for example, of the Minkowski content $\gamma_n^+(A) = \liminf_{r \to 0} \frac{1}{r} \big[\gamma_n(A_r) - \gamma_n(A)\big]$)? The striking answer is that half-spaces H are the extremal sets of the Gaussian isoperimetric problem.



The Gaussian isoperimetric inequality is part of a family of geometric inequalities satisfied by Gaussian measures, described in the parallel note [1]. Due to its dimension-free character,

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it is a main tool in the analysis of infinite-dimensional Gaussian measures and vectors, and and the root of concentration inequalities (cf. [2]). This text reviews the known proofs of the Gaussian isoperimetric inequality.

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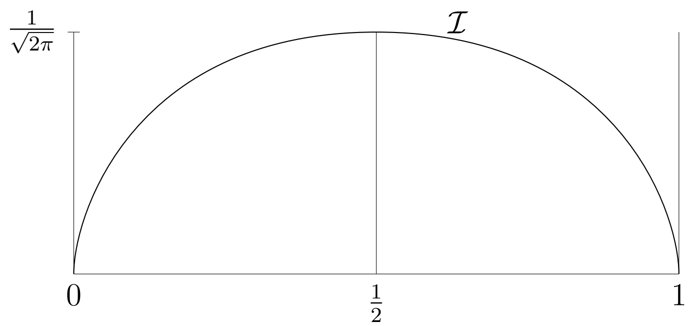
1 The Gaussian isoperimetric inequality

Let γ_n be the standard Gaussian probability measure on the Borel sets of $\mathbb{R}^n$, with density $\varphi_n(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}|x|^2}$, $x \in \mathbb{R}^n$, with respect to the Lebesgue measure. Denote by $\Phi(t) = \int_{-\infty}^{t} \varphi_1(x)dx$, $t \in \mathbb{R}$, the (continuous, strictly increasing) distribution function in dimension one, and define then the Gaussian isoperimetric profile

$$\mathcal{I}(s) = \varphi_1 \circ \Phi^{-1}(s), \quad s \in [0,1]. \tag{1}$$

The function $\mathcal{I}$ is symmetric along the vertical line $s = \frac{1}{2}$, and such that $\mathcal{I}(0) = \mathcal{I}(1) = 0$. It is worthwhile observing that $\mathcal{I}(s) \sim s\sqrt{2\log\left(\frac{1}{s}\right)}$ as $s \to 0$.



Given $r > 0$, $A_r = \{x \in \mathbb{R}^n; \inf_{a\in A} |x-a| \leq r\}$ is the (closed) r-neighborhood of a set A in $\mathbb{R}^n$. The (Gaussian) outer Minkowski content of Borel set A is defined as

$$\gamma_n^+(A) \,=\, \liminf_{r\to 0} \frac{1}{r}\big[\gamma_n(A_r) - \gamma_n(A)\big].$$

Theorem [The Gaussian isoperimetric inequality] *For any Borel set A in $\mathbb{R}^n$,*

$$\gamma_n^+(A) \,\geq\, \mathcal{I}\big(\gamma_n(A)\big). \tag{2}$$

Equality is achieved on the half-spaces $H = \{x \in \mathbb{R}^n; \langle x, u\rangle \leq h\}$ where u is a unit vector and $h \in \mathbb{R}$.

The measure of a half-space is computed in dimension one, $\gamma_n(H) = \Phi(h)$, and its boundary measure is

$$\gamma_n^+(H) \,=\, \liminf_{r\to 0} \frac{1}{r}\big[\Phi(h+r) - \Phi(h)\big] \,=\, \varphi_1(h).$$

The Gaussian isoperimetric inequality thus expresses equivalently that, if H is a half-space such that $\Phi(h) = \gamma_n(H) = \gamma_n(A)$, then

$$\gamma_n^+(A) \,\geq\, \gamma_n^+(H), \tag{3}$$

and half-spaces are the extremal sets of the Gaussian isoperimetric problem.

Integrating along the neighborhoods, (3) is equivalently formulated as

$$\gamma_n(A_r) \,\geq\, \gamma_n(H_r), \quad r > 0, \tag{4}$$

provided that $\gamma_n(A) = (\geq)\ \gamma_n(H)$, or

$$\Phi^{-1}\big(\gamma_n(A_r)\big) \,\geq\, \Phi^{-1}\big(\gamma_n(A)\big) + r, \quad r > 0 \tag{5}$$

(since $\gamma_n(H_r) = \Phi(h+r)$).

Linear (affine) transformations yield the isoperimetric statement for any Gaussian measure. The dimension-free character allows furthermore for an infinite-dimensional formulation on an abstract Wiener space $(E, \mathcal{H}, \mu)$, developed first in [10], as

$$\Phi^{-1}\big(\mu(A + r\mathcal{K})\big) \ \geq \ \Phi^{-1}\big(\mu(A)\big) + r, \quad r \geq 0,$$

where $\mathcal{K}$ is the unit ball of the reproducing kernel Hilbert space $\mathcal{H}$ (cf. [3]). (Here $A + r\mathcal{K} = \{a + rh\,; a \in A, h \in \mathcal{K}\}$, which, in $\mathbb{R}^n$, amounts to A_r for $\mathcal{K}$ the Euclidean unit ball.)

The following sections briefly present the various known proofs of the Gaussian isoperimetric inequality.

2 Limit of spherical isoperimetry

In the neighborhood formulation, the isoperimetric inequality for the (normalized) uniform measure σ_N on the N-sphere $\mathbb{S}^N$ in $\mathbb{R}^{N+1}$, due to P. Lévy [25] and E. Schmidt [31], expresses that whenever A is a Borel set in $\mathbb{S}^N$, and B a spherical cap (geodesic ball) such that $\sigma_N(A) = (\geq)\ \sigma_N(B)$, then

$$\sigma_N(A_r) \ \geq \ \sigma_N(B_r) \tag{6}$$

for any $r \geq 0$, where A_r is the r-neighborhood of A in the geodesic metric.

It is a folklore result, usually quoted as "Poincaré's lemma", that the normalized uniform measure on the sphere $\sqrt{N}\,\mathbb{S}^N$, when projected on a n-dimensional subspace, converges as $N \to \infty$ to the standard n-dimensional Gaussian measure (cf. e.g. [24]). Via this limit, V. Sudakov and B. Tsirel'son [32], and C. Borell [10], independently, put forward the Gaussian isoperimetric inequality from the corresponding one on the sphere, the extremal spherical caps turning into half-spaces.

3 Gaussian symmetrization

Classical proofs of the isoperimetric inequality on the sphere use symmetrization techniques (see e.g. [19]). It is the contribution of A. Ehrhard [16] to have introduced a powerful (Steiner) symmetrization procedure specifically attached to the Gaussian framework, with which he provided a direct independent proof of the Gaussian isoperimetric inequality (along the standard symmetrization scheme). Specifically, given a Borel set A in $\mathbb{R}^n$, and u a direction vector, define the (Gaussian) symmetrized set A^* (in the direction u) such that, for any $x \in (\mathbb{R}u)^\perp$, $A^* \cap (x + \mathbb{R}u) = (-\infty, a]$ where $a \in [-\infty, +\infty]$ is given by

$$\Phi(a) \ = \ \gamma_1\big(A \cap (x + \mathbb{R}u)\big).$$

Then $\gamma_n(A^*) = \gamma_n(A)$, and the task is to show that symmetrization decreases the boundary measure $\gamma_n^+(A^*) \leq \gamma_n^+(A)$. For infinitely many directions u, the resulting symmetrized set is a half-space.

4 Kernel rearrangement inequality

For Borel sets A, B in $\mathbb{R}^n$, and $t > 0$, set

$$K_t(A,B) \,=\, \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathbb{1}_A(x)\mathbb{1}_B\big(e^{-t}x + \sqrt{1-e^{-2t}}\,)\,y\big)d\gamma_n(x)d\gamma_n(y).$$

It has been shown by C. Borell [11], using the Gaussian symmetrization technology of [16, 17], that, whenever H is a half-space with the same Gaussian measure as a Borel set A, then

$$K_t(A,A) \,\leq\, K_t(H,H). \tag{7}$$

A heat flow argument of this inequality is provided in [29], extended in a diffusion process picture in [18]. It is shown in [23, 24] that, for any Borel set A and any $t > 0$,

$$\gamma_n(A) - K_t(A,A) \,=\, K_t(A,A^c) \,\leq\, \frac{\arccos(e^{-t})}{\sqrt{2\pi}}\,\gamma_n^+(A),$$

and that, if H is a half-space,

$$\lim_{t\to 0}\frac{\sqrt{2\pi}}{\arccos(e^{-t})}\,K_t(H,H^c) \,=\, \gamma_n^+(H).$$

Combined with (7), the latter yields that $\gamma_n^+(A) \geq \gamma_n^+(H)$ whenever $\gamma_n(A) = \gamma_n(H)$, that is the Gaussian isoperimetric inequality.

5 Brunn-Minkowski inequality

In [16], A. Ehrhard discovered, using Gaussian symmetrization, an improved form of the Brunn-Minkowski inequality for Gaussian measures

$$\Phi^{-1}\big(\gamma_n(\theta A + (1-\theta)B)\big) \,\geq\, \theta\,\Phi^{-1}\big(\gamma_n(A)\big) + (1-\theta)\,\Phi^{-1}\big(\gamma_n(B)\big) \tag{8}$$

for any $\theta \in [0,1]$ and any convex bodies A, B in $\mathbb{R}^n$. This inequality has been extended to the case of only one convex body in [22], and finally to all Borel sets in [12] by pde methods. New recent proofs include [33, 21, 30].

The inequality (8) applied to B the Euclidean ball with center the origin and radius $\frac{r}{1-\theta}$ yields (5) as $\theta \to 1$.

6 Limit of a two-point inequality

In [8], S. Bobkov showed that for any smooth function $f : \mathbb{R}^n \to [0,1]$,

$$\mathcal{I}\bigg(\int_{\mathbb{R}^n} f\, d\gamma_n\bigg) \le \int_{\mathbb{R}^n} \sqrt{\mathcal{I}(f)^2 + |\nabla f|^2}\, d\gamma_n. \tag{9}$$

Applied to a (smooth) approximation of $f = \mathbb{1}_A$, this inequality yields (2). This functional form is actually equivalent to (2) when considering the level sets of functions defined on $\mathbb{R}^{n+1}$.

The proof of (9) in [8] is based on the two-point inequality

$$\mathcal{I}\left(\tfrac{a+b}{2}\right) \le \tfrac{1}{2}\sqrt{\mathcal{I}(a)^2 + \tfrac{1}{2}|a-b|^2} + \tfrac{1}{2}\sqrt{\mathcal{I}(b)^2 + \tfrac{1}{2}|a-b|^2}$$

for all $a, b \in [0,1]$, and a tensorization argument and the central limit theorem. The stability by product of the functional inequality (9) is indeed a main feature (being true for $n = 1$, it holds for any dimension n).

7 Heat flow monotonicity

A direct heat flow proof of Bobkov's inequality (9) has been presented in [4]. Let

$$p_t(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}}\, e^{-\frac{1}{4t}|x|^2}, \quad t > 0,\ x \in \mathbb{R}^n,$$

be the standard heat kernel, fundamental solution of the heat equation $\partial_t p_t = \Delta p_t$. The convolution semigroup $P_t f(x) = f * p_t(x)$, $t > 0$, solves $\partial_t P_t f = \Delta P_t f$ with initial data f.

At $t = \frac{1}{2}$, p_t is just the standard Gaussian density so that $P_{\frac{1}{2}} f(0) = \int_{\mathbb{R}^n} f d\gamma_n$ (while $P_0 f = f$). In order to verify (9), it suffices therefore to show that, for a smooth function $f : \mathbb{R}^n \to [0,1]$, (at any point),

$$P_s\bigg(\sqrt{\mathcal{I}\left(P_{\frac{1}{2}-s} f\right)^2 + 2s|\nabla P_{\frac{1}{2}-s} f|^2}\bigg), \quad s \in [0, \tfrac{1}{2}],$$

is increasing, which is simply achieved taking its derivative (cf. [4]). A martingale proof along the same line, which includes extensions to path (Wiener) spaces, is provided in [7, 14].

8 Geometric measure theory

A proof of the Gaussian isoperimetric inequality relying on geometric measure theory is presented in the note by F. Morgan [27], with the suitable version of the Heinze-Karcher inequality on weighted manifolds. This inequality provides an upper bound on the volume of a one-sided neighborhood of a hypersurface in terms of its mean curvature and the Ricci curvature of the ambient manifold. In Gauss space, it yields

$$\frac{\gamma_n(A)}{\gamma_n^+(S)} \le \frac{\gamma_n(H)}{\gamma_n^+(H)}$$

where S is a minimizing hypersurface enclosing a set A with $\gamma_n(A) = \gamma_n(H)$. See also E. Milman [26], relying on regularity of isoperimetric minimizers, both in the interior and on the boundary, as emphasized in the early work by M. Gromov [20].

9 Deficit

A stronger version of the isoperimetric inequality examines lower bounds on the deficit

$$\gamma_n^+(A) - \gamma_n(H^+)$$

in terms of a functional measuring the proximity of a half-space $H = H_u = \{x \in \mathbb{R}^n; \langle x, u\rangle \le h\}$ such as $\gamma_n(H_u) = \gamma_n(A)$, with the Borel set A. First steps in this investigation involved a geometric analysis with the Ehrhard symmetrization [15], and a study of the deficit in the kernel rearrangement inequality (7) [28, 29, 18]. A variational method is developed by M. Barchiesi, A. Brancolini and V. Julin [6] providing sharp bounds on the deficit. These authors introduce a technique which is based on an analysis of the first and the second variation conditions of solutions to a suitable minimization problem, providing a direct proof of the sharp deficit bound

$$\gamma_n^+(A) - \gamma_n(H^+) \,\ge\, c\big(\gamma_n(A)\big) \sqrt{\inf_{u \in \mathbb{S}^{n-1}} \gamma_n(A \Delta H_u)}$$

(where $c(\gamma_n(A)) > 0$ only depends on the measure of A).

10 Extension to strongly log-concave measures

The Gauss space and measure is a model example (of positive curvature and infinite dimension in the language of [5]) to which other examples may be compared. A most natural and famous instance is the case of a probability measure $d\mu = e^{-V} dx$ on $\mathbb{R}^n$ whose potential $V : \mathbb{R}^n \to \mathbb{R}$ is more convex than the quadratic potential, that is $V(x) - \frac{c}{2}|x|^2$, $x \in \mathbb{R}^n$, is convex for some $c > 0$. A main result in this setting is that the isoperimetric profile $\mathcal{I}_\mu$ of μ is bounded from below by the Gaussian one. That is, if

$$\mathcal{I}_\mu(s) \,=\, \inf\big\{\mu^+(A)\,;\mu(A) = s\big\}, \quad s \in [0,1],$$

where the infimum is running over all Borel sets A in $\mathbb{R}^n$ (and with a definition of $\mu^+(A)$ similar to $\gamma_n^+(A)$), then

$$\mathcal{I}_\mu \geq \sqrt{c}\,\mathcal{I}. \tag{10}$$

The property (10) has been established in [4] by the heat flow monotonicity method (Section 6). A proof using needle decomposition has been proposed in [9]. A celebrated contraction principle in optimal transport by L. Caffarelli [13], expressing that μ is the $\frac{1}{\sqrt{c}}$-Lipschitz image of γ_n, produces a neat and direct proof of (10) (although not saying anything on the Gaussian case itself). The geometric measure theory approach outlined in Section 7 covers the framework of weighted Riemannian manifolds with (generalized) curvature bounded from below by a positive constant, also covered by the heat flow argument (cf. [4, 5]).

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