"More than fifteen proofs of the logarithmic Sobolev inequality"

In his monograph *Optimal transport, old and new*, Springer (2009), C. Villani mentions on page 559, that: *"At present, there are more than fifteen known proofs of the logarithmic Sobolev inequality"*[1]. It is the purpose of this note to try to identify these proofs.[2]

By "logarithmic Sobolev inequality", it is understood in this note the inequality with respect to the Gaussian measure (or its equivalent form with respect to the Lebesgue measure), the only one considered here. The preceding monograph [20], as well as [2] among others, describe a wealth of logarithmic Sobolev inequalities in various contexts. A main source of references for this post are the surveys [12, 8, 13].

Let $\gamma = \gamma_n$ be the standard Gaussian probability distribution on $\mathbb{R}^n$, with density $\frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}|x|^2}$, $x \in \mathbb{R}^n$, with respect to the Lebesgue measure. The logarithmic Sobolev inequality (for γ) expresses that, for every smooth function $f : \mathbb{R}^n \to \mathbb{R}$ with $\int_{\mathbb{R}^n} f^2 d\gamma = 1$,

$$\int_{\mathbb{R}^n} f^2 \log f^2 \, d\gamma \, \leq \, 2 \int_{\mathbb{R}^n} |\nabla f|^2 d\gamma. \tag{1}$$

(By homogeneity, for every smooth, square integrable, $f : \mathbb{R}^n \to \mathbb{R}$,

$$\int_{\mathbb{R}^n} f^2 \log f^2 \, d\gamma - \int_{\mathbb{R}^n} f^2 \, d\gamma \, \log \left(\int_{\mathbb{R}^n} f^2 \, d\gamma \right) \, \leq \, 2 \int_{\mathbb{R}^n} |\nabla f|^2 d\gamma.)$$

It is a simple matter to check that the inequality is sharp on the exponential functions $f(x) = e^{\langle x,a \rangle - |a|^2}$, $x \in \mathbb{R}^n$, where $a \in \mathbb{R}^n$.

[1] The number 15 seems to be taken from the survey [12].

[2] Any relevant informations, references, omitted or new proofs, are most welcome, and will be incorporated as they appear (to the author's knowledge).

This inequality can take various equivalent forms. For example, for any (smooth strictly positive) probability density f with respect to γ (i.e. $\int_{\mathbb{R}^n} f\, d\gamma = 1$),

$$\int_{\mathbb{R}^n} f \log f \, d\gamma \;\leq\; \frac{1}{2} \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} \, d\gamma. \tag{2}$$

The right-hand side is the so-called Fisher information $\mathrm{I}(\mu \,|\, \gamma)$ of the probability measure $d\mu = f d\gamma$ with respect to γ, so that the latter can also be formulated as

$$\mathrm{H}\big(\mu \,|\gamma\big) \;\leq\; \frac{1}{2}\, \mathrm{I}\big(\mu \,|\gamma\big) \tag{3}$$

where $\mathrm{H}(\mu \,|\gamma) = \int_{\mathbb{R}^n} f \log f d\gamma$ is the relative entropy of μ with respect to γ.

It is an important feature that the inequality and the constants do not depend on the dimension of the underlying state space. By affine transformations, the logarithmic Sobolev inequality may be formulated for arbitrary Gaussian measures. Due to its dimension-free character, infinite dimensional Gaussian measures may also be considered.

After preliminary steps and formulations by

A. Stam. Some inequalities satisfied by the quantities of information of Fisher and Shannon. *Inf. Control* 2, 101–112 (1959)

in a (one-dimensional) information theoretic framework (see **16.**), and

P. Federbush. A partially alternate derivation of a result of Nelson. *J. Math. Phys.* 10, 50–52 (1969)

in a mathematical physics context,

logarithmic Sobolev inequalities have been introduced and emphasized by L. Gross in the fundamental contribution

2. L. Gross. Logarithmic Sobolev inequalities. *Amer. J. Math.* 97, 1061–1083 (1975)

where, in particular, (1) is established.

A further main aspect of this work by L. Gross is the equivalence (in a general Dirichlet form framework) of logarithmic Sobolev inequalities with hypercontractivity properties. For the Gaussian measure γ, let

$$P_t f(x) \;=\; \int_{\mathbb{R}^n} f\big(e^{-t}x + \sqrt{1-e^{-2t}}\, y\big) d\gamma(y), \quad t > 0,\ x \in \mathbb{R}^n, \tag{4}$$

be the so-called Ornstein-Uhlenbeck semigroup (acting on suitable functions $f : \mathbb{R}^n \to \mathbb{R}$). By Jensen's inequality, the operators P_t are contractions in all $\mathrm{L}^p(\gamma)$, $p \geq 1$, spaces. The logarithmic Sobolev inequality (1), holding for all smooth functions f, is equivalent to the hypercontractivity property

$$\|P_t f\|_q \leq \|f\|_p \tag{5}$$

holding for all (measurable) $f : \mathbb{R}^n \to \mathbb{R}$, $1 < p < q < \infty$, $e^{2t} \geq \frac{q-1}{p-1}$ (where $\|\cdot\|_p$ is the $\mathrm{L}^p(\gamma)$ norm). The proof of the equivalence (put forward by L. Gross, and already apparent in the aforementioned work of P. Federbush) amounts to take the time derivative of $\|P_t f\|_{q(t)}$ where $q(t) = 1 + e^{2t}(p-1)$, which yields the logarithmic Sobolev inequality up to a power-type change of function.

The hypercontractive bound (5) was actually put forward first, before the logarithmic Sobolev inequality, by E. Nelson in

1. E. Nelson. A quartic interaction in two dimensions. *Mathematical Theory of Elementary Particles* 69–73. M.I.T. Press, Cambridge (1966); E. Nelson. The free Markov field. *J. Funct. Anal.* 12, 211–227 (1973).

The proof of (5) therein is a reduction to a problem of multipliers for Hermite series, analyzed in the broader context of continuous Fock spaces. Several related papers in the mathematical physics literature appeared in the period (cf. [12, 8]).

For the discussion, it is significant to emphasize the corresponding logarithmic Sobolev inequality and hypercontractivity property on the discrete cube $\{-1, +1\}^n$, and more specifically on the two-point space $\{-1, +1\}$, equipped with the uniform probability measure, where they amount to two-point inequalities. That is, for the logarithmic Sobolev inequality,

$$\frac{1}{2}\big[a^2 \log a^2 + b^2 \log b^2\big] - \frac{1}{2}(a^2 + b^2) \log\Big(\frac{1}{2}(a^2 + b^2)\Big) \leq (a - b)^2 \tag{6}$$

for every $a, b \in \mathbb{R}$, and, for hypercontractivity,

$$\Big(\frac{1}{2}\big[|a + e^{-t}b|^q + |a - e^{-t}b|^q\big]\Big)^{1/q} \leq \Big(\frac{1}{2}\big[|a + b|^p + |a - b|^p\big]\Big)^{1/p} \tag{7}$$

for every $a, b \in \mathbb{R}$, $1 < p < q < \infty$, $e^{2t} \geq \frac{q-1}{p-1}$. Of course, according to Gross' principle, these two families of inequalities are equivalent ((6) being the infinitesimal version of (7)). The point is that actually the two-point inequalities tensorize to the discrete cube $\{-1, +1\}^n$, and then, after a suitable rescaling, yield the Gaussian inequalities via the classical central limit theorem, which is the original proof emphasized by L. Gross in his 1975 article (see e.g. [1] for a detailed discussion). This reasoning has been very fruitful in the origin and proofs of the logarithmic Sobolev inequality and hypercontractivity.

From the same tensorization property, it is enough to prove the logarithmic Sobolev inequality (1) and hypercontractivity (5) in dimension one, and many of the proofs presented below are restricted to this case. Besides, the lists only emphasize proofs yielding optimal constants. In this regard, it may be mentioned that a defective term in the logarithmic Sobolev inequality may be accepted, provided it is dimension-free. More precisely, suppose that it may be established that for every $\varepsilon > 0$, there is a constant $C(\varepsilon) > 0$ such that, for every $n \geq 1$ and all smooth functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $\int_{\mathbb{R}^n} f^2 d\gamma = 1$,

$$\int_{\mathbb{R}^n} f^2 \log f^2 \, d\gamma \, \leq \, (2+\varepsilon) \int_{\mathbb{R}^n} |\nabla f|^2 d\gamma + C(\varepsilon),$$

then (1) holds true. Indeed, given $g : \mathbb{R} \to \mathbb{R}$ with $\int_{\mathbb{R}} g^2 d\gamma_1 = 1$, application of the latter to $f = g^{\otimes n}$ yields

$$\int_{\mathbb{R}} g^2 \log g^2 \, d\gamma_1 \, \leq \, (2+\varepsilon) \int_{\mathbb{R}} g'^2 d\gamma_1 + \frac{C(\varepsilon)}{n}$$

from which the claim follows after letting n tend to infinity and ε to zero.

Logarithmic Sobolev inequalities may be considered in a wide generality (cf. e.g. [20, 2]). Only the Gaussian logarithmic Sobolev inequality and hypercontractivity are considered in this note. The Gaussian logarithmic Sobolev inequality is sometimes considered, and proved, in its Euclidean (dimensional) form, with respect to the Lebesgue measure (obtained after a simple, equivalent, change of function – seemingly first emphasized in [22]),

$$\int_{\mathbb{R}^n} f^2 \log f^2 dx \, \leq \, \frac{n}{2} \log \left(\frac{2}{n\pi e} \int_{\mathbb{R}^n} |\nabla f|^2 dx \right) \tag{8}$$

for every smooth function $f : \mathbb{R}^n \to \mathbb{R}$ such that $\int_{\mathbb{R}^n} f^2 dx = 1$.

The list of proofs is divided into proofs of hypercontractivity (historically first) and proofs of the logarithmic Sobolev inequality. For each of them, a two-line description of the corresponding underlying argument is presented. Some works address at the same time hypercontractivity and logarithmic Sobolev inequality. While only the Gaussian logarithmic Sobolev inequality is classified, it should be mentioned that several proofs apply similarly to the family of log-concave probability measures $d\mu = e^{-V} dx$ on $\mathbb{R}^n$ where $V : \mathbb{R}^n \to \mathbb{R}$ is a potential which is more convex than the quadratic one in the sense that $V(x) - \frac{c}{2}|x|^2$, $x \in \mathbb{R}^n$, is convex for some $c > 0$ (as well as even more general settings cf. [20, 2]).

Only the first references on a given type of proof are emphasized, hopefully not omitting any and addressing proper acknowledgement. It is a bit difficult to settle whether there are more than fifteen different proofs, since a certain amount of overlaps and similar arguments in different contexts may be detected.

1 Hypercontractivity

3. A. Bonami. Étude des coefficients de Fourier des fonctions de $L^p(G)$. *Ann. Inst. Fourier* 20, 335–402 (1970). W. Beckner. Inequalities in Fourier analysis. *Ann. of Math.* 102, 159–182 (1975).

Proofs of the two-point inequality (7).

4. H. Brascamp, E. Lieb. Best constants in Young's inequality, its converse, and its generalization to more than three functions. *Advances in Math.* 20, 151–173 (1976).

Proof of hypercontractivity from a general integral inequality of Hölder type for functions composed along directions in $\mathbb{R}^n$, established by symmetrization techniques[3].

5. J. Neveu. Sur l'espérance conditionnelle par rapport à un mouvement brownien. *Ann. Inst. H. Poincaré Probab. Stat.* 12, 105–109 (1976).

Martingale representation of the conditional expectation with respect to Brownian motion and use of stochastic calculus.

14. J. Epperson. The hypercontractive approach to exactly bounding an operator with complex Gaussian kernel. *J. Funct. Anal.* 87, 1–30 (1989). E. Lieb. Gaussian kernels have only Gaussian maximizers. *Invent. Math.* 102, 179–208 (1990).

Proof of hypercontractivity via the study of Gaussian extremizers of operators given by Gaussian kernels.

21. Y. Hu. A unified approach to several inequalities for Gaussian and diffusion measures. *Séminaire de Probabilités XXXIV, Lecture Notes in Math.* 1729, 329–335. Springer (2000). E. Mossel, J. Neeman. Robust optimality of Gaussian noise stability. *J. Eur. Math. Soc.* 17, 433–482 (2015).

Heat flow proof of both hypercontractivity and the logarithmic Sobolev inequality along the line of the Bakry-Émery proof of logarithmic Sobolev inequality (**10.**).

[3]Numerous alternate proofs of this inequality, known as Brascamp-Lieb inequality, are available nowadays.

2 Logarithmic Sobolev inequality

6. O. Rothaus. Lower bounds for eigenvalues of regular Sturm-Liouville operators and the logarithmic Sobolev inequality. *Duke Math. J.* 45, 351–362 (1978).

Simple use of Jensen's inequality and the fact that the lowest eigenfunction for a Sturm-Liouville boundary value problem with Dirichlet boundary conditions is positive.

7. R. Adams, F. Clarke. Gross's logarithmic Sobolev inequality: a simple proof. *Amer. J. Math.* 101, 1265–1270 (1979).

Solving an optimization problem on the line by calculus of variations.

8. C. Mueller, F. Weissler. Hypercontractivity for the heat semigroup for ultraspherical polynomials and on the n-sphere. *J. Funct. Anal.* 48, 252–283 (1982).

Analytic investigation of multipliers along ultraspherical polynomials yielding at the extremal cases the two-point inequality and the Gaussian logarithmic Sobolev inequality.

9. A. Ehrhard. Sur l'inégalité de Sobolev logarithmique de Gross. *Séminaire de Probabilités XVIII, Lecture Notes in Math. 1059*, 194–196. Springer (1984).

Consequence of a Gaussian Polyá-Szegö energy inequality established by Gaussian symmetrization[4].

10. D. Bakry, M. Émery. Diffusions hypercontractives. *Séminaire de Probabilités XIX, Lecture Notes in Math. 1123*, 177–206. Springer (1985).

Study of the second derivative of entropy along the Ornstein-Uhlenbeck semigroup, outlined, in a simplified form, in Section 3 (a scheme of proof valid in a general Markov triple context cf. [2]).

11. O. Rothaus. Analytic inequalities, isoperimetric inequalities and logarithmic Sobolev inequalities. *J. Funct. Anal.* 64, 296–313 (1985).

Deduction of the (Euclidean) logarithmic Sobolev inequality from the classical Euclidean isoperimetric inequality by the co-area formula.

[4]Further, improved, works in this direction include [7, 18].

12. O. Rothaus. Hypercontractivity and the Bakry-Émery criterion for compact Lie groups. *J. Funct. Anal.* 65, 358–367 (1986).

Use of non-linear equation solved by extremal functions under the curvature condition fulfilled by Gauss space (in a compact setting though).

13. M. Ledoux. Isopérimétrie et inegalités de Sobolev logarithmiques gaussiennes. *Comptes Rendus Acad. Sci. Paris* 306, 79–82 (1988). E. Pelliccia, G. Talenti. A proof of a logarithmic Sobolev inequality. *Calc. Var. Partial Differential Equations* 1, 237–242 (1993).

Deduction of the (Gaussian) logarithmic Sobolev inequality from the Gaussian isoperimetric inequality by the co-area formula.

15. E. Carlen, M. Loss. Extremals of functionals with competing symmetries. *J. Funct. Anal.* 88, 437–456 (1990). W. Beckner. Sobolev inequalities, the Poisson semigroup, and analysis on the sphere $\mathbb{S}^n$. *Proc. Nat. Acad. Sci. U.S.A.* 89, 4816–4819 (1992).

Proof by duality and general rearrangement inequalities, of both hypercontractivity and logarithmic Sobolev inequality, exploiting competition between rearrangement on the sphere and Gauss space and the familiar Riesz rearrangement (partly inspired by E. Lieb's proof [17] of the sharp Hardy-Littlewood-Sobolev inequalities).

16. E. Carlen. Superadditivity of Fisher's information and logarithmic Sobolev inequalities. *J. Funct. Anal.* 101, 194–211 (1991).

Consequence of an inequality due to N. Blachman [4] and A. Stam [19] in information theory, itself a direct consequence of superadditivity of Fisher information. Proof of a strengthened form of the logarithmic Sobolev inequality involving a Fourier transform contribution, itself a reinterpretation of an inequality due to W. Beckner [3] and I. Hirschmann [14]. The information-theoretic background and approach to the logarithmic Sobolev inequality, in particular the connection to the Shannon-Stam entropy power inequality, are discussed in [10] and [9].

17. G. Toscani. Sur l'inégalité logarithmique de Sobolev. *C. R. Acad. Sci. Paris, Math.* 324, 689–694 (1997).

Proof of the Euclidean logarithmic Sobolev inequality via the Fokker-Planck equation, similar in nature to the Bakry-Émery argument (**10.**), leading to a deficit term identifying the extremal functions[5].

[5]see also [15] about the latter point.

18. W. Beckner. Geometric asymptotics and the logarithmic Sobolev inequality. *Forum Math.* 11, 105–137 (1999).

The Euclidean logarithmic Sobolev inequality is developed as a geometric asymptotic estimate with respect to Lebesgue measure from the classical isoperimetric inequality in $\mathbb{R}^n$ in its functional form with sharp constants.

19. W. Beckner (as quoted in [16], p. 146).

Deduction from S. Bobkov's functional form of the Gaussian isoperimetric inequality [5, 2].

20. M. Capitaine, E. P. Hsu, M. Ledoux. Martingale representation and a simple proof of logarithmic Sobolev inequalities on path spaces *Electron. Comm. Probab.*2, 71–81 (1999).

Proof by stochastic calculus on Brownian paths, starting from the Clark-Ocone-Haussmann formula. Going back to unpublished works by B. Maurey and J. Neveu, and actually close in spirit to the Neveu proof of hypercontractivity.

22. S. Bobkov, M. Ledoux. From Brunn-Minkowski to Brascamp-Lieb and to logarithmic Sobolev inequalities *Geom. Funct. Anal.* 10, 1028–1052 (2000).

Deduction from the functional form (Prékopa-Leindler) of the Brunn-Minkowski inequality at the edge of the parameters.

23. F. Otto, C. Villani. Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality. *J. Funct. Anal.* 173, 361–400 (2000).

Proof of a strengthened form of the logarithmic Sobolev inequality linking entropy, Fisher information and Wasserstein distance (HWI inequality) by optimal transport and pde tools.

24. S. Bobkov, I. Gentil, M. Ledoux. Hypercontractivity of Hamilton-Jacobi equations. *J. Math. Pures Appl.* 80, 669–696 (2001).

Proof of the Otto-Villani HWI inequality by means of the logarithmic form of Wang's Harnack inequality [21], itself following from the Bakry-Émery scheme **10.**

25. M. Del Pino, J. Dolbeault. Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions. *J. Math. Pures Appl.* 81, 847–875 (2002); M. Del Pino, J. Dolbeault. The optimal Euclidean L^p-Sobolev logarithmic inequality. *J. Funct. Anal.* 197, 151–161 (2003).

The Euclidean logarithmic Sobolev inequality appears as an extreme point in the proof of best constants in a family of Gagliardo-Nirenberg inequalities achieved by non-linear analysis.

26. D. Cordero-Erausquin. Some applications of mass transport to Gaussian-type inequalities. *Arch. Ration. Mech. Anal.* 161, 257–269 (2002).

Direct mass transportion methods (including the Otto-Villani HWI inequality).

27. J. Lehec. Representation formula for the entropy and functional inequalities. *Ann. Inst. Henri Poincaré Probab. Stat.* 49, 885–899 (2013).

Proof from a stochastic representation formula for Gaussian relative entropy in Wiener space, relying on stochastic calculus and Girsanov's theorem.

28.[6] G. Wang, C. Xia, X. Zhang. The logarithmic Sobolev inequality on non-compact self-shrinkers (2024).

Proof for the logarithmic Sobolev inequality in the Euclidean space by using the Alexandrov-Bakelman-Pucci (ABP) pde method emphasized by S. Brendle in [6] to establish logarithmic Sobolev inequalities on subspaces and submanifolds. (See also [11] for earlier work in this line of investigation.)

3 The simplest proof?

In [12], L. Gross presents the 1978 proof by O. Rothaus as the simplest proof of the logarithmic Sobolev inequality. In 2010, he agreed that, probably, the simplest proof is the one by D. Bakry and M. Émery (1985) **10.** This section thus outlines this argument.

Let

$$h_t(x) \,=\, \frac{1}{(4\pi t)^{\frac{n}{2}}}\, e^{-\frac{1}{4t}|x|^2}, \quad t>0,\ x\in\mathbb{R}^n,$$

be the standard heat kernel, fundamental solution of the heat equation $\partial_t h_t = \Delta p_t$. The convolution semigroup $H_t f(x) = f * h_t(x)$, $t>0$, solves $\partial_t H_t f = \Delta H_t f = H_t \Delta f$ with initial data f. At $t=\frac{1}{2}$, h_t is just the standard Gaussian density so that $H_{\frac{1}{2}} f(0) = \int_{\mathbb{R}^n} f d\gamma$ (while $P_0 f = f$).

[6] Added in 2024.

For $f > 0$ smooth, $t > 0$, at any point, by the heat equation,

$$\begin{aligned} H_t(f \log f) - H_t f \log H_t f & \\ &= \int_0^t \frac{d}{ds} H_s\big(H_{t-s} f \log H_{t-s} f\big) ds \\ &= \int_0^t H_s\Big(\Delta\big(H_{t-s} f \log H_{t-s} f\big) - \Delta H_{t-s} f \log H_{t-s} f - \Delta H_{t-s} f\Big) ds \\ &= \int_0^t H_s\Big(\frac{|\nabla H_{t-s} f|^2}{H_{t-s} f}\Big) ds \end{aligned}$$

(since $\Delta(g \log g) - \Delta g \log g - \Delta g = \frac{|\nabla g|^2}{g}$ for any smooth positive function g). Now, for any $u > 0$, $\nabla H_u f = H_u(\nabla f)$, so that

$$|\nabla H_u f|^2 \ \le\ \big[H_u(|\nabla f|)\big]^2 \ \le\ H_u\Big(\frac{|\nabla f|^2}{f}\Big) H_u f$$

by the Cauchy-Schwarz inequality for the integral kernel H_u. With $u = t - s$, it follows that

$$\frac{|\nabla H_{t-s} f|^2}{H_{t-s} f} \ \le\ H_{t-s}\Big(\frac{|\nabla f|^2}{f}\Big),$$

and hence

$$H_t(f \log f) - H_t f \log H_t f \ \le\ \int_0^t H_s\Big(H_{t-s}\Big(\frac{|\nabla f|^2}{f}\Big)\Big) ds \ =\ t\, H_t\Big(\frac{|\nabla f|^2}{f}\Big)$$

from the semigroup property. For $t = \frac{1}{2}$, at $x = 0$ (for example), the latter amounts to (2). Thus, the logarithmic Sobolev inequality is (just) a consequence of the Cauchy-Schwarz inequality.

References

[1] D. Bakry. L'hypercontractivité et son utilisation en théorie des semigroupes. *École d'Été de Probabilités de Saint-Flour 1992, Lecture Notes in Math.* 1581, 1–114. Springer (1994).

[2] D. Bakry, I. Gentil, M. Ledoux. *Analysis and geometry of Markov diffusion operators.* Grundlehren der mathematischen Wissenschaften 348. Springer (2014).

[3] W. Beckner. Inequalities in Fourier Analysis. *Ann. of Math.* 102, 159–182 (1975).

[4] N. Blachman. The convolution inequality for entropy powers. *IEEE Trans. Inform. Theory* 2, 267–271 (1965).

[5] S. Bobkov. An isoperimetric inequality on the discrete cube, and an elementary proof of the isoperimetric inequality in Gauss space. *Ann. Probab.* 25, 206–214 (1997).

[6] S. Brendle. The logarithmic Sobolev inequality for a submanifold in Euclidean space. *Comm. Pure Appl. Math. 75*, 449–454 (2022).

[7] E. Carlen, M. Loss. Extremals of functionals with competing symmetries. *J. Funct. Anal.* 88, 437–456 (1990).

[8] E. B. Davies, L. Gross, B. Simon. Hypercontractivity: a bibliographic review. *Ideas and methods in quantum and statistical physics, Oslo, 1988*, 370–389. Cambridge Univ. Press (1992).

[9] A. Dembo. Information inequalities and uncertainty principles. Tech. Report, Department of Statistics, Stanford University (1990).

[10] A. Dembo, T. Cover, J. Thomas. Information theoretic inequalities. *IEEE Trans. Inform. Theory* 37, 1501-1518 (1991).

[11] K. Ecker. Logarithmic Sobolev inequalities on submanifolds of Euclidean space. *J. Reine Angew. Math.* 522, 105–118 (2000).

[12] L. Gross. Logarithmic Sobolev inequalities and contractivity properties of semigroups. *Dirichlet Forms, Varenna, 1992, Lecture Notes in Math.* 1563, 54–88. Springer (1993).

[13] L. Gross. Hypercontractivity, logarithmic Sobolev inequalities, and applications: a survey of surveys. *Diffusion, Quantum Theory, and Radically Elementary Mathematics, Math. Notes* 47, 45–73. Princeton Univ. Press (2006).

[14] I. Hirchsmann. A note on entropy. *Amer. J. Math.* 79, 152–156 (1957).

[15] M. Ledoux. On an integral criterion for hypercontractivity of diffusion semigroups and extremal functions *J. Funct. Anal.* 105, 444–465 (1992).

[16] M. Ledoux. Concentration of measure and logarithmic Sobolev inequalities. *Séminaire de Probabilités XXXIII, Lecture Notes in Math.* 1709, 120–216. Springer (1999)).

[17] E. Lieb. Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities. *Ann. of Math.* 118, 349–374 (1983)

[18] J. Martín, M. Milman. Isoperimetry and symmetrization for logarithmic Sobolev inequalities. *J. Funct. Anal.* 256, 149–178 (2009).

[19] A. Stam. Some inequalities satisfied by the quantities of information of Fisher and Shannon. *Inf. Control* 2, 101–112 (1959).

[20] C. Villani. *Optimal transport, old and new.* Grundlehren der mathematischen Wissenschaften 338. Springer (2009).

[21] F.-Y. Wang. Logarithmic Sobolev inequalities on noncompact Riemannian manifolds. *Probab. Theory Relat. Fields* 109, 417–424 (1997).

[22] F. Weissler. Logarithmic Sobolev inequalities for the heat-diffusion semigroup. *Trans. Amer. Math. Soc.* 237, 255–269 (1978).