A baby version of the Tracy-Widom asymptotics

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Abstract

The note specifies the mean behavior of the eigenvalue counting function at the edge of the semi-circle law studied by J. Gustavsson, in the form of a Tracy-Widom-type asymptotics involving the Airy function.

Let $X = X^n$ be a $n \times n$ random matrix from the Gaussian Unitary Ensemble (GUE), with thus complex Gaussian entries with mean zero and variance one, independent save for the condition that the matrix is Hermitian, with (real) eigenvalues $\lambda_1^n \leq \cdots \leq \lambda_n^n$. As is classical, the (renormalized) spectral measure $\frac{1}{n} \sum_{j=1}^n \delta_{\frac{1}{\sqrt{n}} \lambda_j^n}$ converges, as $n \to \infty$, to the semi-circle distribution with density $\rho(x) = \frac{1}{2\pi} \sqrt{4 - x^2}\, \mathbb{1}_{[-2,+2]}(x)$, $x \in \mathbb{R}$. For general references on random matrices and the results displayed here, see e.g. [6, 1, 9]...

The normalized largest eigenvalue $\frac{1}{\sqrt{n}} \lambda_n^n$ converges almost surely to 2. The fluctuations around the edge are described by the limit

$$\lim_{n \to \infty} \mathbb{P}\big(\lambda_n^n \leq 2\sqrt{n} + n^{-1/6} t\big) \, = \, F_{\mathrm{TW}}(t), \quad t \in \mathbb{R}, \tag{1}$$

where F_{TW} is the distribution function of the Tracy-Widom law, firstly expressed as the Fredholm determinant

$$F_{\mathrm{TW}}(t) \, = \, \det\Big(\big[\mathrm{Id} - K_{\mathrm{Ai}}\big]_{\mathrm{L}^2([t,\infty))}\Big), \quad t \in \mathbb{R},$$

of the integral operator associated to the Airy kernel

$$K_{\mathrm{Ai}}(x,y) \, = \, \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x - y}, \quad x, y \in \mathbb{R},$$

with Ai the special Airy function solution of $\mathrm{Ai}'' = x\mathrm{Ai}$ with the asymptotics $\mathrm{Ai}(x) \sim \frac{e^{-\frac{2}{3}x^{\frac{3}{2}}}}{2\sqrt{\pi}x^{\frac{1}{4}}}$ as $x \to \infty$.

It is part of the main contribution [8] to provide an alternate analytic description of F_{TW} in terms of some Painlevé equation as

$$F_{\mathrm{TW}}(t) \, = \, \exp\bigg(- \int_t^\infty (x-t) q(x)^2 dx \bigg), \quad t \in \mathbb{R},$$

where $q = q(x)$ is the solution to the Painlevé II ordinary differential equation with infinite boundary conditon given by the Airy function

$$q'' \, = \, xq + 2q^3, \qquad q(x) \, \sim \, \mathrm{Ai}(x) \quad x \to \infty.$$

By symmetry, the normalized smallest eigenvalue $\frac{1}{\sqrt{n}}\,\lambda_1^n$ converges to -2 with fluctuations around this value given similarly by the Tracy-Widom distribution.

Denote by

$$\mathcal{N}_s^n \, = \, \sum_{j=1}^n \mathbb{1}_{\{\frac{1}{\sqrt{n}}\lambda_j^n \geq s\}}$$

the number of (renormalized) eigenvalues of the GUE matrix X^n above $s \in \mathbb{R}$. By convergence of the empirical measure to the semi-circle law ρ,

$$\lim_{n\to\infty} \frac{1}{n}\,\mathbb{E}\big(\mathcal{N}_s^n\big) \, = \, \rho\big([s,\infty)\big).$$

The standard orthogonal polynomial approach to invariant matrix ensembles (see [6, 1]) describes the expected eigenvalue counting function by the one-point correlation function as

$$\mathbb{E}\big(\mathcal{N}_s^n\big) \, = \, \int_s^\infty \sum_{k=0}^{n-1} H_k^2 \, d\gamma \tag{2}$$

where $(H_k)_{k\in\mathbb{N}}$ is the family of Hermite polynomials which form an orthonormal basis of the Hilbert space $\mathrm{L}^2(\gamma)$ of the standard normal distribution $d\gamma(x) = \frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}x^2}dx$. Based on this representation and the asymptotic behavior of Hermite polynomials, J. Gustavsson [2, Lemma 2.2]) showed that whenever $s = s_n = 2 - \delta_n$, $\delta_n > 0$, $\delta_n \to 0$, then

$$\mathbb{E}\big(\mathcal{N}_{s_n}^n\big) \, = \, \frac{2}{3\pi}\, n\, \delta_n^{3/2} + O(1). \tag{3}$$

The result is however not qualitatively informative at the critical rate $\delta_n \sim n^{-2/3}$ corresponding to the Tracy-Widom fluctuations (1). With some further details, the following asymptotics may be produced.

Theorem. *For any $t \in \mathbb{R}$,*

$$\lim_{n\to\infty} \mathbb{E}\big(\mathcal{N}^n_{2+n^{-2/3}t}\big) \;=\; \int_t^\infty (x-t)\mathrm{Ai}(x)^2 dx.$$

The asymptotics of the Airy function for large positive values shows an exponential decay so that the integral is well defined. By symmetry, there is an analogous statement around -2. It is expected that a similar regime and limiting representation arises for large families of invariant or Wigner random matrix ensembles.

For the matter of comparison with (1), observe that $\mathbb{E}(\mathcal{N}^n_s) = \sum_{j=1}^n \mathbb{P}(\lambda^n_j \geq s\sqrt{n}\,)$, $s \in \mathbb{R}$, so that the theorem states that

$$\lim_{n\to\infty} \sum_{j=1}^n \mathbb{P}\big(\lambda^n_j \geq 2\sqrt{n} + n^{-1/6}t\big) \;=\; \int_t^\infty (x-t)\mathrm{Ai}(x)^2 dx, \quad t \in \mathbb{R}.$$

From the Gaussian behavior of the bulk eigenvalues [2], only a portion of the largest eigenvalues is actually relevant in the limit, specifically

$$\lim_{n\to\infty} \sum_{j=j_n}^n \mathbb{P}\big(\lambda^n_j \geq 2\sqrt{n} + n^{-1/6}t\big) \;=\; \int_t^\infty (x-t)\mathrm{Ai}(x)^2 dx$$

for any sequence of integers $j_n \to \infty$ with $j_n \leq \delta n$ for any $\delta < 1$ (and presumably as soon as $n - j_n \to \infty$).

Proof. The proof relies similarly on the orthogonal polynomial representation (2). A variation on the classical Christoffel-Darboux formula (see e.g. [5, Lemma 5.1]) shows that for $u \in \mathbb{R}$,

$$\int_u^\infty \sum_{k=0}^{n-1} H_k^2\, d\gamma \;=\; u\sqrt{n} \int_{-\infty}^u H_n H_{n-1}\, d\gamma + \sqrt{n} \int_u^\infty x H_n H_{n-1}\, d\gamma.$$

Since by orthogonality $\int_{\mathbb{R}} H_n H_{n-1} d\gamma = 0$,

$$\int_u^\infty \sum_{k=0}^{n-1} H_k^2\, d\gamma \;=\; \sqrt{n} \int_u^\infty (x-u) H_n H_{n-1}\, d\gamma.$$

Hence, from (2) and the preceding, for $t \in \mathbb{R}$, and $n \geq 1$,

$$\begin{aligned}\mathbb{E}\big(\mathcal{N}^n_{2+n^{-2/3}t}\big) \;&=\; \int_{2\sqrt{n}+n^{-1/6}t}^\infty \sum_{k=0}^{n-1} H_k^2\, d\gamma \\ &=\; \sqrt{n} \int_{2\sqrt{n}+n^{-1/6}t}^\infty \big(x - 2\sqrt{n} - n^{-1/6}t\big) H_n H_{n-1}\, d\gamma.\end{aligned}$$

Next, for $c > t$ to be specified,

$$\begin{aligned}\mathbb{E}\big(\mathcal{N}^n_{2+n^{-2/3}t}\big) \;=\; & \int_{2\sqrt{n}+n^{-1/6}c}^{\infty} \sum_{k=0}^{n-1} H_k^2 \, d\gamma \\ & + \sqrt{n} \int_{2\sqrt{n}+n^{-1/6}t}^{2\sqrt{n}+n^{-1/6}c} \big(x - 2\sqrt{n} - n^{-1/6}t\big) H_n H_{n-1} \, d\gamma \\ & + (c-t)\, n^{1/3} \int_{2\sqrt{n}+n^{-1/6}c}^{\infty} H_n H_{n-1} \, d\gamma.\end{aligned} \tag{4}$$

After a change of variables,

$$\begin{aligned}&\int_{2\sqrt{n}+n^{-1/6}t}^{2\sqrt{n}+n^{-1/6}c} \big(x - 2\sqrt{n} - n^{-1/6}t\big) H_n H_{n-1} \, d\gamma \\ &\qquad = \frac{1}{n^{1/3}} \int_t^c (x-t) H_n\big(2\sqrt{n} + n^{-1/6}x\big) H_{n-1}\big(2\sqrt{n} + n^{-1/6}x\big) e^{-\frac{1}{2}[2\sqrt{n}+n^{-1/6}x]^2} \, \frac{dx}{\sqrt{2\pi}} \, .\end{aligned}$$

The standard Plancherel-Rotach asymptotics as $n \to \infty$ of the n-Hermite polynomials around $2\sqrt{n}$ (cf. [7, 1]) indicates that, uniformly on compact sets,

$$H_n\big(2\sqrt{n} + n^{-1/6}x\big) e^{-\frac{1}{4}[2\sqrt{n}+n^{-1/6}x]^2} \;=\; (2\pi)^{1/4} n^{-1/12} \mathrm{Ai}(x) + O\big(n^{-2/3}\big).$$

Checking the normalizations, the middle term in (4) produces in the limit

$$\int_t^c (x-t)\mathrm{Ai}(x)^2 dx.$$

Provided $c > 0$ is large enough, the exponential decay of the Airy function at $+\infty$ eventually yields the formula of the theorem.

It remains however to handle the first and third terms on the right-hand side of (4) and show that they are uniformly (in n) negligible as c is large enough. The classical Harer-Zagier recursion formula for the moments of the GUE [4, 3] expresses that the sequence $M_p^n = \int_{\mathbb{R}} x^{2p} \sum_{k=0}^{n-1} H_k^2 d\gamma$, $p \in \mathbb{N}$, solves

$$M_p^n \;=\; 4n \, \frac{2p-1}{2p+2} \, M_{p-1}^n + 4p(p-1) \, \frac{2p-1}{2p+2} \cdot \frac{2p-3}{2p} \, M_{p-2}^n \, , \quad M_0^n = n, \; M_1^n = n^2.$$

By induction, for every $p \in \mathbb{N}$,

$$M_p^n \;\leq\; n^{p+1} \bigg(1 + \frac{p^2}{n^2}\bigg)^p \chi_p$$

where

$$\chi_p \;=\; 4 \, \frac{2p-1}{2p+2} \, \chi_{p-1} \;=\; \frac{(2p)!}{p!(p+1)!}$$

are the $2p$-th moments of the semi-circle distribution. Hence, for every $n, p \geq 1$,

$$\int_{2\sqrt{n}+n^{-1/6}c}^{\infty} \sum_{k=0}^{n-1} H_k^2 \, d\gamma \leq \frac{M_p^n}{(2\sqrt{n}+n^{-1/6}c)^{2p}} \leq \frac{n^{p+1}\big(1+\frac{p^2}{n^2}\big)^p}{(2\sqrt{n}+n^{-1/6}c)^{2p}} \, \chi_p.$$

By Stirling's formula, $\chi_p \lesssim 2^{2p}p^{-3/2}$ so that for p of the order of $n^{2/3}$, the preceding provides a bound of the order of e^{-c}, uniformly in n.

A similar analysis may be performed on the moments $\int_{\mathbb{R}} x^{2p} H_n^2 d\gamma$, $p \in \mathbb{N}$, which also solve a three-term recursion formula (see [5, (5.6)]), and from which it may be shown that, uniformly in n,

$$\bigg| \int_{2\sqrt{n}+n^{-1/6}c}^{\infty} H_n H_{n-1} \, d\gamma \bigg| \lesssim \frac{e^{-c}}{n^{1/3}}$$

for $c > 0$ large enough. Reporting these estimates in (4) concludes the proof of the theorem.

□

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