The Fourth-Moment Theorem

By the classical moment theorem, a sequence of random variables $(F_n)_{n\in\mathbb{N}}$ on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$, having moments of all orders, converges in distribution to a standard normal variable Z if (and only if)

$$\mathbb{E}\big(F_n^p\big) \to \mathbb{E}\big(Z^p\big)$$

for every integer p.

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent standard normal variables. For a fixed integer $d \geq 1$, consider

$$F_n = \sum_{i_1,\dots,i_d=1}^{N_n} a_{i_1,\dots,i_k} X_{i_1} \cdots X_{i_d},$$

where $a_{i_1,\dots,i_d}$ are real numbers vanishing on diagonals and symmetric in the indices, and $(N_n)_{n\in\mathbb{N}}$ is a sequence of integers strictly increasing to infinity as $n \to \infty$. In a striking contribution, D. Nualart and G. Peccati showed in 2005 [13] that in order for such a sequence $(F_n)_{n\in\mathbb{N}}$ to converge weakly to a standard normal Z, it is actually enough that the first 2 moments

$$\mathbb{E}\big(F_n^2\big) \to 1 \qquad \text{and} \qquad \mathbb{E}\big(F_n^4\big) \to 3$$

converge to the second and fourth moments of the standard normal. (Equivalently, the F_n's may be normalized in $\mathrm{L}^2(\mathbb{P})$ so that the fourth moment characterizes the weak convergence.)

The analysis developed in [13] actually holds for homogeneous chaos (multiple stochastic integrals) on the infinite dimensional Wiener space, with a proof relying on multiplication formulas for homogeneous chaos and stochastic calculus. This post exposes a (short and simple) proof of this result following [4] (see also [6]) based on a spectral analysis of the

Drafted by M. L. **v1** April 2024

Ornstein-Uhlenbeck operator applied to such homogeneous polynomials. As in some of the alternate proofs (see e.g. [12, 7, 9, 10]...), the first step of the argument makes use of a version of Stein's inequality in this context (cf. [3]).

Table of contents

1 Homogeneous polynomials and a target inequality

Let $d\gamma_N(x) = \frac{1}{(2\pi)^{N/2}}\, e^{-|x|^2/2}dx$ on $\mathbb{R}^N$ be the standard Gaussian measure on $\mathbb{R}^N$. For a fixed $1 \leq d \leq N$, let $F : \mathbb{R}^N \to \mathbb{R}$ be defined by

$$F \,=\, F(x) \,=\, \sum_{i_1,\ldots,i_d=1}^{N} a_{i_1,\ldots,i_k}\, x_{i_1} \cdots x_{i_d}, \quad x = (x_1, \ldots, x_N) \in \mathbb{R}^N, \tag{1}$$

where $a_{i_1,\ldots,i_d}$ are real numbers vanishing on diagonals and symmetric in the indices. Such a function F will be called homogeneous of degree d. The Fourth-Moment Theorem is thus reformulated equivalently as the distributional convergence of F_n on $\mathbb{R}^{N_n}$, $n \in \mathbb{N}$, $N_n \to \infty$, under γ_{N_n}, towards the standard Gaussian distribution on the real line if and only if

$$\int_{\mathbb{R}^{N_n}} F_n^2\, d\gamma_{N_n} \to\, 1 \qquad \text{and} \qquad \int_{\mathbb{R}^{N_n}} F_n^4\, d\gamma_{N_n} \to\, 3.$$

With respect to the introduction, the proof developed below is thus presented in the context of functions with respect to the standard Gaussian measure γ_N, and emphasizes more precisely inequalities for each fixed N. The target will actually to prove that, for ν the distribution of F under γ_N,

$$\bigg| \int_{\mathbb{R}} \varphi\, d\nu - \int_{\mathbb{R}} \varphi\, d\gamma_1 \bigg| \,\leq\, C_\varphi\, d \left(\frac{1}{3} \int_{\mathbb{R}^N} F^4 d\gamma_N - 2 \int_{\mathbb{R}^N} F^2 d\gamma_N + 1 \right)^{1/2} \tag{2}$$

for any smooth function $\varphi : \mathbb{R} \to \mathbb{R}$, with $C_\varphi > 0$ only depending on φ. From this family of inequalities, the Fourth-Moment Theorem immediately follows.

While the proof is outlined below in a finite dimensional setting, the principle behind it may be extended to an infinite dimensional Wiener chaos framework (cf. [9]) along the corresponding infinite dimensional spectral analysis, or a direct finite dimensional approximation procedure on the dimension free inequality (2).

2 Reminders on the Ornstein-Uhlenbeck operator and Hermite polynomials

This section briefly brings together elements from the posts [1] and [2] (see also [11, 9, 5]...).

The Ornstein-Uhlenbeck operator is acting on smooth functions $f : \mathbb{R}^N \to \mathbb{R}$ as

$$\mathrm{L}f \,=\, \Delta f - x \cdot \nabla f.$$

The basic integration by parts formula with respect to γ_N expresses that for every smooth functions $f, g : \mathbb{R}^N \to \mathbb{R}$,

$$\int_{\mathbb{R}^N} f(-\mathrm{L}g) d\gamma_N \,=\, \int_{\mathbb{R}^N} \nabla f \cdot \nabla g \, d\gamma_N. \tag{3}$$

According to the integration by parts formula (3), the Ornstein-Uhlenbeck operator L has γ_N as invariant and reversible probability measure, and generates the symmetric bilinear carré du champ operator

$$\Gamma(f, g) \,=\, \nabla f \cdot \nabla g \,=\, \frac{1}{2}\big[\mathrm{L}(fg) - f\,\mathrm{L}g - g\,\mathrm{L}f\big]$$

acting on functions f, g in a suitable domain $\mathcal{A}$ of smooth functions on $\mathbb{R}^N$. For simplicity in the notation below, set $\Gamma(f) = \Gamma(f, f) = |\nabla f|^2$ which is always non-negative. The operator L is a diffusion operator in the sense that for every smooth function $\varphi : \mathbb{R} \to \mathbb{R}$, and every $f \in \mathcal{A}$,

$$\mathrm{L}\varphi(f) \,=\, \varphi'(f)\,\mathrm{L}f + \varphi''(f)\,\Gamma(f). \tag{4}$$

Alternatively, Γ is a derivation in the sense that $\Gamma(\varphi(f), g) = \varphi'(f)\,\Gamma(f, g)$.

The spectrum of the operator $-\mathrm{L}$ is $\mathbb{N}$, with eigenfunctions given by the Hermite polynomials. For every $\lambda \in \mathbb{R}$, the expansion in $x \in \mathbb{R}$,

$$e^{\lambda x - \frac{1}{2}\lambda^2} \,=\, \sum_{k=0}^{\infty} \frac{\lambda^k}{\sqrt{k!}}\, h_k(x)$$

defines the one-dimensional Hermite polynomials h_k, $k \in \mathbb{R}$. For every $k \in \mathbb{N}$, h_k is a polynomial of degree k. For example $h_0(x) = 1$, $h_1(x) = x$, $h_2(x) = \frac{1}{\sqrt{2}}(x^2 - 1)$. It is easily checked, on the generating series for example, that $\mathrm{L}h_k = -k\, h_k$, $k \in \mathbb{N}$. The Hermite polynomials are orthogonal in $\mathrm{L}^2(\gamma_1)$, orthonormal in the preceding formulation, and the complete system h_k, $k \in \mathbb{N}$, therefore defines an orthonormal basis (of eigenvectors of L) of $\mathrm{L}^2(\gamma_1)$.

Multi-dimensional Hermite polynomials on $\mathbb{R}^N$ are defined as products of one-dimensional polynomials with multi-index. Namely, for $\underline{k} = (k_1, \ldots, k_N) \in \mathbb{N}^N$ and $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$,

$$H_{\underline{k}}(x) \,=\, h_{k_1}(x_1) \cdots h_{k_N}(x_N).$$

For every $\underline{k} = (k_1, \ldots, k_N) \in \mathbb{N}^N$, $\mathrm{L}H_{\underline{k}} \,=\, -k\, H_{\underline{k}}$ with $k = k_1 + \cdots + k_N$, $\underline{k} = (k_1, \ldots, k_N)$. As in the real case, the family $H_{\underline{k}}$, $\underline{k} \in \mathbb{N}^N$, defines an orthonormal basis of the Hilbert space $\mathrm{L}^2(\gamma_N)$. The so-called orthogonal chaos decomposition (Fourier-Hermite expansion) of a function in $\mathrm{L}^2(\gamma_N)$ takes the form

$$f \,=\, \sum_{k \in \mathbb{N}^N} f_{\underline{k}}\, H_{\underline{k}} \,=\, \sum_{k=0}^{\infty} \bigg(\sum_{|\underline{k}|=k} f_{\underline{k}}\, H_{\underline{k}} \bigg)$$

where the $f_{\underline{k}}$'s are real numbers, and $|\underline{k}| = k_1 + \cdots + k_N$. The sums under parentheses actually represents the so-called homogeneous Wiener chaos of order k, which may be considered more generally.

Since $h_1(x) = x$, $x \in \mathbb{R}$, the homogeneous polynomial F from (1) is an eigenfunction of order d of L, that is

$$-\mathrm{L}F \,=\, dF. \tag{5}$$

3 The first step: Stein's lemma

The following statement from [8, 9] presents the application of Stein's method in the context of an eigenfunction of the Ornstein-Uhlenbeck operator. The outcome is that, for an eigenfunction F of $-\mathrm{L}$ with eigenvalue $d \geq 1$, the proximity of $\Gamma(F) = |\nabla F|^2$ to a constant forces the law of F under γ_N to be close to the (one-dimensional) standard normal. The underlying principle relies on the chain rule formula for the diffusion operator L,

$$\mathrm{L}(\varphi \circ F) \,=\, \varphi'(F)\, \mathrm{L}F + \varphi''(F)\, \Gamma(F) \,=\, -dF \varphi'(F) + \varphi''(F)\, \Gamma(F)$$

along a smooth function $\varphi : \mathbb{R} \to \mathbb{R}$. Therefore, if $\Gamma(F) = d$, $\mathrm{L}(\varphi \circ F) = d\, (\mathrm{L}_1 \varphi)(F)$ where L_1 is the one-dimensional Ornstein-Uhlenbeck operator with invariant measure γ_1, and thus the law of F is γ_1.

Proposition 1. *Let F be an eigenfunction of $-\mathrm{L}$ with eigenvalue $d \geq 1$. Denote by ν the distribution of F under γ_N. Given $\varphi : \mathbb{R} \to \mathbb{R}$ integrable with respect to ν and γ_1, let ψ be a smooth solution of the associated Stein equation $\varphi - \int_{\mathbb{R}} \varphi \, d\gamma_1 = \psi' - x\psi$. Then,*

$$\left| \int_{\mathbb{R}} \varphi \, d\nu - \int_{\mathbb{R}} \varphi \, d\gamma_1 \right| \leq \frac{C_\varphi}{d} \bigg(\int_{\mathbb{R}^N} \big[\Gamma(F) - d \big]^2 d\gamma_N \bigg)^{1/2}$$

where $C_\varphi = \|\psi'\|_\infty^2$.

Proof. Since ν is the distribution of F under γ_N, by the Stein equation,

$$\int_{\mathbb{R}} \varphi \, d\nu - \int_{\mathbb{R}} \varphi \, d\gamma_1 \, = \, \int_{\mathbb{R}^N} \varphi(F) d\gamma_N - \int_{\mathbb{R}} \varphi \, d\gamma_1 \, = \, \int_{\mathbb{R}^N} \big[\psi'(F) - F\psi(F) \big] d\gamma_N .$$

Now $-\mathrm{L}F = dF$ so that

$$\psi'(F) - F\psi(F) \, = \, \psi'(F) + d^{-1} \, \mathrm{L}F \, \psi(F)$$

and hence, after integration by parts with respect to the operator L (3) and the use of the diffusion property (4),

$$\int_{\mathbb{R}} \varphi \, d\nu - \int_{\mathbb{R}} \varphi \, d\gamma_1 \, = \, \int_{\mathbb{R}^N} \psi'(F) \big[1 - d^{-1} \Gamma(F) \big] d\gamma_N .$$

Together with the Cauchy-Schwarz inequality,

$$\left| \int_{\mathbb{R}} \varphi \, d\nu - \int_{\mathbb{R}} \varphi \, d\gamma_1 \right| \leq \bigg(\int_{\mathbb{R}^N} \psi'(F)^2 d\gamma_N \bigg)^{1/2} \bigg(\int_{\mathbb{R}^N} \big[1 - d^{-1} \Gamma(F) \big]^2 d\gamma_N \bigg)^{1/2}$$

which amounts to the claim. □

4 The second step: chaos decomposition

The preceding step indicates that it is enough to show that, for an eigenfunction F, $\Gamma(F) = |\nabla F|^2$ is close to its mean under a condition on the fourth moment of F. The next statement achieves this goal for the specific homogeneous polynomials (1). Together with Proposition 1, this result yields the announced claim (2). The proof of the proposition below is thus taken from [4].

Proposition 2. *Let F be an homogeneous polynomial of degree d as in* (1)*. Then*

$$\int_{\mathbb{R}^N} \big[\Gamma(F) - d \big]^2 d\gamma_N \, \leq \, d^2 \bigg(\frac{1}{3} \int_{\mathbb{R}^N} F^4 d\gamma_N - 2 \int_{\mathbb{R}^N} F^2 d\gamma_N + 1 \bigg).$$

Proof. As a first simple, but powerful, observation,

$$\Gamma(F) \;=\; \frac{1}{2}\,\mathrm{L}(F^2) - F\,\mathrm{L}F \;=\; \frac{1}{2}\,\mathrm{L}(F^2) + d\,F^2 \;=\; \frac{1}{2}\,(\mathrm{L} + 2d\,\mathrm{Id})(F^2)$$

since F is an eigenfunction of $-\mathrm{L}$ with eigenvalue $-d$, i.e. $-\mathrm{L}F = dF$. In particular

$$\Gamma(F) - dF^2 \;=\; \frac{1}{2}\,\mathrm{L}(F^2) \qquad \text{and} \qquad \Gamma(F) - d \;=\; (\mathrm{L} + 2d\,\mathrm{Id})(F^2 - 1), \tag{6}$$

and $\int_{\mathbb{R}^N} \Gamma(F) d\gamma_N = d \int_{\mathbb{R}^N} F^2 d\gamma_N$.

As follows from the expression (1) of F, F^2 may be represented as an orthogonal sum

$$F^2 \;=\; \sum_{k=0}^{2d} F_k$$

of chaos elements of degree less than or equal to $2d$. Indeed,

$$F^2 \;=\; \sum_{i_1,\ldots,i_d=1;j_1,\ldots,j_d}^{N} a_{i_1,\ldots,i_k}\, a_{j_1,\ldots,j_k}\, x_{i_1} \cdots x_{i_d}\, x_{j_1} \cdots x_{j_d},$$

and if $x_{i_k} = x_{j_\ell}$, since $h_2(x) = \frac{1}{\sqrt{2}}(x^2 - 1)$, the sum decomposes into 2 orthogonal elements of degree less than or equal to $2d$. Using that each F_k is an eigenfunction of $-\mathrm{L}$ with eigenvalue k, $0 \le k \le 2d$,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \mathrm{L}(F^2)(\mathrm{L}{+}2d\,\mathrm{Id})(F^2 - 1) d\gamma_N \\
&\qquad = \int_{\mathbb{R}^N} \mathrm{L}(F^2)\big[\mathrm{L}(F^2) + 2dF^2 - 2d)\big] d\gamma_N \\
&\qquad = \int_{\mathbb{R}^N} \bigg(\sum_{k=0}^{2d} (-k) F_k\bigg)\bigg(\sum_{k=0}^{2d} (-k + 2d) F_k - 2d\bigg) d\gamma_N \\
&\qquad = -\sum_{k=0}^{2d} (2d - k) k \int_{\mathbb{R}^N} F_k^2 d\gamma_N \\
&\qquad \le\; 0.
\end{aligned}$$

Hence, by (6),

$$\int_{\mathbb{R}^N} \big[\Gamma(F) - dF^2\big]\big[\Gamma(F) - d\big] d\gamma_N \;\le\; 0.$$

As a consequence,

$$\begin{aligned}
\int_{\mathbb{R}^N} \big[\Gamma(F) - d\big]^2 d\gamma_N \;&=\; \int_{\mathbb{R}^N} \Big[\Gamma(F) - dF^2\Big]\big[\Gamma(F) - d\big] d\gamma_N \\
&\qquad + \int_{\mathbb{R}^N} \big[dF^2 - d\big]\big[\Gamma(F) - d\big] d\gamma_N \\
&\le\; \int_{\mathbb{R}^N} \big[dF^2 - d\big]\big[\Gamma(F) - d\big] d\gamma_N.
\end{aligned} \tag{7}$$

Now, since $\int_{\mathbb{R}^N} \Gamma(F) d\gamma_N = d \int_{\mathbb{R}^N} F^2 d\gamma_N$,

$$\begin{aligned}
\int_{\mathbb{R}^N} \big[dF^2 - d\big]\big[\Gamma(F) - d\big] d\gamma_N \\
&= d \int_{\mathbb{R}^N} F^2\, \Gamma(F) d\gamma_N - d^2 \int_{\mathbb{R}^N} F^2 d\gamma_N - d \int_{\mathbb{R}^N} \Gamma(F) d\gamma_N + d^2 \\
&= d \int_{\mathbb{R}^N} F^2\, \Gamma(F) d\gamma_N - 2d^2 \int_{\mathbb{R}^N} F^2 d\gamma_N + d^2.
\end{aligned}$$

By the diffusion property (4), the integration by parts formula (3), and the eigenfunction property (5),

$$3 \int_{\mathbb{R}^N} F^2\, \Gamma(F) d\gamma_N \, = \int_{\mathbb{R}^N} \Gamma\Big(F^3, F\Big) d\gamma_N \, = \int_{\mathbb{R}^N} F^3 (-\mathrm{L}F) d\gamma_N \, = \, d \int_{\mathbb{R}^N} F^4 d\gamma_N$$

so that

$$\int_{\mathbb{R}^N} \big[dF^2 - d\big]\big[\Gamma(F) - d\big] d\gamma_N \, = \, \frac{d^2}{3} \int_{\mathbb{R}^N} F^4 d\gamma_N - 2d^2 \int_{\mathbb{R}^N} F^2 d\gamma_N + d^2.$$

Together with (7), the inequality of Proposition 2 follows. □

References

[1] Mehler kernel, and the Ornstein-Uhlenbeck operator. *The Gaussian Blog.*

[2] Hermite polynomials. *The Gaussian Blog.*

[3] Stein's method for normal approximation. *The Gaussian Blog.*

[4] E. Azmoodeh, S. Campese, G. Poly. Fourth Moment Theorems for Markov diffusion generators. *J. Funct. Anal.* 266, 2341–2359 (2014).

[5] D. Bakry, I. Gentil, M. Ledoux. *Analysis and geometry of Markov diffusion operators.* Grundlehren der mathematischen Wissenschaften 348. Springer (2014).

[6] M. Ledoux, Chaos of a Markov operator and the fourth moment condition. *Ann. Probab.* 40, 2439–2459 (2012).

[7] I. Nourdin. Yet another proof of the Nualart–Peccati criterion. *Electron. Commun. Probab.* 16, 467–481 (2011).

[8] I. Nourdin, G. Peccati. Stein's method on Wiener chaos. *Probab. Theor. Rel. Fields* 145, 75–118 (2009).

[9] I. Nourdin, G. Peccati. *Normal approximations with Malliavin calculus: from Stein's method to universality*. Cambridge University Press (2012).

[10] I. Nourdin, J. Rosinski. Asymptotic independence of multiple Wiener-Itô integrals and the resulting limit laws. *Ann. Probab.* 42, 497–526 (2014).

[11] D. Nualart. *The Malliavin calculus and related topics.* Second Edition. Springer (2006).

[12] D. Nualart, S. Ortiz-Latorre. Central limit theorems for multiple stochastic integrals and Malliavin calculus. *Stochastic Process. Appl.* 118, 614–628 (2008).

[13] D. Nualart, G. Peccati. Central limit theorems for sequences of multiple stochastic integrals. *Ann. Probab.* 33, 177–193 (2005).