A proof of the
Law of the Iterated Logarithm

The Law of the Iterated Logarithm (LIL)is the third fundamental limit law of probability theory, after the Law of Large Numbers and the Central Limit Theorem.

The LIL describes, with a surprising accuracy, the almost sure behavior closest to the weak convergence of the Central Limit Theorem. The iterated logarithm is due to the conjonction of a block-type argument along geometric sequences and exponential inequalities of Gaussian type for sums of independent random variables.

Let X be a random variable on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and let $(X_n)_{n\geq 1}$ be a sequence of independent copies of X (on $(\Omega, \mathcal{A}, \mathbb{P})$); set $S_n = X_1 + \cdots + X_n$, $n \geq 1$. The standard Law of Large Numbers expresses that if X is integrable,

$$\lim_{n\to\infty} \frac{S_n}{n} = \mathbb{E}(X) \qquad \text{almost surely,}$$

while the Central Limit Theorem quantifies the fluctuations in the form

$$\lim_{n\to\infty} \frac{S_n}{\sqrt{n}} = G \qquad \text{in distribution}$$

whenever $\sigma^2 = \mathbb{E}(X^2) < \infty$ and $\mathbb{E}(X) = 0$, where G is a normal random variable with mean zero and variance σ^2.

Interpolating the preceding behaviors, the LIL expresses in its classical standard form that, if $\sigma = \sqrt{\mathbb{E}(X^2)} < \infty$ and $\mathbb{E}(X) = 0$,

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log(n)}} = \sigma \qquad \text{almost surely.} \tag{1}$$

The sequence $\sqrt{2n\log\log(n)} = \sqrt{2n\log(\log(n))}$ is rigorously defined only for $n \geq 3$, but to ease the notation, it will be written that $\log\log(n) = 1$ for $n = 1, 2$.

With $-X$ instead of X, it is also true that

$$\liminf_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log(n)}} = -\sigma \qquad \text{almost surely.} \tag{2}$$

As a consequence of (1) and (2),

$$\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2n\log\log(n)}} = \sigma \qquad \text{almost surely,} \tag{3}$$

and the sequence $\left(\frac{S_n}{\sqrt{2n\log\log(n)}}\right)_{n\geq 1}$ is in particular almost surely bounded.

This statement has been established by P. Hartman and A. Wintner [4] after the founding works by A. Khintchine [6] and A. Kolmogorov [7], surprisingly technical and modern for their time. Most of the more recent proofs, including this one, actually essentially follow the same pattern.

A more precise form, due to V. Strassen [10], describes that

$$\lim_{n\to\infty} d\left(\frac{S_n}{\sqrt{2n\log\log(n)}}, [-\sigma, +\sigma]\right) = 0 \tag{4}$$

and

$$C\left(\left(\frac{S_n}{\sqrt{2n\log\log(n)}}\right)_{n\geq 1}\right) = [-\sigma, +\sigma] \tag{5}$$

almost surely. In (4), $d(\cdot, [-\sigma, +\sigma]) = \inf_{a\in[-\sigma,+\sigma]} |\cdot - a|$ is the distance to the set $[-\sigma, +\sigma]$, and in (5), for a sequence $(a_n)_{n\geq 1}$ of real numbers, $C((a_n)_{n\geq 1})$ is the cluster set of the limiting points of the sequence. V. Strassen establishes this result by embedding a sequence of independent identically distributed random variables into the Brownian trajectories and a version of the LIL for this specific Gaussian process.

Property (4) is actually an immediate consequence of (3) (even only the upper bound on the limsup, in the form of (6) below, is enough). If (1) and (2) ensure that $+\sigma$ et $-\sigma$ are almost sure limit points of the sequence $\left(\frac{S_n}{\sqrt{2n\log\log(n)}}\right)_{n\geq 1}$, the strength of (5) expresses that all points of the interval $[-\sigma, +\sigma]$ are also in the cluster set.

The aim of this note is to present a simple and complete proof of (1) and (5). The various arguments are drawn from the classical proofs of the LIL, for example from the books and articles [7, 5, 9, 8, 2, 1] among (many) others. The proof, while not difficult, requires nevertheless some care and precision.

The first sections are devoted to the proof of (1), splitted into upper and lower bounds. The last section establishes Strassen's form of the LIL via the multi-dimensional extension.

By homogeneity (work with $\frac{X}{\sigma}$), it may be supposed that $\sigma = 1$ (the case $\sigma = 0$ is of no interest).

1 Proof of the upper bound in (1)

The purpose of this section is to establish that, for X centered with variance 1,

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log(n)}} \leq 1 \qquad \text{almost surely.} \tag{6}$$

The proof relies on the application of the Borel-Cantelli lemma via a block decomposition along geometric sequences and an exponential inequality for sums of independent random variables.

1.1 Truncation and Borel-Cantelli lemma

The principle of the proof couples a truncation argument and the Borel-Cantelli lemma. The result (6) will be achieved whenever for all $\varepsilon > 0$ and $\rho > 1$,

$$\sum_{\ell=1}^{\infty} \mathbb{P}\Big(\max_{1\leq n\leq n_\ell} T_n^\varepsilon \geq (1+2\varepsilon)\sqrt{2n_\ell\log\log(n_\ell)}\Big) < \infty \tag{7}$$

with n_ℓ the integer part of ρ^ℓ, $\ell \geq 0$, and

$$\lim_{n\to\infty} \frac{S_n - T_n^\varepsilon}{\sqrt{2n\log\log(n)}} = 0 \qquad \text{almost surely,} \tag{8}$$

where

$$T_n^\varepsilon = Y_1^\varepsilon + \cdots + Y_n^\varepsilon, \quad n \geq 1,$$

and

$$Y_k^\varepsilon = X_k \mathbb{1}_{\{|X_k|\leq c\sqrt{k/\log\log(k)}\}} - \mathbb{E}\Big(X_k \mathbb{1}_{\{|X_k|\leq c\sqrt{k/\log\log(k)}\}}\Big), \quad k \geq 1,$$

where $c = c(\varepsilon) > 0$ is to be specified later on in the proof (see the condition (10)).

Indeed, by the Borel-Cantelli lemma, the convergence of the series in (7) ensures that for almost every $\omega \in \Omega$, starting from some integer $\ell_0 = \ell_0(\omega)$, for all $\ell \geq \ell_0$,

$$\max_{1\leq n\leq n_\ell} T_n^\varepsilon(\omega) \leq (1+2\varepsilon)\sqrt{2n_\ell\log\log(n_\ell)}.$$

Hence

$$\max_{n_{\ell-1}<n\leq n_\ell} \frac{T_n^\varepsilon(\omega)}{\sqrt{2n\log\log(n)}} \leq (1+2\varepsilon)\rho$$

since $\sqrt{2n_\ell\log\log(n_\ell)} \leq \rho\sqrt{2n\log\log(n)}$ when $n_{\ell-1} < n \leq n_\ell$, at least for ℓ large enough, so that, almost surely,

$$\limsup_{n\to\infty} \frac{T_n^\varepsilon}{\sqrt{2n\log\log(n)}} \leq (1+2\varepsilon)\rho.$$

By (8), it also holds true that

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log(n)}} \leq (1+2\varepsilon)\rho \qquad \text{almost surely.}$$

Choosing ε et ρ of the form $\frac{1}{p}$ et $1+\frac{1}{q}$ where p et q are integers so to ensure a countable union of negligible sets, property (6) will be established.

1.2 Use of the Kronecker lemma

This section establishes the convergence (8) by the classical Kronecker lemma. The latter will ensure the conclusion whenever the series $\sum_{k=1}^{\infty} \frac{X_k - Y_k^\varepsilon}{\sqrt{2k\log\log(k)}} < \infty$ converges almost surely. By definition of the Y_k^ε's and centering of the X_k's, this will be the case if

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{2k\log\log(k)}}\, \mathbb{E}\Big(|X_k|\, \mathbb{1}_{\{|X_k|>c\sqrt{k/\log\log(k)}\}}\Big) < \infty.$$

By equidistribution of the X_k's and the Fubini-Tonelli theorem,

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{2k\log\log(k)}}\, \mathbb{E}\Big(|X_k|\, \mathbb{1}_{\{|X_k|>c\sqrt{k/\log\log(k)}\}}\Big) = \mathbb{E}\bigg(|X| \sum_{k=1}^{\infty} \frac{1}{\sqrt{2k\log\log(k)}}\, \mathbb{1}_{\{|X|>c\sqrt{k/\log\log(k)}\}}\bigg).$$

An easy upper bound shows that for $N \geq 1$,

$$\sum_{k=1}^{N} \frac{1}{\sqrt{2k\log\log(k)}} \leq C\sqrt{\frac{N}{\log\log(N)}}$$

for some numerical constant $C > 0$. Since $|X| > c\sqrt{k/\log\log(k)}$ amounts to say that $k < c' X^2 \log\log(X^2)$ for some c' only depending on c, for N of the order of $X^2\log\log(X^2)$, it follows that

$$\mathbb{E}\bigg(|X| \sum_{k=1}^{\infty} \frac{1}{\sqrt{2k\log\log(k)}}\, \mathbb{1}_{\{|X|>c\sqrt{k/\log\log(k)}\}}\bigg) \leq C'\, \mathbb{E}(X^2) < \infty.$$

The announced conclusion is established.

1.3 A maximal inequality

In order to handle (7), it is necessary to get rid of the maximum in the probability by means of a maximal inequality. Several tools and inequalities are available, for example the classical Ottaviani inequality.

Lemma 1. *Let $Y_1, \dots, Y_n$ be independent random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and set $T_k = Y_1 + \cdots + Y_k$, $k = 1, \dots, n$. For $s, t > 0$,*

$$\mathbb{P}(T_n \geq t) \;\geq\; \min_{1 \leq k \leq n} \mathbb{P}\big(|T_n - T_k| < s\big)\, \mathbb{P}\big(\max_{1 \leq k \leq n} T_k \geq s + t\big).$$

Proof. Consider the subsets in $\mathcal{A}$ defined by

$$A_k \;=\; \big\{T_1 < s+t, \dots, T_{k-1} < s+t, T_k \geq s+t\big\}, \quad k = 1, \dots, n,$$

which form a partition of $\{\max_{1 \leq k \leq n} T_k \geq s + t\}$. Then

$$\mathbb{P}\big(T_n \geq t\big) \;\geq\; \sum_{k=1}^{n} \mathbb{P}\big(T_n \geq t, A_k\big) \;\geq\; \sum_{k=1}^{n} \mathbb{P}\big(|T_n - T_k| < s, A_k\big)$$

since $T_n = (T_n - T_k) + T_k$ and $T_k \geq s + t$ on A_k. By independence, the measurable set A_k only depending on the variables $Y_1, \dots, Y_k$ and $T_n - T_k = Y_{k+1} + \cdots + Y_n$,

$$\begin{aligned}\mathbb{P}\big(T_n \geq t\big) \;&\geq\; \sum_{k=1}^{n} \mathbb{P}\big(|T_n - T_k| < s\big)\, \mathbb{P}(A_k) \\ &\geq\; \min_{1 \leq k \leq n} \mathbb{P}\big(|T_n - T_k| < s\big) \sum_{k=1}^{n} \mathbb{P}(A_k) \\ &\geq\; \min_{1 \leq k \leq n} \mathbb{P}\big(|T_n - T_k| < s\big)\, \mathbb{P}\big(\max_{1 \leq k \leq n} T_k \geq s + t\big).\end{aligned}$$

The conclusion follows. □

This lemma may be applied to the random variables $T_1^\varepsilon, \dots, T_{n_\ell}^\varepsilon$ which are involved in (7). By Chebychev's inequality, for every $n = 1, \dots, n_\ell$,

$$\mathbb{P}\big(|T_{n_\ell}^\varepsilon - T_n^\varepsilon| \geq s\big) \;\leq\; \frac{1}{s^2} \sum_{k=n+1}^{n_\ell} \mathbb{E}\big((Y_k^\varepsilon)^2\big) \;\leq\; \frac{n_\ell}{s^2}$$

since $\mathbb{E}((Y_k^\varepsilon)^2) \leq \mathbb{E}(X_k^2) = 1$. Therefore, if $s = \varepsilon \sqrt{2 n_\ell \log\log(n_\ell)}$, these probabilities will be, for example, less than or equal to $\frac{1}{2}$ for every $\ell \geq 1$ large enough so that

$$\min_{1 \leq n \leq n_\ell} \mathbb{P}\big(|T_{n_\ell}^\varepsilon - T_n^\varepsilon| < s\big) \;\geq\; \frac{1}{2}\,.$$

Hence, from the lemma, for ℓ large enough,

$$\begin{aligned}&\mathbb{P}\Big(\max_{1 \leq n \leq n_\ell} T_n^\varepsilon \geq (1 + 2\varepsilon)\sqrt{2 n_\ell \log\log(n_\ell)}\Big) \\ &\qquad\qquad \leq\; 2\, \mathbb{P}\Big(T_{n_\ell}^\varepsilon \geq (1 + \varepsilon)\sqrt{2 n_\ell \log\log(n_\ell)}\Big).\end{aligned}$$

As a conclusion, (7) holds true as soon as

$$\sum_{\ell=1}^{\infty} \mathbb{P}\Big(T_{n_\ell}^\varepsilon \geq (1 + \varepsilon)\sqrt{2 n_\ell \log\log(n_\ell)}\Big) \;<\; \infty. \tag{9}$$

1.4 Exponential inequality

The proofs of the LIL rely in essence on exponential inequalities of Gaussian type for sums of independent random variables, at the origin of the iterated logarithm. There are numerous and diverse such exponential inequalities (cf. [7, 9, 8]...). The one below, taken from [1], is easy to access.

Lemma 2. *Let $Y_1, \ldots, Y_n$ be independent centered random variables such that $|Y_k| \leq C$ almost surely for every $k = 1, \ldots, n$, where $C > 0$ is a fixed positive constant. Set $T_n = Y_1 + \cdots + Y_n$. Then, for every $\alpha^2 \geq \max_{1 \leq k \leq n} \mathbb{E}(Y_k^2)$ and $t > 0$,*

$$\mathbb{P}(T_n \geq t) \leq \exp\Big(-\frac{t^2}{2\alpha^2 n}\big[2 - e^{Ct/\alpha^2 n}\big]\Big).$$

Proof. For every $x \in \mathbb{R}$, $e^x \leq 1 + x + \frac{x^2}{2} e^{|x|}$ so that, by centering, for every $\lambda > 0$ and every $k = 1, \ldots, n$,

$$\mathbb{E}(e^{\lambda Y_k}) \leq 1 + \frac{\lambda^2}{2}\mathbb{E}\big(Y_k^2 e^{\lambda|Y_k|}\big) \leq 1 + \frac{\alpha^2\lambda^2}{2} e^{\lambda C} \leq \exp\Big(\frac{\alpha^2\lambda^2}{2} e^{\lambda C}\Big)$$

by the hypotheses. As a consequence, by independence,

$$\mathbb{E}(e^{\lambda T_n}) = \prod_{k=1}^{n} \mathbb{E}(e^{\lambda Y_k}) \leq \exp\Big(\frac{\alpha^2\lambda^2 n}{2} e^{\lambda C}\Big),$$

and by Markov's inequality

$$\mathbb{P}(T_n \geq t) \leq \exp\Big(-\lambda t + \frac{\alpha^2\lambda^2 n}{2} e^{\lambda C}\Big).$$

The choice of $\lambda = \frac{t}{\alpha^2 n}$ leads to the announced claim. □

1.5 End of the proof of (6)

On the basis of the exponential inequality of the preceding paragraph, the proof of (9) is almost immediate. The inequality of Lemma 2 applied to the sample of independent centered random variables $Y_1^\varepsilon, \ldots, Y_{n_\ell}^\varepsilon$ (for each fixed $\ell \geq 1$) for which

$$|Y_k^\varepsilon| \leq C = 2c\sqrt{\frac{n_\ell}{\log\log(n_\ell)}}, \qquad \mathbb{E}\big((Y_k^\varepsilon)^2\big) \leq \mathbb{E}(X_k^2) = 1, \quad k = 1, \ldots, n_\ell,$$

and to $t = (1+\varepsilon)\sqrt{2n_\ell \log\log(n_\ell)}$ leads to

$$\mathbb{P}\bigg(\sum_{k=1}^{n_\ell} Y_k^\varepsilon \geq (1+\varepsilon)\sqrt{2n_\ell\log\log(n_\ell)}\bigg) \leq \exp\Big(-(1+\varepsilon)^2\big[2 - e^{2\sqrt{2}c(1+\varepsilon)}\big]\log\log(n_\ell)\Big).$$

Choose then $c = c(\varepsilon) > 0$ small enough in order that

$$(1+\varepsilon)^2 \left[2 - e^{2\sqrt{2}c(1+\varepsilon)}\right] \, > \, 1, \tag{10}$$

so that the right-hand side of the previous inequality defines the general term of a convergent series in $\ell \geq 1$ since $\log\log(n_\ell) \sim \log(\ell)$.

The proof of the convergence of the series (9), and thus of the upper bound (6) in the limsup (1), is therefore completed.

2 Proof of the lower bound in (1)

The object of this section is to establish that, for X centered of variance 1,

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log(n)}} \geq 1 \qquad \text{almost surely.} \tag{11}$$

The proof rests on the Borel-Cantelli lemma in its independent part and on an moderate deviation inequality.

2.1 Use of the Borel-Cantelli lemma

It will be enough to show that for every $\varepsilon > 0$, there exists a sequence of integers $(n_\ell)_{\ell\geq 1}$ increasing to infinity such that

$$\limsup_{\ell\to\infty} \frac{S_{n_\ell}}{\sqrt{2n_\ell\log\log(n_\ell)}} \geq 1-\varepsilon \qquad \text{almost surely.} \tag{12}$$

Thus $\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log(n)}} \geq 1-\varepsilon$ almost surely, and consider next a countable union of negligible sets with $\varepsilon = \frac{1}{p}$, $p \geq 1$ integer.

If the subsequence is chosen so that $\lim_{\ell\to\infty} \frac{n_{\ell-1}}{n_\ell} = 0$, it will actually be enough to prove that

$$\limsup_{\ell\to\infty} \frac{S_{n_\ell} - S_{n_{\ell-1}}}{\sqrt{2n_\ell\log\log(n_\ell)}} \geq 1-\varepsilon \qquad \text{almost surely} \tag{13}$$

since according to (6) the sequence $\left(\frac{S_{n_{\ell-1}}}{\sqrt{2n_{\ell-1}\log\log(n_{\ell-1})}}\right)_{\ell\geq 1}$ is almost surely bounded.

By independence of the random variables $S_{n_\ell} - S_{n_{\ell-1}}$, $\ell \geq 1$, property (13) will hold true if

$$\sum_{\ell=1}^{\infty} \mathbb{P}\left(\frac{S_{n_\ell} - S_{n_{\ell-1}}}{\sqrt{2n_\ell\log\log(n_\ell)}} \geq 1-\varepsilon\right) \, = \, \infty \tag{14}$$

according to the Borel-Cantelli lemma (in its independent version thus).

2.2 A moderate deviation inequality

The proof of (14) will be established by a minoration inequality in the form of an asymptotics of moderate deviations (from the central limit theorem).

Recall $S_n = X_1 + \cdots + X_n$, $n \geq 1$, where $(X_n)_{n\geq 1}$ is a sequence of independent identically distributed random variables, with mean zero and variance one.

Consider $u > 0$ and integers $p, q \geq 1$. Then, by independence and equidistribution,

$$\mathbb{P}\bigg(\frac{S_{pq}}{pq} \geq u\bigg) \,\geq\, \bigg[\mathbb{P}\bigg(\frac{S_p}{p} \geq u\bigg)\bigg]^q$$

since

$$S_{pq} \,=\, (X_1 + \cdots + X_p) + (X_{p+1} + \cdots + X_{2p}) + \cdots + (X_{p(q-1)+1} + \cdots + X_{pq}).$$

If $p = p_\ell$, $q = q_\ell$, $\ell \geq 1$, are sequences of integers tending to infinity, then, for $u = \frac{t}{\sqrt{p}}$ where $t > 0$ is fixed, by the central limit theorem,

$$\liminf_{\ell\to\infty} \frac{1}{q_\ell}\, \log \mathbb{P}\bigg(\frac{S_{p_\ell q_\ell}}{\sqrt{p_\ell} q_\ell} \geq t\bigg) \,\geq\, \log \mathbb{P}(G \geq t) \tag{15}$$

where G is standard normal $\mathcal{N}(0,1)$.

Let then ε be fixed, $0 < \varepsilon < \frac{1}{3}$. For $\alpha > 0$ to be specified below as a function of ε, set, for each $\ell \geq 1$,

$$p_\ell \,=\, \bigg[\frac{\alpha\,\ell^\ell}{2\log\log(\ell^\ell)}\bigg], \qquad q_\ell \,=\, \bigg[\frac{2}{\alpha}\,\log\log(\ell^\ell)\bigg]$$

(integer parts). Define then a sequence of integers $(n_\ell)_{\ell\geq 1}$ by the recurrence relation

$$n_\ell \,=\, p_\ell\, q_\ell + n_{\ell-1}, \quad \ell \geq 2, \quad n_1 \,=\, 1.$$

It is easy to see that $n_\ell \sim \ell^\ell$ as $\ell \to \infty$. Indeed, letting $v_\ell = \frac{n_\ell}{\ell^\ell}$, $\ell \geq 1$,

$$v_\ell \,=\, u_\ell + \frac{(\ell-1)^{\ell-1}}{\ell^\ell}\, v_{\ell-1}$$

where the sequence $u_\ell = \frac{p_\ell q_\ell}{\ell^\ell}$, $\ell \geq 1$, is converging to 1. The sequence $(v_\ell)_{\ell\geq 1}$ is thus bounded, and converges also to 1. In particular,

$$\sqrt{p_\ell}\, q_\ell \,\sim\, \frac{1}{\sqrt{\alpha}}\sqrt{2n_\ell \log\log(n_\ell)} \qquad \text{and} \qquad q_\ell \,\sim\, \frac{2}{\alpha}\,\log\log(n_\ell).$$

Together with these observations, as an application of (15) for $t = (1-2\varepsilon)\sqrt{\alpha}$,

$$\liminf_{\ell\to\infty} \frac{1}{\log\log(n_\ell)} \log \mathbb{P}\bigg(\frac{S_{n_\ell} - S_{n_{\ell-1}}}{\sqrt{2n_\ell \log\log(n_\ell)}} \geq 1 - 3\varepsilon \bigg) \geq \frac{2}{\alpha} \liminf_{\ell\to\infty} \frac{1}{q_\ell} \log \mathbb{P}\bigg(\frac{S_{p_\ell q_\ell}}{\sqrt{p_\ell q_\ell}} \geq (1-2\varepsilon)\sqrt{\alpha} \bigg) \geq \frac{2}{\alpha} \log \mathbb{P}\big(G \geq (1-2\varepsilon)\sqrt{\alpha}\, \big).$$

The traditional lower bound

$$\mathbb{P}(G \geq u) \geq \Big(\frac{1}{u} - \frac{1}{u^3} \Big) \frac{1}{\sqrt{2\pi}}\, e^{-\frac{u^2}{2}}, \quad u > 0,$$

leads to observe that for α large enough (depending on $\varepsilon < \frac{1}{3}$),

$$\frac{2}{\alpha} \log \mathbb{P}\big(G \geq (1-2\varepsilon)\sqrt{\alpha}\, \big) \geq -(1-2\varepsilon),$$

so that, as a consequence,

$$\liminf_{\ell\to\infty} \frac{1}{\log\log(n_\ell)} \log \mathbb{P}\bigg(\frac{S_{n_\ell} - S_{n_{\ell-1}}}{\sqrt{2n_\ell \log\log(n_\ell)}} \geq 1 - 3\varepsilon \bigg) \geq -(1-2\varepsilon). \tag{16}$$

2.3 End of the proof of (11)

The lower bound (16) from the preceding paragraph easily ensures the divergence of the series (14), and will conclude in this way the proof of (11). Indeed, for every ℓ large enough,

$$\mathbb{P}\bigg(\frac{S_{n_\ell} - S_{n_{\ell-1}}}{\sqrt{2n_\ell \log\log(n_\ell}} \geq 1 - 3\varepsilon \bigg) \geq e^{-(1-\varepsilon)\log\log(n_\ell))}.$$

Since $n_\ell \sim \ell^\ell$, the right-hand side defines the general term of a divergent series, so that, $\varepsilon > 0$ being arbitrary, (14) holds true.

On the other hand $\frac{n_{\ell-1}}{n_\ell} \to 0$, as expected to conclude the proof. The almost sure inequality (12) is established, and therefore also (11).

3 Multi-dimensional extension and Strassen's form

Strassen's result (5) on the cluster set

$$C\bigg(\Big(\frac{S_n}{\sqrt{2n\log\log(n)}} \Big)_{n\geq 1} \bigg) = [-1, +1] \qquad \text{almost surely,}$$

is presented here via the extension of the real LIL (1) to random vectors. It is possible to provide a direct proof on the basis of the moderate deviation argument of the last section as developed in [1].

The LIL is rather easily extended to random vectors in the following statement. Let $X = (X^{(1)}, \dots, X^{(d)})$ be a random vector on $(\Omega, \mathcal{A}, \mathbb{P})$ with values in $\mathbb{R}^d$, such that $\mathbb{E}(\|X\|^2) < \infty$ and $\mathbb{E}(X) = 0$, where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$. It is supposed that the covariance matrix $\Gamma = \big(\mathbb{E}((X^{(i)}(X^{(j)})\big)_{1\leq i,j\leq d}$ is non-degenerate, with square root $\Gamma = A\,{}^tA$.

Let, as before, on $(\Omega, \mathcal{A}, \mathbb{P})$, be $(X_n)_{n\geq 1}$ a sequence of independent copies of X, and $S_n = X_1 + \cdots + X_n$, $n \geq 1$. Then, with

$$K \,=\, A(B^d) \,=\, \big\{Ax\,;\|x\| \leq 1\big\}$$

where B^d is the closed Euclidian unit ball in $\mathbb{R}^d$, the Strassen formulation (4) and (5) in $\mathbb{R}^d$ expresses that

$$\lim_{n\to\infty} d\bigg(\frac{S_n}{\sqrt{2n\log\log(n)}}\,, K\bigg) \,=\, 0 \tag{17}$$

and

$$C\bigg(\bigg(\frac{S_n}{\sqrt{2n\log\log(n)}}\bigg)_{n\geq 1}\bigg) \,=\, K \tag{18}$$

almost surely (with the corresponding notation).

The proof of these properties, following [3], may be presented as a consequence of the one-dimensional result together with compactness and projection arguments. Working with $A^{-1}X$, it may be assumed that the covariance matrix of X is the identity matrix and that $K = B^d$.

It is worthwhile observing to start with (and will be freely used below) that $\mathbb{P}(\Omega_1) = 1$ where $\Omega_1 = \big\{\sup_{n\geq 1} \frac{\|S_n\|}{\sqrt{2n\log\log(n)}} < \infty\big\}$ (by the LIL coordinatewise).

Again by the real LIL, for each $y \in \mathbb{R}^d$,

$$\limsup_{n\to\infty} \frac{\langle y, S_n\rangle}{\sqrt{2n\log\log(n)}} \,=\, \sqrt{\mathbb{E}(\langle y, X\rangle^2)} \,=\, \|y\| \qquad \text{almost surely.} \tag{19}$$

By countable density in $\mathbb{R}^d$, there exists a measurable set Ω_2 with $\mathbb{P}(\Omega_2) = 1$ on which this property holds true for every $y \in \mathbb{R}^d$.

As in the one-dimensional setting, property (17) will be achieved as soon as

$$\limsup_{n\to\infty} \frac{\|S_n\|}{\sqrt{2n\log\log(n)}} \,\leq\, 1 \qquad \text{almost surely.} \tag{20}$$

If this is not the case, there is an measurable set Ω_0 of positive probability on which the limsup is strictly greater than 1. Let then $\omega_0 \in \Omega_0 \cap \Omega_1 \cap \Omega_2$; there exists $\eta > 0$ and an infinite subsequence of integers $(n_k)_{k\geq 1}$ such that, for every $k \geq 1$,

$$\frac{\|S_{n_k}(\omega_0)\|}{\sqrt{2n_k \log\log(n_k)}} \ \geq \ 1 + 3\eta.$$

There exists then a sequence $(z_{n_k})_{k\geq 1}$ of norm 1 elements of $\mathbb{R}^d$ such that

$$\frac{\langle z_{n_k}, S_{n_k}(\omega_0)\rangle}{\sqrt{2n_k \log\log(n_k)}} \ \geq \ 1 + 2\eta$$

for all $k \geq 1$. By compactness, the sequence $(z_{n_k})_{k\geq 1}$ admits a limit point $z = z(\omega_0) \in \mathbb{R}^d$ of norm 1, and thus, for infinitely many integers k,

$$\frac{\langle z(\omega_0), S_{n_k}(\omega_0)\rangle}{\sqrt{2n_k \log\log(n_k)}} \ \geq \ 1 + \eta.$$

This is however in contradiction with (19) applied to $y = z(\omega_0)$. The claim (20) follows. Due to (19), this limsup is actually equal to 1.

Concerning (18), if y is of norm 1 in $\mathbb{R}^d$,

$$\left\| \frac{S_n}{\sqrt{2n \log\log(n)}} - y \right\|^2 = \left\| \frac{S_n}{\sqrt{2n \log\log(n)}} \right\|^2 + 1 - 2\,\frac{\langle y, S_n\rangle}{\sqrt{2n \log\log(n)}}$$

so that, following (20) and (19),

$$\liminf_{n\to\infty} \left\| \frac{S_n}{\sqrt{2n \log\log(n)}} - y \right\| = 0 \qquad \text{almost surely.}$$

Therefore, any point of the unit sphere S^{d-1} of $\mathbb{R}^d$ is almost surely a limit point of the sequence $\left(\frac{S_n}{\sqrt{2n\log\log(n)}}\right)_{n\geq 1}$. By countable density,

$$C\left(\left(\frac{S_n}{\sqrt{2n \log\log(n)}}\right)_{n\geq 1}\right) \supset S^{d-1} \qquad \text{almost surely.} \tag{21}$$

But this result holds true in any dimension d. It may thus be applied to a random vector $\widetilde{X} = (X, Z)$ in $\mathbb{R}^{d+1}$ where Z is real centered with variance 1, independent from X. Since by the projection

$$\pi_{d+1,d} : (x, z) \in \mathbb{R}^{d+1} = \mathbb{R}^d \times \mathbb{R} \ \mapsto \ x \in \mathbb{R}^d,$$

$\pi_{d+1,d}(S^d) = B^d$ and, with the corresponding notation, $\pi_{d+1,d}(\widetilde{S_n}) = S_n$, the inclusion (21) in $\mathbb{R}^{d+1}$ thus projected on $\mathbb{R}^d$ yields the conclusion with B^d in place of S^{d-1}. The almost sure inequality (20) ensuring the reverse inclusion, the conclusion (18) follows.

The application of this result in $\mathbb{R}^2$ projected on $\mathbb{R}$ yields in particular part (5) of Strassen's form on the cluster set of the real LIL.

All of these statements therefore conclude a complete proof of the LIL.

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