Some geometric inequalities for Gaussian measures

Abstract

The note emphasizes some inequalities for Gaussian measures of geometric flavour. Its pattern is modeled on the 2002 review article [10] by R. Latała, with the remarkable feature that all the conjectures exposed therein have now (2014) been solved.

Gaussian measures share some surprising geometric inequalities. The following reviews the answers of the last decade to several conjectured inequalities, the Ehrhard inequality, the S and B-inequalities and the Gaussian correlation inequality. The first section is a brief reminder of the Gaussian isoperimetric inequality in order to put some of the results in perspective.

Let γ_n be the standard Gaussian measure on the Borel sets of $\mathbb{R}^n$, with density $\frac{1}{(2\pi)^{\frac{n}{2}}}\, e^{-\frac{1}{2}|x|^2}$, $x \in \mathbb{R}^n$, with respect to the Lebesgue measure.

1 The Gaussian isoperimetric inequality

Given $r > 0$, let $A_r = \{x \in \mathbb{R}^n; \inf_{a\in A} |x - a| \le r\}$ be the (closed) r-neighborhood of a set A in $\mathbb{R}^n$. The (Gaussian) outer Minkowski content of Borel set A is given by

$$\gamma^+(A) \,=\, \liminf_{r\to 0} \frac{1}{r} \big[\gamma(A_r) - \gamma(A)\big].$$

A half-space in $\mathbb{R}^n$ is defined as $H = \{x \in \mathbb{R}^n; \langle x, u\rangle \le h\}$ where u is a unit vector and $h \in \mathbb{R}$.

Theorem 1 (The Gaussian isoperimetric inequality [2, 15])**.** *For any Borel set A in $\mathbb{R}^n$, if H is a half-space such that $\gamma(A) = \gamma(H)$, then*

$$\gamma^+(A) \,\ge\, \gamma^+(H). \tag{1}$$

The measure of a half-space is computed in dimension one as $\gamma(H) = \Phi(h)$ where $\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-t}^{\infty} e^{-\frac{1}{2}x^2} dx$, $t \in \mathbb{R}$, is the distribution function of the standard normal law on the real line (with the convention $\Phi(-\infty) = 0$, $\Phi(+\infty) = 1$). Integrating along the neighborhoods, (1) is equivalently formulated as

$$\gamma(A_r) \geq \gamma(H_r), \quad r > 0,$$

provided that $\gamma(A) = (\geq)\ \gamma(H)$, or

$$\Phi^{-1}\big(\gamma(A_r)\big) \geq \Phi^{-1}\big(\gamma(A)\big) + r, \quad r > 0 \tag{2}$$

since $\gamma(H_r) = \Phi(h + r)$.

Linear (affine) transformations yield the isoperimetric statement for any Gaussian measure. The dimension-free character allows furthermore for an infinite-dimensional formulation on an abstract Wiener space $(E, \mathcal{H}, \mu)$ as

$$\Phi^{-1}\big(\mu(A + r\mathcal{K})\big) \geq \Phi^{-1}\big(\mu(A)\big) + r, \quad r \geq 0,$$

where $\mathcal{K}$ is the unit ball of the associated reproducing kernel Hilbert space $\mathcal{H}$, and $A + r\mathcal{K} = \{a + rh\,; a \in A, h \in \mathcal{K}\}$ (see [11]). (Due to the linear structure, on the Euclidean space $\mathbb{R}^n$, $A_r = A + rB(0,1)$ where $B(0,1)$ is the (closed) Euclidean unit ball.)

2 The Ehrhard inequality

The classical Brunn-Minkowski inequality in Euclidean space states that for any Borel sets A and B in $\mathbb{R}^n$,

$$\mathrm{vol}_n\big(\theta A + (1-\theta)B\big) \geq \theta\, \mathrm{vol}_n(A) + (1-\theta)\, \mathrm{vol}_n(B), \quad \theta \in [0,1]. \tag{3}$$

(If A and B are subsets of $\mathbb{R}^n$, $A + B = \{a + b\,; a \in A, b \in B\}$.) This remarkable and powerful geometric inequality, with numerous consequences and applications, may be used in particular to recover the standard isoperimetric inequality in $\mathbb{R}^n$. The task is to show that, for fixed volume, balls are the extremal sets of the isoperimetric problem. That is, in the integrated form, whenever $\mathrm{vol}_n(A) = (\geq)\ \mathrm{vol}_n(B)$ where B is some ball,

$$\mathrm{vol}_n\big(A + B(0,r)\big) \geq \mathrm{vol}_n\big(B + B(0,r)\big)$$

for every $r > 0$. Now, if $B = B(0, r_0)$ for some r_0, the choice in (3) of $B = B(0, \frac{\theta r}{1-\theta})$ such that $\theta = \frac{r_0}{r_0 + r} \in (0,1)$, yields on the left-hand side $\theta^n\, \mathrm{vol}_n(A + B(0,r))$ while, by the choice of θ, the right-hand side is equal to

$$\begin{aligned}\theta \operatorname{vol}_n\big(B(0,r_0)\big) + (1-\theta)\operatorname{vol}_n\big(B\big(0,\tfrac{\theta r}{1-\theta}\big)\big)& \\ &= \theta\, r_0^n \operatorname{vol}_n\big(B(0,1)\big) + (1-\theta)\,\frac{\theta^n r^n}{(1-\theta)^n}\operatorname{vol}_n\big(B(0,1)\big)\\ &= \theta^n (r_0+r)^n \operatorname{vol}_n\big(B(0,1)\big)\\ &= \theta^n \operatorname{vol}_n\big(B(0,r_0+r)\big)\\ &= \theta^n \operatorname{vol}_n\big(B(0,r_0)+B(0,r)\big)\end{aligned}$$

from which the claim follows.

Gaussian measures satisfy a similar property, in the form of the log-concavity inequality

$$\log\gamma_n\big(\theta A + (1-\theta)B\big) \;\geq\; \theta\log\gamma_n(A) + (1-\theta)\log\gamma_n(B), \quad \theta\in[0,1].$$

This inequality extends to any Gaussian measure on a separable Banach space E, and any Borel sets A and B in E (cf. [5]). However, this log-concavity property does not imply the Gaussian isoperimetry.

In 1983, A. Ehrhard [5] emphasized an improved form of log-concavity of Gaussian measures through the inverse Φ^{-1} of the distribution function Φ the standard normal distribution, which in particular covers the isoperimetric inequality.

Theorem 2 (The Ehrhard inequality)**.** *For any Borel sets A, B in $\mathbb{R}^n$, and any $\theta\in[0,1]$,*

$$\Phi^{-1}\big(\gamma_n(\theta A + (1-\theta)B)\big) \;\geq\; \theta\,\Phi^{-1}\big(\gamma_n(A)\big) + (1-\theta)\,\Phi^{-1}\big(\gamma_n(B)\big).$$

Again, Theorem 2 extends to any Gaussian measure on a separable Banach space.

It is not difficult to see how Ehrhard's inequality includes isoperimetry. Indeed, applying it to $\frac{1}{\theta}A$ and to $B=\frac{r}{1-\theta}B(0,1)$, $r>0$, $\theta\in(0,1)$, where $B(0,1)$ is the (closed) Euclidean unit ball, yields

$$\Phi^{-1}\big(\gamma_n(A+rB(0,1))\big) \;\geq\; \theta\,\Phi^{-1}\big(\gamma_n\big(\tfrac{1}{\theta}\,A\big)\big) + (1-\theta)\,\Phi^{-1}\big(\gamma_n\big(\tfrac{r}{1-\theta}\,B(0,1)\big)\big).$$

As $\theta\to 1$,

$$\Phi^{-1}\big(\gamma_n(A+rB(0,1))\big) \;\geq\; \Phi^{-1}\big(\gamma_n(A)\big) + r,$$

which is one form of Gaussian isoperimetry (2).

Theorem 2 was established for convex sets by A. Ehrhard [5] using symmetrization scheme in Gauss space that he introduced to this task. It was extended to the case of only one of the sets A, B to be convex (good enough to recover isoperimetry) in [8]. C. Borell [3] finally proved the full result using pde tools on the functional version, in the form of the following Prékopa-Leindler-type inequality. If $f, g, h : \mathbb{R}^n \to [0,1]$ are measurable and $\theta \in [0,1]$ are such that

$$\Phi^{-1}\big(h(\theta x + (1-\theta)y)\big) \;\geq\; \theta\, \Phi^{-1}\big(f(x)\big) + (1-\theta)\, \Phi^{-1}\big(g(y)\big),$$

for all $x, y \in \mathbb{R}^n$, then

$$\Phi^{-1}\bigg(\int_{\mathbb{R}^n} h d\gamma_n\bigg) \geq \theta\, \Phi^{-1}\bigg(\int_{\mathbb{R}^n} f d\gamma_n\bigg) + (1-\theta)\Phi^{-1}\bigg(\int_{\mathbb{R}^n} g d\gamma_n\bigg).$$

Applied to $f = \mathbb{1}_A$ and $g = \mathbb{1}_B$ yields the statement in Theorem 2 (and this functional form is actually equivalent to it when considering the level sets of functions defined on $\mathbb{R}^{n+1}$).

The proof in [3][1] is based on a parabolic maximum principle applied to the second order differential operator on $\mathbb{R}^n \times \mathbb{R}^n$,

$$\mathcal{E} = \Delta_x + \Delta_y + 2\sum_{i=1}^{n} \partial_{x_i}\partial_{y_i}$$

and the functional

$$C(t,x,y) \;=\; U_h\big(t, \theta x + (1-\theta)y\big) - \theta\, U_f(t,x) - (1-\theta)\, U_g(t,y), \quad t \geq 0,\; x, y \in \mathbb{R}^n,$$

where, for $q = h, f, g$, $U_q = \Phi^{-1}(u_q)$ and

$$u_q(t,x) \;=\; \int_{\mathbb{R}^n} q\big(x + \sqrt{t}\, z\big) d\gamma_n(z).$$

3 The S-inequality

The S-*inequality* is a type of isoperimetric inequality with respect to homotheties, with strips as extremal sets.

Theorem 3 (The S-inequality)**.** *Let A be a symmetric closed convex set in $\mathbb{R}^n$, and let $S = \{x \in \mathbb{R}^n; |\langle x, u\rangle| \leq s\}$, u unit vector, $s \geq 0$, be a strip such that $\gamma_n(A) = \gamma_n(S)$. Then*

$$\gamma_n(tA) \;\geq\; \gamma_n(tS) \quad \textit{for } t \geq 1,$$

and

$$\gamma_n(tA) \;\leq\; \gamma_n(tS) \quad \textit{for } 0 \leq t \leq 1.$$

[1] Alternate proofs, posterior to the note, have been presented in: R. van Handel, The Borell-Ehrhard game, *Probab. Theory Related Fields* 170, 555–585 (2018), and: J. Neeman, G. Paouris, An interpolation proof of Ehrhard's inequality, *Geometric aspects of functional analysis, Lecture Notes in Math.* 2266, 263–278. Springer (2020).

This theorem has been established by R. Latała and K. Oleszkiewicz [9], relying on technical arguments and some clever real-line inequalities. It was observed by S. Szarek (cf. [9]) that from the S-inequality the moment comparison of Gaussian measures on Banach spaces is the same than in the real case. That is, if μ is a centered Gaussian measure on a real separable Banach space E with norm $\|\cdot\|$, then

$$\frac{\left(\int_E \|x\|^q d\mu\right)^{1/q}}{\left(\int_{\mathbb{R}} |x|^q d\gamma_1\right)^{1/q}} \leq \frac{\left(\int_E \|x\|^p d\mu\right)^{1/p}}{\left(\int_{\mathbb{R}} |x|^p d\gamma_1\right)^{1/p}}$$

for any $0 \leq p \leq q$.

4 The B-inequality

The *B-inequality* for Gaussian measure is another statement about convex sets. It has been established by D. Cordero-Erausquin, M. Fradelizi and B. Maurey [4].

Theorem 4 (The B-inequality)**.** *Let A be a symmetric closed convex set in $\mathbb{R}^n$. For every $\alpha, \beta > 0$,*

$$\gamma_n\big(\sqrt{\alpha\beta}A\big) \geq \sqrt{\gamma_n(\alpha A)\,\gamma_n(\beta A)}.$$

In an equivalent formulation, the map $t \mapsto \gamma_n(e^t A)$ is log-concave on $\mathbb{R}$.

A interesting feature of the proof of [4] is that it is connected to (but lies much deeper than) the Gaussian Poincaré inequality for functions f which are orthogonal to constants and to linear functions, for which the constant is improved from 1 to $\frac{1}{2}$ as

$$\mathrm{Var}_{\gamma_n}(f) \leq \frac{1}{2}\int_{\mathbb{R}^n} |\nabla f|^2 d\gamma_n.$$

This is in particular clear on the Hermite expansion proof of the Gaussian Poincaré inequality (cf. [1]).

5 The Gaussian correlation inequality

Theorem 5 (The Gaussian correlation inequality)**.** *For any symmetric convex sets A, B in $\mathbb{R}^n$,*

$$\gamma_n(A \cap B) \geq \gamma_n(A)\,\gamma_n(B). \tag{4}$$

The same result holds true for any centered Gaussian measure on a Banach space E, and symmetric convex sets in E.

The *Gaussian correlation inequality* has aroused great interest over the last 50 years[2]. In dimension 2, the result goes back to L. Pitt [12]. When one of the sets A or B is a symmetric strip, the inequality was proved independently by C. Khatri [7] and Z. Šidák [14]. It was extended to the case when one of the sets is a symmetric ellipsoid by G. Hargé [6]. The final step was achieved in a striking short contribution by T. Royen in 2014 [13].

As in earlier proofs, T. Royen uses an interpolation argument to establish that for any centered Gaussian random vector $X = (X_1, \ldots, X_n)$ in $\mathbb{R}^n$ and any $1 \leq m < n$,

$$\mathbb{P}\big(\max_{1\leq i\leq n} |X_i| \leq 1\big) \ \geq \ \mathbb{P}\big(\max_{1\leq i\leq m} |X_i| \leq 1\big)\, \mathbb{P}\big(\max_{m<i\leq n} |X_i| \leq 1\big).$$

Working rather with the vector $(X_1^2, \ldots, X_n^2)$, his main new ingredient is a clever use of the Laplace transform of Gamma distributions to control the signs in the derivative along the interpolation.

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