A simple proof of the Nualart-Peccati Fourth-Moment Theorem

Abstract

D. Nualart and G. Peccati showed in 2005 [10] that for a sequence of Wiener chaos of fixed degree, the convergence to a Gaussian law is equivalent to the convergence of the fourth moments. The note presents the recent simple and short proof by E. Azmoodeh, S. Campese and G. Poly [1] of this result.

In the striking contribution [10], D. Nualart and G. Peccati discovered that the fourth moment of homogeneous polynomial chaos on Wiener space is enough to characterize convergence towards the Gaussian distribution. Specifically, and in a simplified (finite dimensional) setting, let $F : \mathbb{R}^N \to \mathbb{R}$, $1 \leq d \leq N$, be defined by

$$F = F(x) = \sum_{i_1,\ldots,i_d=1}^{N} a_{i_1,\ldots,i_k} x_{i_1} \cdots x_{i_d}, \quad x = (x_1, \ldots, x_N) \in \mathbb{R}^N, \tag{1}$$

where $a_{i_1,\ldots,i_d}$ are real numbers vanishing on diagonals and symmetric in the indices. Such a function F will be called homogeneous of degree d. Let now F_n on $\mathbb{R}^{N_n}$, $n \in \mathbb{N}$, $N_n \to \infty$, be a sequence of such homogeneous polynomials of fixed degree d. The theorem of D. Nualart and G. Peccati expresses that the distributions F_n under the standard Gaussian measures $d\gamma_{N_n}(x) = \frac{1}{(2\pi)^{N_n/2}} e^{-|x|^2/2} dx$ on $\mathbb{R}^{N_n}$ converge as $n \to \infty$ towards the standard Gaussian distribution on the real line if and only if

$$\int_{\mathbb{R}^{N_n}} F_n^2 \, d\gamma_{N_n} \to 1 \qquad \text{and} \qquad \int_{\mathbb{R}^{N_n}} F_n^4 \, d\gamma_{N_n} \to 3,$$

1 and 3 being respectively the second and fourth moments of the standard normal. (Equivalently, the F_n's may be normalized in $\mathrm{L}^2(\gamma_{N_n})$ so that the fourth-moment characterizes the weak convergence.)

The analysis developed in [10] actually holds for homogeneous chaos (multiple stochastic inetegrals) on the infinite dimensional Wiener space, and the equivalence is further described in terms of convergence of contractions. The proof of [10] relies on multiplication formulas for homogeneous chaos and stochastic calculus.

Since [10] was published, other proofs have been developed, including [4, 6, 7]... In particular, the work by D. Nualart and S. Ortiz-Latorre [9] introduces a technological advance with a new proof based on Malliavin operators and the use of integration by parts on Wiener space emphasizing that the convergence of $(F_n)_{n\in\mathbb{N}}$ to a Gaussian distribution is also equivalent to the fact that

$$\mathrm{Var}_{\gamma_{N_n}}\big(|\nabla F_n|^2\big) \to 0,$$

where $\mathrm{Var}_{\gamma_{N_n}}$ is the variance with respect to γ_{N_n}. This reduction was actually analyzed in the work by I. Nourdin and G. Peccati [5] as an effect of the so-called Stein method in this context. By this methodology, the latter authors provide furthermore quantitative bounds in the Nualart-Peccati theorem via the inequality (under $\int_{\mathbb{R}^{N_n}} F_n^2 d\gamma_{N_n} = 1$)

$$\mathrm{Var}_{\gamma_{N_n}}\big(|\nabla F_n|^2\big) \leq C_d\bigg(\int_{\mathbb{R}^{N_n}} F_n^4\, d\gamma_{N_n} - 3\bigg)$$

where $C_d > 0$ only depends on d.

Virtually all the aforementioned proofs make crucial use of the product formula for multiple stochastic integrals, itself a form of the multiplication formula for Hermite polynomials (and rely on a rather rigid structure of the underlying probability space). Building upon the Stein method and the reduction to the analysis of $\mathrm{Var}_{\gamma_{N_n}}\big(|\nabla F_n|^2\big)$, E. Azmoodeh, S. Campese and G. Poly [1] (see also the earlier [3]) recently produced a new simple proof avoiding the use of product formulas. Their spectral argument rather relies on the eigenfunction properties of the Ornstein-Uhlenbeck operator with invariant measure the Gaussian distribution, and a clever argument on the chaotic decomposition of the square of an homogeneous polynomial. One success of the approach is its ability to cover, with the same flexibility, settings away from the original Gaussian model (not expanded here).

The aim of the note is to present, in a concise exposition, the new argument by E. Azmoodeh, S. Campese and G. Poly [1]. The exposition is introduced by some classical facts on the Ornstein-Uhlenbeck operator, and its eigenfunctions consisting of the Hermite polynomials, and the chaos decomposition on a function on the Gauss space, and a brief reminder of Stein's method in this context following [5, 6].

1 Preliminaries on the Ornstein-Uhlenbeck operator

Recall the standard Gaussian measure γ_N on $\mathbb{R}^N$ with density $\frac{1}{(2\pi)^{N/2}} e^{-|x|^2/2}$ with respect to the Lebesgue measure. General references on the content of this section are e.g. the monographs [8, 6, 2].

The basic integration by parts formula with respect to γ_N expresses that for every smooth functions $f, g : \mathbb{R}^N \to \mathbb{R}$,

$$\int_{\mathbb{R}^N} f(-\mathrm{L}g) d\gamma_N = \int_{\mathbb{R}^N} \nabla f \cdot \nabla g \, d\gamma_N \tag{2}$$

where

$$\mathrm{L}f = \Delta f - x \cdot \nabla f$$

is the Ornstein-Uhlenbeck operator.

According to the integration by parts formula (2), the Ornstein-Uhlenbeck operator L has γ_N as invariant and reversible probability measure, and generates the symmetric bilinear carré du champ operator

$$\Gamma(f, g) = \nabla f \cdot \nabla g = \frac{1}{2}\big[\mathrm{L}(fg) - f\,\mathrm{L}g - g\,\mathrm{L}f\big]$$

acting on functions f, g in a suitable domain $\mathcal{A}$ of smooth functions on $\mathbb{R}^N$. For simplicity in the notation below, set $\Gamma(f) = \Gamma(f, f) = |\nabla f|^2$ which is always non-negative. The operator L is a diffusion operator in the sense that for every smooth function $\varphi : \mathbb{R} \to \mathbb{R}$, and every $f \in \mathcal{A}$,

$$\mathrm{L}\varphi(f) = \varphi'(f)\,\mathrm{L}f + \varphi''(f)\,\Gamma(f). \tag{3}$$

Alternatively, Γ is a derivation in the sense that $\Gamma(\varphi(f), g) = \varphi'(f)\,\Gamma(f, g)$.

The spectrum of the operator $-\mathrm{L}$ is $\mathbb{N}$, with eigenfunctions given by the Hermite polynomials. For every $\lambda \in \mathbb{R}$, the expansion in $x \in \mathbb{R}$,

$$e^{\lambda x - \frac{1}{2}\lambda^2} = \sum_{k=0}^{\infty} \frac{\lambda^k}{\sqrt{k!}}\, h_k(x)$$

defines the one-dimensional Hermite polynomials h_k, $k \in \mathbb{R}$. For every $k \in \mathbb{N}$, h_k is a polynomial of degree k. For example $h_0(x) = 1$, $h_1(x) = x$, $h_2(x) = \frac{1}{\sqrt{2}}(x^2 - 1)$. It is easily checked, on the generating series for example, that $\mathrm{L}h_k = -k\,h_k$, $k \in \mathbb{N}$. The Hermite polynomials are orthogonal in $\mathrm{L}^2(\gamma_1)$, orthonormal in the preceding formulation, and the complete system h_k, $k \in \mathbb{N}$, therefore defines an orthonormal basis (of eigenvectors of L) of $\mathrm{L}^2(\gamma_1)$.

Multi-dimensional Hermite polynomials on $\mathbb{R}^N$ are defined as products of one-dimensional polynomials with multi-index. Namely, for $\underline{k} = (k_1, \dots, k_N) \in \mathbb{N}^N$ and $x = (x_1, \dots, x_N) \in \mathbb{R}^N$, set

$$H_{\underline{k}}(x) \,=\, h_{k_1}(x_1) \cdots h_{k_N}(x_N).$$

For every $\underline{k} = (k_1, \dots, k_N) \in \mathbb{N}^N$, $\mathrm{L} H_{\underline{k}} \,=\, -k\, H_{\underline{k}}$ with $k = k_1 + \cdots + k_N$, $\underline{k} = (k_1, \dots, k_N)$. As in the real case, the family $H_{\underline{k}}$, $\underline{k} \in \mathbb{N}^N$, defines an orthonormal basis of the Hilbert space $\mathrm{L}^2(\gamma_N)$. The so-called orthogonal chaos decomposition (Fourier-Hermite expansion) of a function in $\mathrm{L}^2(\gamma_N)$ takes the form

$$f \,=\, \sum_{k \in \mathbb{N}^N} f_{\underline{k}}\, H_{\underline{k}} \,=\, \sum_{k=0}^{\infty} \bigg(\sum_{|\underline{k}|=k} f_{\underline{k}}\, H_{\underline{k}} \bigg)$$

where the $f_{\underline{k}}$'s are real numbers, and $|\underline{k}| = k_1 + \cdots + k_N$. The sums under parentheses actually represents the so-called homogeneous Wiener chaos of order k, which may be considered more generally.

Since $h_1(x) = x$, $x \in \mathbb{R}$, the homogeneous polynomial F from (1) is an eigenfunction of order d of L, that is

$$-\mathrm{L} F \,=\, dF. \tag{4}$$

2 The first step: Stein's lemma

The following statement from [5] (see also [6]) presents the application of Stein's method in the context of an eigenfunction of the Ornstein-Uhlenbeck operator. The outcome is that, for an eigenfunction F of $-\mathrm{L}$ with eigenvalue $d \geq 1$, the proximity of $\Gamma(F) = |\nabla F|^2$ to a constant forces the law of F under γ_N to be close to the (one-dimensional) standard normal. The underlying principle relies on the chain rule formula for the diffusion operator L,

$$\mathrm{L}(\varphi \circ F) \,=\, \varphi'(F)\, \mathrm{L}F + \varphi''(F)\, \Gamma(F) \,=\, -dF\varphi'(F) + \varphi''(F)\, \Gamma(F)$$

along a smooth function $\varphi : \mathbb{R} \to \mathbb{R}$. Therefore, if $\Gamma(F) = d$, $\mathrm{L}(\varphi \circ F) = d\, (\mathrm{L}_1 \varphi)(F)$ where L_1 is the one-dimensional Ornstein-Uhlenbeck operator with invariant measure γ_1, and thus the law of F is γ_1.

Proposition 1. *Let F be an eigenfunction of $-\mathrm{L}$ with eigenvalue $d \geq 1$. Denote by ν the distribution of F under γ_N. Given $\varphi : \mathbb{R} \to \mathbb{R}$ integrable with respect to ν and γ_1, let ψ be a smooth solution of the associated Stein equation $\varphi - \int_{\mathbb{R}} \varphi \, d\gamma_1 = \psi' - x\psi$. Then,*

$$\bigg| \int_{\mathbb{R}} \varphi \, d\nu - \int_{\mathbb{R}} \varphi \, d\gamma_1 \bigg| \,\leq\, \frac{C_\varphi}{d} \bigg(\int_{\mathbb{R}^N} \big[\Gamma(F) - d \big]^2 d\gamma_N \bigg)^{1/2}$$

where $C_\varphi = \|\psi'\|_\infty^2$.

Proof. Since ν is the distribution of F under γ_N, by the Stein equation,

$$\int_{\mathbb{R}} \varphi\, d\nu - \int_{\mathbb{R}} \varphi\, d\gamma_1 \,=\, \int_{\mathbb{R}^N} \varphi(F) d\gamma_N - \int_{\mathbb{R}} \varphi\, d\gamma_1 \,=\, \int_{\mathbb{R}^N} \big[\psi'(F) - F\psi(F)\big] d\gamma_N.$$

Now $-\mathrm{L}F = dF$ so that

$$\psi'(F) - F\psi(F) \,=\, \psi'(F) + d^{-1}\,\mathrm{L}F\,\psi(F)$$

and hence, after integration by parts with respect to the operator L (2) and the use of the diffusion property (3),

$$\int_{\mathbb{R}} \varphi\, d\nu - \int_{\mathbb{R}} \varphi\, d\gamma_1 \,=\, \int_{\mathbb{R}^N} \psi'(F)\big[1 - d^{-1}\Gamma(F)\big] d\gamma_N.$$

Together with the Cauchy-Schwarz inequality,

$$\bigg| \int_{\mathbb{R}} \varphi\, d\nu - \int_{\mathbb{R}} \varphi\, d\gamma_1 \bigg| \leq \bigg(\int_{\mathbb{R}^N} \psi'(F)^2 d\gamma_N \bigg)^{1/2} \bigg(\int_{\mathbb{R}^N} \big[1 - d^{-1}\Gamma(F)\big]^2 d\gamma_N \bigg)^{1/2}$$

which amounts to the claim. □

3 The second step: chaos decomposition

The preceding step indicates that in order to tackle the Fourth-Moment Theorem it is enough to show that, for an eigenfunction F, $\Gamma(F)$ is close to its mean under a condition on the (second and) fourth moment of F. The next statement achieves this goal for the specific homogeneous polynomials (1) as considered in the Nualart-Peccati theorem. The proof is thus taken from [1].

Proposition 2. *Let F be an homogeneous polynomial of degree d as in* (1)*. Then*

$$\int_{\mathbb{R}^N} \big[\Gamma(F) - d\big]^2 d\gamma_N \,\leq\, d^2 \bigg(\frac{1}{3} \int_{\mathbb{R}^N} F^4 d\gamma_N - 2 \int_{\mathbb{R}^N} F^2 d\gamma_N + 1 \bigg).$$

Proof. A first simple, but powerful, observation is the spectral representation of the Gamma operator for an eigenfunction F of $-\mathrm{L}$ with eigenvalue d (i.e. $-\mathrm{L}F = dF$) expressed as

$$\Gamma(F) \,=\, \frac{1}{2}\,\mathrm{L}(F^2) - F\,\mathrm{L}F \,=\, \frac{1}{2}\,\mathrm{L}(F^2) + d\,F^2 \,=\, \frac{1}{2}\,(\mathrm{L} + 2d\,\mathrm{Id})(F^2). \tag{5}$$

In particular $\int_{\mathbb{R}^N} \Gamma(F) d\gamma_N = d \int_{\mathbb{R}^N} F^2 d\gamma_N$.

From (5), the clever argument from [1] relies on the decomposition

$$\begin{aligned}\int_{\mathbb{R}^N} \big[\Gamma(F) - d\big]^2 d\gamma_N &= \frac{1}{4}\int_{\mathbb{R}^N} \big[(\mathrm{L} + 2d\,\mathrm{Id})(F^2 - 1)\big]^2 d\gamma_N \\ &= \frac{1}{4}\int_{\mathbb{R}^N} \mathrm{L}(F^2 - 1)(\mathrm{L} + 2d\,\mathrm{Id})(F^2 - 1) d\gamma_N \\ &\quad + \frac{d}{2}\int_{\mathbb{R}^N} (F^2 - 1)(\mathrm{L} + 2d\,\mathrm{Id})(F^2 - 1) d\gamma_N.\end{aligned} \tag{6}$$

Now, as follows from the expression (1) of F, F^2 may be represented as an orthogonal sum

$$F^2 = \sum_{k=0}^{2d} F_k$$

of chaos elements of degree less than or equal to $2d$. Indeed,

$$F^2 = \sum_{i_1,\dots,i_d=1; j_1,\dots,j_d=1}^{N} a_{i_1,\dots,i_k}\, a_{j_1,\dots,j_k}\, x_{i_1} \cdots x_{i_d}\, x_{j_1} \cdots x_{j_d},$$

and if $x_{i_k} = x_{j_\ell}$ for some k, ℓ, since $h_2(x) = \frac{1}{\sqrt{2}}(x^2 - 1)$, the sum decomposes into 2 orthogonal elements of degree less than or equal to $2d$. Using that each F_k is an eigenfunction of $-\mathrm{L}$ with eigenvalue k, $0 \le k \le 2d$,

$$\begin{aligned}&\int_{\mathbb{R}^N} \mathrm{L}(F^2)(\mathrm{L}{+}2d\,\mathrm{Id})(F^2 - 1) d\gamma_N \\ &\qquad = \int_{\mathbb{R}^N} \mathrm{L}(F^2)\big[\mathrm{L}(F^2) + 2dF^2 - 2d)\big] d\gamma_N \\ &\qquad = \int_{\mathbb{R}^N} \bigg(\sum_{k=0}^{2d}(-k)F_k\bigg)\bigg(\sum_{k=0}^{2d}(-k + 2d)F_k - 2d\bigg) d\gamma_N \\ &\qquad = -\sum_{k=0}^{2d}(2d - k)k \int_{\mathbb{R}^N} F_k^2 d\gamma_N \\ &\qquad \le 0.\end{aligned}$$

Hence from (6), as a first step

$$\int_{\mathbb{R}^N} \big[\Gamma(F) - d\big]^2 d\gamma_N \le \frac{d}{2}\int_{\mathbb{R}^N} (F^2 - 1)(\mathrm{L} + 2d\,\mathrm{Id})(F^2 - 1) d\gamma_N. \tag{7}$$

Now, using (5) backwards,

$$\begin{aligned}
\int_{\mathbb{R}^N} (F^2-1)(\mathrm{L}{+}2d\,\mathrm{Id})(F^2-1)d\gamma_N \\
&= 2\int_{\mathbb{R}^N} (F^2-1)\big[\Gamma(F)-d\big]\,d\gamma_N \\
&= 2\int_{\mathbb{R}^N} F^2\,\Gamma(F)d\gamma_N - 2d\int_{\mathbb{R}^N} F^2 d\gamma_N - 2\int_{\mathbb{R}^N}\Gamma(F)d\gamma_N + 2d \\
&= 2\int_{\mathbb{R}^N} F^2\,\Gamma(F)d\gamma_N - 4d\int_{\mathbb{R}^N} F^2 d\gamma_N + 2d
\end{aligned}$$

since $\int_{\mathbb{R}^N}\Gamma(F)d\gamma_N = d\int_{\mathbb{R}^N} F^2 d\gamma_N$. By the diffusion property (3), the integration by parts formula (2), and the eigenfunction property (4),

$$3\int_{\mathbb{R}^N} F^2\,\Gamma(F)d\gamma_N \,=\, \int_{\mathbb{R}^N}\Gamma\Big(F^3,F)d\gamma_N \,=\, \int_{\mathbb{R}^N} F^3(-\mathrm{L}F)d\gamma_N \,=\, d\int_{\mathbb{R}^N} F^4 d\gamma_N$$

so that

$$\int_{\mathbb{R}^N}(F^2-1)(\mathrm{L}+2d\,\mathrm{Id})(F^2-1)d\gamma_N \,=\, \frac{2d}{3}\int_{\mathbb{R}^N} F^4 d\gamma_N - 4d\int_{\mathbb{R}^N} F^2 d\gamma_N + 2d.$$

By (7), the inequality of Proposition 2 follows. □

Proposition 1 and 2 then immediately lead to the bound, for ν the distribution of F under γ_N,

$$\bigg|\int_{\mathbb{R}}\varphi\,d\nu - \int_{\mathbb{R}}\varphi\,d\gamma_1\bigg| \,\leq\, C_\varphi\, d\left(\frac{1}{3}\int_{\mathbb{R}^N} F^4 d\gamma_N - 2\int_{\mathbb{R}^N} F^2 d\gamma_N + 1\right)^{1/2} \tag{8}$$

for any smooth function $\varphi:\mathbb{R}\to\mathbb{R}$. From this family of inequalities, the Nualart-Peccati Fourth-Moment theorem immediately follows.

While the preceding simple proof is outlined in a finite dimensional setting, the principle behind it may be extended to an infinite dimensional Wiener chaos framework along the corresponding infinite dimensional spectral analysis, or a direct finite dimensional approximation procedure on the dimension free inequality (8).

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