Mehler kernel, and the Ornstein-Uhlenbeck operator

Let

$$h_t(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \, e^{-\frac{1}{4t}|x|^2}, \quad t > 0, \; x \in \mathbb{R}^n,$$

be the standard heat kernel, fundamental solution of the heat equation

$$\partial_t h_t = \Delta p_t$$

with Δ the standard Laplacian. The convolution semigroup $H_t f(x) = f * h_t(x)$, $t > 0$, solves $\partial_t H_t f = \Delta H_t f = H_t \Delta f$ with initial data f. At $t = \frac{1}{2}$, h_t is just the standard Gaussian density so that $H_{\frac{1}{2}} f(0) = \int_{\mathbb{R}^n} f d\gamma_n$ (while $H_0 f = f$), where $d\gamma_n(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}|x|^2} d\lambda_n(x)$ is the standard Gaussian measure on the Borel sets of $\mathbb{R}^n$.

While the Gaussian density is the central piece of the heat kernel definition, invariance of the heat semigroup $(H_t)_{t \geq 0}$ is still with respect to the Lebesgue measure (in the sense that $\int_{\mathbb{R}^n} h_t(x) d\lambda_n(x) = 1$). There is a related Gaussian kernel, the Mehler kernel,

$$p_t(x, y) = \frac{1}{(1 - e^{-2t})^{\frac{n}{2}}} \exp\bigg(- \frac{e^{-2t}}{2(1 - e^{-2t})} \big[|x|^2 + |y|^2 - 2e^t x \cdot y \big] \bigg), \tag{1}$$

$t > 0$, $x, y \in \mathbb{R}^n$, which has the advantage to be invariant with respect to γ_n, i.e.

$$\int_{\mathbb{R}^n} p_t(x, y) d\gamma_n(y) = 1$$

(for every $x \in \mathbb{R}^n$).

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The Mehler kernel induces the Ornstein-Uhlenbeck semigroup, with infinitesimal generator the drifted Laplacian $\mathrm{L} = \Delta - x \cdot \nabla$. The spectrum of the operator $-\mathrm{L}$ is $\mathbb{N}$, and the eigenvectors are the Hermite polynomials (cf. [1]).

It is the purpose of this post to briefly present some general aspects and results on the Mehler kernel and the Ornstein-Uhlenbeck operator. Standard references include [6, 5, 7, 8, 4]...

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1 Mehler kernel and Ornstein-Uhlenbeck operator

The Mehler kernel, as given in (1), satisfies the basic semigroup property with respect to γ_n,

$$\int_{\mathbb{R}^n} p_s(x,z)\, p_t(z,y) d\gamma_n(z) \;=\; p_{s+t}(x,y) \tag{2}$$

for all $s, t > 0$ and $x, y \in \mathbb{R}^n$. As such, it generates the Ornstein-Uhlenbeck semigroup

$$P_t f(x) \;=\; \int_{\mathbb{R}^n} f(z)\, p_t(x,z) d\gamma_n(z), \quad t > 0,\ x \in \mathbb{R}^n, \tag{3}$$

for any suitable measurable function $f : \mathbb{R}^n \to \mathbb{R}$, with the natural extension $P_0 = \mathrm{Id}$. After the change of variable $e^{-t}x + \sqrt{1 - e^{-2t}}\, y = z$ in (3), it takes the form

$$P_t f(x) \;=\; \int_{\mathbb{R}^n} f\big(e^{-t}x + \sqrt{1 - e^{-2t}}\, y\big) d\gamma_n(y), \quad t \geq 0,\ x \in \mathbb{R}^n, \tag{4}$$

known as Mehler's integral formula.

The family $(P_t)_{t\geq 0}$ defines a Markov semigroup, symmetric in $\mathrm{L}^2(\gamma_n)$ and invariant with respect to γ_n, that is

$$\int_{\mathbb{R}^n} f P_t g \, d\gamma_n = \int_{\mathbb{R}^n} g P_t f d\gamma_n \quad \text{and} \quad \int_{\mathbb{R}^n} P_t f d\gamma_n = \int_{\mathbb{R}^n} f d\gamma_n.$$

These properties are actually a reformulation of the rotational invariance of Gaussian measures, expressing that under $\gamma_n \otimes \gamma_n$, the couples

$$\big(x\sin(\theta) + y\cos(\theta), x\cos(\theta) - y\sin(\theta)\big),$$

with $e^{-t} = \sin(\theta)$, are distributed as (x, y).

The infinitesimal generator

$$\mathrm{L} = \lim_{t\to 0} \frac{1}{t}\,[P_t - P_0]$$

of the Markov semigroup $(P_t)_{t\geq 0}$ is the drifted Laplacian $\mathrm{L} = \Delta - x\cdot\nabla$. This can be checked for instance on the Mehler formula (4) since

$$\begin{aligned}
\frac{d}{dt}\,P_t f &= \int_{\mathbb{R}^n} \Big(- e^{-t}x + \frac{e^{-2t}}{\sqrt{1-e^{-2t}}}\,y\Big)\cdot \nabla f\big(e^{-t}x + \sqrt{1-e^{-2t}}\,y\big) d\gamma_n(y) \\
&= -e^{-t}\int_{\mathbb{R}^n} x\cdot\nabla f\big(e^{-t}x + \sqrt{1-e^{-2t}}\,y\big) d\gamma_n(y) \\
&\quad + e^{-2t}\int_{\mathbb{R}^n} \Delta f\big(e^{-t}x + \sqrt{1-e^{-2t}}\,y\big) d\gamma_n(y) \\
&= \mathrm{L}P_t f
\end{aligned}$$

where the last steps follows from integration by parts in the y variable.

The semigroup $(P_t)_{t\geq 0}$ is invariant with respect to γ_n ($\int_{\mathbb{R}^n} \mathrm{L} f d\gamma_n = 0$), and fulfills the basic integration by parts formula by with respect to γ_n

$$\int_{\mathbb{R}^n} f(-\mathrm{L}g) d\gamma_n = \int_{\mathbb{R}^n} \nabla f\cdot\nabla g\, d\gamma_n \tag{5}$$

for every smooth functions $f, g : \mathbb{R}^n \to \mathbb{R}$.

The semigroup $(P_t)_{t\geq 0}$ is a contraction in all $\mathrm{L}^p(\mu)$-spaces with norms $\|\cdot\|_p$, $1 \leq p \leq \infty$. The hypercontractivity property [2] on the other hand expresses that whenever $1 < p < q < \infty$ and $e^{2t} \geq \frac{q-1}{p-1}$,

$$\|P_t f\|_q \leq \|f\|_p. \tag{6}$$

The Ornstein-Uhlenbeck semigroup $(P_t)_{t\geq 0}$ is ergodic, $\lim_{t\to\infty} P_t f = \int_{\mathbb{R}^n} f d\gamma_n$. The convergence in the $\mathrm{L}^2(\gamma_n)$-norm is exponential on mean-zero functions $f : \mathbb{R}^n \to \mathbb{R}$ as a consequence of the Gaussian Poincaré inequality (cf. Section 6).

2 Spectrum of the Ornstein-Uhlenbeck operator

The spectrum of the operator $-\mathrm{L}$ is $\mathbb{N}$, with eigenfunctions given by the Hermite polynomials $H_{\underline{k}}$, $\underline{k} \in \mathbb{N}^n$,

$$\mathrm{L} H_{\underline{k}} \, = \, -k \, H_{\underline{k}} \tag{7}$$

with $k = k_1 + \cdots + k_n$, $\underline{k} = (k_1, \ldots, k_n)$.

This may be seen in various ways. For example, by the Mehler formula (4), the action of P_t on the multi-dimensional generating function $f_\lambda(x) = e^{\lambda \cdot x - \frac{1}{2}|\lambda|^2}$, $x, \lambda \in \mathbb{R}^n$, of the family of Hermite polynomials, is given by

$$P_t f_\lambda(x) \, = \, \int_{\mathbb{R}^n} e^{\lambda \cdot (e^{-t}x + \sqrt{1-e^{-2t}}y) - \frac{1}{2}|\lambda|^2} d\gamma_n(y) \, = \, f_{e^{-t}\lambda}(x).$$

Therefore $P_t H_{\underline{k}} = e^{-kt} H_{\underline{k}}$, $t \geq 0$, where $k = k_1 + \cdots + k_n$, $\underline{k} = (k_1, \ldots, k_n)$, and hence $\mathrm{L} H_{\underline{k}} = -k H_{\underline{k}}$.

As a consequence of the integration by parts formual (7), for any (smooth) function $f : \mathbb{R}^n \to \mathbb{R}$, and any $\underline{k} \in \mathbb{N}^n$,

$$k \int_{\mathbb{R}^n} f \, H_{\underline{k}} \, d\gamma_n \, = \, -\int_{\mathbb{R}^n} f \, \mathrm{L} H_{\underline{k}} \, d\gamma_n \, = \, \int_{\mathbb{R}^n} \nabla f \cdot \nabla H_{\underline{k}} \, d\gamma_n,$$

which is a generalized form of the basic integration by parts formula

$$\int_{\mathbb{R}^n} x f \, d\gamma_n \, = \, \int_{\mathbb{R}^n} \nabla f \, d\gamma_n$$

(as vector integrals), corresponding to the choice of the first eigenfunctions $H_{\underline{k}}$, $k = 1$.

3 Differential formulas

The following differential formulas on the Mehler kernel are fundamental in Gaussian calculus of variation, and directly follow from the Mehler formula (4).

Whenever $f : \mathbb{R}^n \to \mathbb{R}$ is smooth enough, $t > 0$, $x \in \mathbb{R}^n$,

$$\nabla P_t f(x) \, = \, e^{-t} \int_{\mathbb{R}^n} \nabla f \big(e^{-t} x + \sqrt{1 - e^{-2t}} \, y \big) d\gamma_n(y) \, = \, e^{-t} \, P_t(\nabla f)(x), \tag{8}$$

$$\nabla P_t f(x) \, = \, \frac{e^{-t}}{\sqrt{1 - e^{-2t}}} \int_{\mathbb{R}^n} y \, f \big(e^{-t} x + \sqrt{1 - e^{-2t}} \, y \big) d\gamma_n(y), \tag{9}$$

the second resulting from integration by parts.

4 Ornstein-Uhlenbeck process

The Ornstein-Uhlenbeck process $\{X_t^x \,; t \geq 0, x \in \mathbb{R}^n\}$ with associated semigroup

$$P_t f(x) \,=\, \mathbb{E}\big(f(X_t^x)\big) \,=\, \mathbb{E}\big(f(X_t) \,|\, X_0 = x)\big), \quad t \geq 0, \; x \in \mathbb{R}^n,$$

admits the explicit representation

$$X_t^x \,=\, e^{-t}\bigg(x + \sqrt{2}\int_0^t e^s B_s\bigg)$$

where $(B_s)_{s\geq 0}$ is a standard Brownian motion in $\mathbb{R}^n$. This process is the solution of the stochastic differential equation

$$dX_t \,=\, \sqrt{2}\, dB_t - X_t dt.$$

The law of X_t given $X_0 = x$ is normal with mean $e^{-t}x$ and covariance $\sqrt{1 - e^{-2t}}\,\mathrm{Id}$, from which the Mehler formula (4) is recovered, and if X_0 is distributed as $\mathcal{N}(0, \mathrm{Id})$, so is X_t (invariance). For $s, t \geq 0$,

$$\mathrm{Cov}(X_s, X_t) \,=\, e^{-|s-t|}\,(1 - e^{-2(s\wedge t)}).$$

5 Harmonic oscillator

The Ornstein-Uhlenbeck operator is closely related to another famous and well-studied operator, the harmonic oscillator in $\mathbb{R}^n$, given on smooth functions f by

$$\mathrm{H}f \,=\, \Delta f - \frac{1}{4}\,|x|^2 f. \tag{10}$$

The harmonic oscillator H is thus adding a potential to the Laplace operator. It is still symmetric with respect to the Lebesgue measure, and represents the simplest model of quantum mechanics. Denoting by $U_0 = e^{-\frac{1}{4}|x|^2}$, $x \in \mathbb{R}^n$, the ground state function for which $\mathrm{H}U_0 = -\frac{n}{2}\,U_0$, the (ground state) transformation

$$f \,\mapsto\, \frac{n}{2}\,f + \frac{1}{U_0}\,\mathrm{H}(U_0 f)$$

yields the Ornstein-Uhlenbeck operator L since

$$\mathrm{H}(U_0 f) \,=\, -\frac{n}{2}\,U_0 f + U_0\,\Delta f + 2\,\nabla U_0 \cdot \nabla f.$$

The transformation $f \mapsto U_0 f$ therefore carries over the analysis of the harmonic oscillator H into the analysis of the Ornstein-Uhlenbeck operator L in terms of Hermite polynomials.

6 A proof of the Gaussian Poincaré inequality

The Gaussian Poincaré inequality

$$\mathrm{Var}_{\gamma_n}(f) \le \int_{\mathbb{R}^n} |\nabla f|^2 d\gamma_n, \tag{11}$$

for functions f in $\mathrm{L}^2(\gamma_n)$ as well as their gradients, is presented in the post [3]. A quick proof may be provided by interpolation along the Ornstein-Uhlenbeck semigroup. Namely, for a smooth function $f : \mathbb{R}^n \to \mathbb{R}$,

$$\mathrm{Var}_{\gamma_n}(f) = \int_{\mathbb{R}^n} f^2 d\gamma_n - \bigg(\int_{\mathbb{R}^n} f \, d\gamma_n \bigg)^2 = \int_0^\infty \bigg(\frac{d}{dt} \int_{\mathbb{R}^n} (P_t f)^2 d\gamma_n \bigg) dt.$$

Now, by the integration by parts formula (5),

$$\frac{d}{dt} \int_{\mathbb{R}^n} (P_t f)^2 d\gamma_n = 2 \int_{\mathbb{R}^n} P_t f \, \mathrm{L} P_t f d\gamma_n = 2 \int_{\mathbb{R}^n} |\nabla P_t f|^2 d\gamma_n.$$

Using that $\nabla P_t f = e^{-t} P_t(\nabla f)$, $t \ge 0$ (8), and that P_t is a contraction in $\mathrm{L}^2(\gamma_n)$, it follows that

$$\begin{aligned} \mathrm{Var}_{\gamma_n}(f) &= 2 \int_0^\infty e^{-2t} \bigg(\int_{\mathbb{R}^n} \big| P_t(\nabla f) \big|^2 d\gamma_n \bigg) dt \\ &\le 2 \int_0^\infty e^{-2t} \bigg(\int_{\mathbb{R}^n} |\nabla f|^2 d\gamma_n \bigg) dt \\ &= \int_{\mathbb{R}^n} |\nabla f|^2 d\gamma_n. \end{aligned}$$

The Gaussian Poincaré inequality induces (is actually equivalent to) the exponential decay for mean-zero functions f in $\mathrm{L}^2(\gamma_n)$,

$$\| P_t f \|_2 \le e^{-t} \| f \|_2 , \quad t \ge 0. \tag{12}$$

Namely,

$$\begin{aligned} \frac{d}{dt} e^{2t} \| P_t f \|_2^2 &= e^{2t} \bigg(2 \int_{\mathbb{R}^n} (P_t f)^2 d\gamma_n + 2 \int_{\mathbb{R}^n} P_t f \, \mathrm{L} P_t f \, d\gamma_n \bigg) \\ &= e^{2t} \bigg(2 \int_{\mathbb{R}^n} (P_t f)^2 d\gamma_n - 2 \int_{\mathbb{R}^n} |\nabla P_t f|^2 \, d\gamma_n \bigg) \end{aligned}$$

where integration by parts has been used. Hence the Poincaré inequality (11) applied to $P_t f$ ensures that $e^{2t} \| P_t f \|_2^2$, $t \ge 0$, is decreasing, which amounts to (12).

This exponential decay may also be viewed spectrally, as a spectral gap. Namely, in dimension one for simplicity, if a mean-zero function f is Fourier-Hermite expanded as $f = \sum_{k\geq 1} a_k h_k$, then, for every $t \geq 0$,

$$P_t f \, = \sum_{k\geq 1} e^{-kt} a_k h_k.$$

Taking the $\mathrm{L}^2(\gamma_1)$-norm,

$$\|P_t f\|_2^2 \, = \sum_{k\geq 1} e^{-2kt} a_k^2 \, \leq \, e^{-2t} \sum_{k\geq 1} a_k^2 \, = \, e^{-2t} \, \|f\|_2^2.$$

References

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[2] Logarithmic Sobolev and transportation inequalities. *The Gaussian Blog.*

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