The Gaussian product conjecture

Abstract

Let $(X_1, \ldots, X_n)$ be a centered Gaussian vector in $\mathbb{R}^n$ defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The *Gaussian product conjecture (inequality)* expresses that for any integers $p_1, \ldots, p_n \in \mathbb{N}$,

$$\mathbb{E}\bigg(\prod_{k=1}^{n} X_k^{2p_k} \bigg) \geq \prod_{k=1}^{n} \mathbb{E}\big(X_k^{2p_k}\big).$$

Despite a lot of interest, the general case of this conjecture is still widely open. The post is devoted to a brief exposition of the state of the art.

1 The known results

Proposition 1. *The conjecture is true when $n = 2$.*

Proof. One possible argument is reminiscent of the proof of Slepian's inequality (cf. e.g. [9]). By homogeneity, it may be assumed that $\mathbb{E}(X_1^2) = \mathbb{E}(X_2^2) = 1$. Let then Y_1 and Y_2 be independent standard normal variables, independent of the couple (X_1, X_2). For integers $q_1, q_2 \geq 1$, consider the function of $t \in [0, 1]$,

$$\phi(t) = \mathbb{E}\Big(\big(\sqrt{t}\, X_1 + \sqrt{1-t}\, Y_1\big)^{q_1} \big(\sqrt{t}\, X_2 + \sqrt{1-t}\, Y_2\big)^{q_2} \Big).$$

Since moments of real Gaussian variables only depend on the variance, the task is to show that when $q_1 = 2p_1$ and $q_2 = 2p_2$ are even integers, ϕ is increasing so that

$$\mathbb{E}\big(X_1^{2p_1} X_2^{2p_2}\big) = \phi(1) \geq \phi(0) = \mathbb{E}\big(Y_1^{2p_1}\big)\, \mathbb{E}\big(Y_2^{2p_2}\big) = \mathbb{E}\big(X_1^{2p_1}\big)\, \mathbb{E}\big(X_2^{2p_2}\big).$$

Now

$$\begin{aligned} 2\phi'(t) = {} & q_1\, \mathbb{E}\bigg(\Big(\frac{X_1}{\sqrt{t}} - \frac{Y_1}{\sqrt{1-t}} \Big) \big(\sqrt{t}\, X_1 + \sqrt{1-t}\, Y_1\big)^{q_1 - 1} \big(\sqrt{t}\, X_2 + \sqrt{1-t}\, Y_2\big)^{q_2} \bigg) \\ & + q_2\, \mathbb{E}\bigg(\Big(\frac{X_2}{\sqrt{t}} - \frac{Y_2}{\sqrt{1-t}} \Big) \big(\sqrt{t}\, X_1 + \sqrt{1-t}\, Y_1\big)^{q_1} \big(\sqrt{t}\, X_2 + \sqrt{1-t}\, Y_2\big)^{q_2 - 1} \bigg). \end{aligned}$$

The integration by parts formula with respect to X_1 expresses that whenever $\varphi : \mathbb{R} \to \mathbb{R}$ is smooth,

$$\mathbb{E}\big(X_1 \varphi(X_1)\big) \,=\, \mathbb{E}\big(\varphi'(X_1)\big).$$

Therefore

$$\begin{aligned}
&\mathbb{E}\Big(X_1\big(\sqrt{t}\,X_1 + \sqrt{1-t}\,Y_1\big)^{q_1-1}\big(\sqrt{t}\,X_2 + \sqrt{1-t}\,Y_2\big)^{q_2}\Big) \\
&\qquad\qquad = \sqrt{t}\,(q_1-1)\mathbb{E}\Big(\big(\sqrt{t}\,X_1 + \sqrt{1-t}\,Y_1\big)^{q_1-2}\big(\sqrt{t}\,X_2 + \sqrt{1-t}\,Y_2\big)^{q_2}\Big) \\
&\qquad\qquad\quad + \sqrt{t}\,q_2\,\mathbb{E}(X_1X_2)\,\mathbb{E}\Big(\big(\sqrt{t}\,X_1 + \sqrt{1-t}\,Y_1\big)^{q_1-1}\big(\sqrt{t}\,X_2 + \sqrt{1-t}\,Y_2\big)^{q_2-1}\Big),
\end{aligned}$$

and with respect to Y_1,

$$\begin{aligned}
&\mathbb{E}\Big(Y_1\big(\sqrt{t}\,X_1 + \sqrt{1-t}\,Y_1\big)^{q_1-1}\big(\sqrt{t}\,X_2 + \sqrt{1-t}\,Y_2\big)^{q_2}\Big) \\
&\qquad\qquad = \sqrt{1-t}\,(q_1-1)\mathbb{E}\Big(\big(\sqrt{t}\,X_1 + \sqrt{1-t}\,Y_1\big)^{q_1-2}\big(\sqrt{t}\,X_2 + \sqrt{1-t}\,Y_2\big)^{q_2}\Big).
\end{aligned}$$

(Note that when $q_1 = 1$, $(q_1-1)(\sqrt{t}\,X_1 + \sqrt{1-t}\,Y_1)^{q_1-2}$ is understood as 0.) Together with the corresponding identities with respect to X_2 and Y_2, it readily follows that

$$2\phi'(t) \,=\, (q_1+q_2)\,\mathbb{E}(X_1X_2)\,\mathbb{E}\Big(\big(\sqrt{t}\,X_1 + \sqrt{1-t}\,Y_1\big)^{q_1-1}\big(\sqrt{t}\,X_2 + \sqrt{1-t}\,Y_2\big)^{q_2-1}\Big).$$

Repeating the argument with the couple of integers (q_1-1, q_2-1), it follows that when q_1 and q_2 are even, $\phi''(t) \geq 0$, $t \in [0,1]$. But $\phi'(0) = 0$ since Y_1 and Y_2 are independent and centered, so ϕ is increasing, which is the claim. □

Proposition 2. *The conjecture is true when* $p_1 = \cdots = p_n = 1$.

This proposition is established in [4], as consequence of a general linear algebra inequality between Hafnians and permanents. An alternate proof is provided by the more general Theorem 5 below from [11], from which the following sketch is taken.

Proof. By homogeneity, it may be assumed that $\mathbb{E}(X_1^2) = \cdots = \mathbb{E}(X_n^2) = 1$, so that the inequality to establish is that

$$\mathbb{E}\bigg(\prod_{k=1}^n X_k^2\bigg) \,\geq\, 1.$$

The proof relies on the interpolation scheme of Proposition 1, although it will be convenient to develop it with respect to the standard Gaussian measure $d\gamma_n(x) = \frac{1}{(2\pi)^{\frac{n}{2}}}\, e^{-\frac{1}{2}|x|^2} d\lambda_n(x)$. Namely, if $\Sigma = M^{\top}M$ is the covariance matrix of the law of the centered Gaussian vector $(X_1, \ldots, X_n)$, the latter is distributed as MG where G has law γ_n. Denoting by $v_1, \ldots, v_n$

the rows of M, which are unit vectors in $\mathbb{R}^n$ by the chosen normalization, the inequality to be proved amounts to

$$\int_{\mathbb{R}^n} \prod_{k=1}^n \langle v_k, x\rangle^2 d\gamma_n(x) \,\geq\, 1.$$

Set then, for $k = 1, \dots, n$,

$$f_k(x, y; t) \,=\, \sqrt{t}\,\langle v_k, x\rangle + \sqrt{1-t}\, y_k, \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}^n,\ t \in [0, 1].$$

The aim will be to show that the function of $t \in [0, 1]$,

$$\phi(t) \,=\, \int_{\mathbb{R}^n \times \mathbb{R}^n} \prod_{k=1}^n f_k(x, y; t)^2 d\gamma_n(x) d\gamma_n(y)$$

is increasing, so that

$$\int_{\mathbb{R}^n} \prod_{k=1}^n \langle v_k, x\rangle^2 d\gamma_n(x) \,=\, \phi(1) \,\geq\, \phi(0) \,=\, \int_{\mathbb{R}^n} \prod_{k=1}^n y_k^2\, d\gamma_n(y) \,=\, 1.$$

Arguing as in the proof of Proposition 1,

$$\begin{aligned} \phi'(t) \,&=\, 2 \sum_{\ell=1}^n \int_{\mathbb{R}^n \times \mathbb{R}^n} \partial_t f_\ell(x, y; t) f_\ell(x, y; t) \prod_{k \neq \ell} f_k(x, y; t)^2 d\gamma_n(x) d\gamma_n(y) \\ &=\, \sum_{\ell=1}^n \int_{\mathbb{R}^n \times \mathbb{R}^n} \bigg(\frac{\langle v_\ell, x\rangle}{\sqrt{t}} - \frac{y_\ell}{\sqrt{1-t}} \bigg) f_\ell(x, y; t) \prod_{k \neq \ell} f_k(x, y; t)^2 d\gamma_n(x) d\gamma_n(y) \\ &=\, 2 \sum_{\ell \neq j} \langle v_\ell, v_j\rangle \int_{\mathbb{R}^n \times \mathbb{R}^n} f_\ell(x, y; t) f_j(x, y; t) \prod_{k \neq \ell, j} f_k(x, y; t)^2 d\gamma_n(x) d\gamma_n(y) \end{aligned}$$

by integration by parts with respect to x and y in the last step.

Consider next the Ornstein-Uhlenbeck operator $\mathrm{L} = \Delta - \langle z, \nabla\rangle$ on $\mathbb{R}^n \times \mathbb{R}^n$, invariant and symmetric with respect to $\gamma_n \otimes \gamma_n$, and with Hermite polynomials as eigenvectors (see [2]). For smooth functions f, g, $\mathrm{L}(fg) = f\,\mathrm{L}g + g\,\mathrm{L}f + 2\nabla f \cdot \nabla g$. Each function f_k is an eigenfunction with eigenvalue 1 of $-\mathrm{L}$. It therefore follows that, with $F = \prod_{k=1}^n f_k$,

$$\begin{aligned} \mathrm{L}F \,&=\, \sum_{\ell=1}^n \mathrm{L} f_\ell \prod_{k \neq \ell} f_k + \sum_{\ell \neq j}^n \langle \nabla f_\ell, \nabla f_j\rangle \prod_{k \neq \ell, j} f_k \\ &=\, -nF + t \sum_{\ell \neq j}^n \langle v_\ell, v_j\rangle \prod_{k \neq \ell, j} f_k. \end{aligned}$$

From the expression of ϕ', the task is therefore to show that

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} F \, \mathrm{L} F d\gamma_n d\gamma_n + n \int_{\mathbb{R}^n \times \mathbb{R}^n} F^2 d\gamma_n d\gamma_n \, \geq \, 0. \tag{1}$$

The clever observation of [11] is that the product $F = \prod_{k=1}^n f_k$ may be expanded as

$$F \, = \, \sum_{\ell=0}^n W_\ell$$

where each W_ℓ is a linear combination of Hermite polynomials in $\mathbb{R}^n \times \mathbb{R}^n$ of degree ℓ. Since $\mathrm{L} W_\ell = -\ell \, W_\ell$, $\ell = 0, 1, \ldots, n$, and the W_ℓ are orthogonal in $\mathrm{L}^2(\gamma_n \otimes \gamma_n)$,

$$\begin{aligned}
\int_{\mathbb{R}^n \times \mathbb{R}^n} F \, \mathrm{L} F d\gamma_n d\gamma_n \, &= \, \sum_{\ell=0}^n \int_{\mathbb{R}^n \times \mathbb{R}^n} F \, \mathrm{L} W_\ell \, d\gamma_n d\gamma_n \\
&= \, - \sum_{\ell=0}^n \ell \int_{\mathbb{R}^n \times \mathbb{R}^n} F \, W_\ell \, d\gamma_n d\gamma_n \\
&= \, - \sum_{\ell=0}^n \ell \int_{\mathbb{R}^n \times \mathbb{R}^n} W_\ell^2 \, d\gamma_n d\gamma_n \\
&\geq \, -n \sum_{\ell=0}^n \int_{\mathbb{R}^n \times \mathbb{R}^n} W_\ell^2 \, d\gamma_n d\gamma_n \, = \, -n \int_{\mathbb{R}^n \times \mathbb{R}^n} F^2 d\gamma_n d\gamma_n
\end{aligned}$$

from which (1) follows. As a result, it holds true that $\phi'(t) \geq 0$ as requested. □

Proposition 3. *The conjecture is true when $n = 3$.*

The full statement is established in [7], after partial results [8, 14, 15]. By means of integration by parts, the proof in [7] shows that the map

$$\Sigma \, \mapsto \, \mathbb{E}\big(X_1^{2p_1} X_2^{2p_2} X_3^{2p_3} \big)$$

from the covariance matrices Σ of the Gaussian vector (X_1, X_2, X_3) is minimized at the identity matrix.

In addition to the preceding results, the Gaussian product inequality has been shown to hold under additional assumptions on the covariance structure of the Gaussian vector, and extended outside the Gaussian setting (cf. [5, 12, 14, 3, 16, 6]...).

2 The complex version

J. Arias-de-Reyna proved in [1] that the complex version of the conjecture holds true.

Theorem 4 (The complex Gaussian product inequality)**.** *If $(Z_1, \ldots, Z_n)$ is a centered Gaussian vector in $\mathbb{C}^n$, then for any integers $p_1, \ldots, p_n \in \mathbb{N}$,*

$$\mathbb{E}\bigg(\prod_{k=1}^{n} |Z_k|^{2p_k}\bigg) \,\geq\, \prod_{k=1}^{n} \mathbb{E}\big(|Z_k|^{2p_k}\big).$$

By complex Gaussian vector, it is understood that for each $k = 1, \ldots, n$, $Z_k = X_k + iY_k$, and that the vector $(X_1, \ldots, X_n, Y_1, \ldots, Y_n)$ is jointly Gaussian in $\mathbb{R}^{2n}$. The proof in [1] is based on a permanent inequality by E. Lieb [10] together with integral representations and relations between Gaussian variables and permanents.

It is a standard fact that the monomials $z \in \mathbb{C} \mapsto z^k$, $k \in \mathbb{N}$, form a complete orthogonal system for the complex Hilbert space $\mathrm{L}^2(\gamma^{\mathbb{C}})$ where

$$d\gamma^{\mathbb{C}}(z) \,=\, \frac{1}{2\pi}\, e^{-\frac{1}{2}|z|^2} d\lambda^{\mathbb{C}}(z) \,=\, \frac{1}{2\pi}\, e^{-\frac{1}{2}(x^2+y^2)} d\lambda_2(x,y)$$

is the standard Gaussian measure on $\mathbb{C}$. As such, the following statement obtained by D. Malicet, I. Nourdin, G. Peccati, G. Poly in [11] is a natural version of the Gaussian product conjecture with the monomials z^k replaced by the Hermite polynomials h_k, $k \in \mathbb{N}$, orthonormal basis of the real Hilbert space $\mathrm{L}^2(\gamma_1)$ with respect to the standard normal distribution γ_1 on $\mathbb{R}$ (cf. [2]).

Theorem 5. *Let $(X_1, \ldots, X_n)$ be a centered Gaussian vector in $\mathbb{R}^n$. For any integers $p_1, \ldots, p_n \in \mathbb{N}$,*

$$\mathbb{E}\bigg(\prod_{k=1}^{n} h_{p_k}(X_k)^2\bigg) \,\geq\, \prod_{k=1}^{n} \mathbb{E}\big(h_{p_k}(X_k)^2\big).$$

This theorem is shown in [11] to be a particular case of an inequality for squares of elements belonging to the Wiener chaos of the Gaussian vector. The proof of Theorem 5, relying on integration by parts with respect to the Ornstein-Uhlenbeck operator, goes along the lines put forward to cover Proposition 2 in the previous section.

3 A polarization problem

The Gaussian product inequality in the complex case (Theorem 4) is closely connected to a famous polarization problem (a discussion taken from [1]), and its validity actually implies, after the use of polar coordinates, the following theorem.

Theorem 6 (The complex polarization problem)**.** *For any collection $v_1, \ldots, v_n$ of unit vectors in $\mathbb{C}^n$, $n \geq 2$, there exists a unit vector $u \in \mathbb{C}^n$ such that*

$$\big|\langle u, v_1\rangle \cdots \langle u, v_n\rangle\big| \,\geq\, n^{-\frac{n}{2}}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{C}^n$. As a consequence, for every complex Hilbert space $\mathcal{H}$ of dimension $n \geq 2$, the n-th linear polarization constant $c_n(\mathcal{H})$ of $\mathcal{H}$, defined as the least $m > 0$ such that for all unit vectors $v_1, \ldots, v_n$ in $\mathcal{H}$ there exists a unit vector $u \in \mathcal{H}$ such that

$$\left|\langle u, v_1 \rangle_{\mathcal{H}} \cdots \langle u, v_n \rangle_{\mathcal{H}}\right| \geq m$$

is equal to $n^{-\frac{n}{2}}$.

Now of course, the issue is about the real version of this theorem (with the same statement). This real version has been established up to $n \leq 5$ [13]. It is shown in [4] that it would be implied by a solution to the Gaussian product conjecture.

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