The random continued fractions of Dyson and their extensions

La Sapientia, January 25th, 2010.

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56 years ago Freeman J. Dyson * the Princeton physicist and number theorist, considers N particles of mass m_j on a one dimensional chain such that K_j is the elastic modulus of the spring between particles j and $j+1$. If $x_j(t)$ is the *displacement* from the equilibrium of particle j at time t the equations of the motion of the system are, with $x_0 = x_{N+1} = 0$, with $j = 1, \dots, N$:

$$m_j x_j'' = K_j(x_{j+1} - x_j) + K_{j-1}(x_{j-1} - x_j)$$

Therefore the numbers $m_1, \dots, m_N, K_0, \dots, K_N$ are given. Introducing $\lambda_{2j-1} = \frac{K_j}{m_j}$, $\lambda_{2j} = \frac{K_j}{m_{j+1}}$ we get constants $\lambda_1, \dots, \lambda_{2N-2}$.

*'The dynamics of a disordered linear chain' *Phys. Rev.* 1953 **92**, 1331-1338. I learnt about it by reading the deep papers by Jens Marklov, Yves Tourigny and Lech Wolowski 'Explicit invariant measures for products of random matrices' *Trans. Amer. Math Soc.* **360** (2008) 3391-3427, and Alain Comtet and Yves Tourigny 'Excursions of diffusion processes and continued fractions' arXiv 0906.4651v1[math.PR] 25 Jun. 2009

We introduce new functions

$u_{2j-1} = \sqrt{m_j} x_j, u'_{2j} = \sqrt{\lambda_{2j}} u_{2j+1} - \sqrt{\lambda_{2j-1}} u_{2j-1}$

and the above system is transformed in the linear differential system $u' = Au$ or

$$\begin{bmatrix} u'_1 \\ u'_2 \\ \vdots \\ \vdots \\ u'_{2N-1} \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{\lambda_1} & 0 & 0 & \dots \\ -\sqrt{\lambda_1} & 0 & \sqrt{\lambda_2} & 0 & \dots \\ 0 & -\sqrt{\lambda_2} & 0 & \sqrt{\lambda_3} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_{2N-1} \end{bmatrix}$$

Introduce the Hermitian matrix $H = iA$. Since A is antisymmetric with odd order, H has zero as eigenvalue and the other eigenvalues $\pm w_j$ with $j = 1, 2, \dots, N-1$ are real and go by pairs. Therefore

$$u(t) = e^{itH} u(0) = v_0 + \sum_{j=1}^{N-1} v_j \cos(w_j t)$$

where v_j are constant vectors of $\mathbb{R}^{2N-1}$.

At last, some continued fractions. Dyson at this point lets $N \to \infty$ such that the empirical distribution

$$\mu_N(dt) = \frac{1}{N-1} \sum_{j=1}^{N-1} \delta_{w_j^2}(dt)$$

converges weakly towards a probability $\mu(dt)$ and links μ to a continued fraction in the following way : Defining

$$\Omega_\mu(x) = \int_0^\infty \log(1 + tx)\mu(dt) = \lim_{N\to\infty} \Omega_{\mu_N}(x)$$

by a remarkable combinatorial argument, Dyson shows that if

$$H_j(x) = \frac{x\lambda_j}{1 + \frac{x\lambda_{j+1}}{1+\frac{x\lambda_{j+2}}{\cdots}}}$$

then

$$\Omega_\mu(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{2N-1} \log(1 + H_j(x))$$

after having explained how to recover μ from the knowledge of Ω_μ by nowadays familiar arguments about Stieltjes transforms.

At last, some randomness.

Finally Dyson observes that if $\lambda_1, \dots, \lambda_n, \dots$ are iid random variables with distribution ν , all the H_j have the same distribution μ_x (although they are dependent). A surprizing result is that $\Omega_\mu(x)$ is not random and is exactly equal to

$$\Omega_\mu(x) = \int_0^\infty \log(1+t)\mu_x(dt).$$

Of course recovering μ from μ_x is very difficult but Freeman J. Dyson is able to work on an example that we are going to generalize.

What is a Kummer distribution of type 2 ?

Given $a, p > 0$ and $b \in \mathbb{R}$ the Kummer distributions $K^{(2)}(a, b, p)$ are the members of the natural exponential family generated by the density

$$\frac{x^{a-1}}{(1+x)^{a+b}}\mathbf{1}_{(0,\infty)}(x).$$

Denote by $C(a, b, p) = \int_0^\infty \frac{x^{a-1}}{(1+x)^{a+b}} e^{-px} dx$ its Laplace transform. It satisfies actually an extraordinary formula : if $a, a+b, p > 0$ then

$$(*) \qquad C(a, b, p) =$$

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\, {}_1F_1(a\,;1-b\,;p) + \Gamma(-b)p^b\, {}_1F_1(a+b\,;1+b\,;p)$$

where $p \mapsto {}_1F_1(a\,;b\,;p) = \sum_{n=0}^\infty \frac{(a)_n p^n}{n!(b)_n}$ is the confluent entire function defined for $b \notin -\mathbb{N}$.
It implies

$$C(a+b; -b; p) = C(a; b; p)\frac{\Gamma(a+b)}{\Gamma(a)}p^{-b}.$$

Why extraordinary ?

1. Because nobody knows an elementary proof, or a probabilistic one.
2. Because if

$$Z \sim \gamma_{b,p}(dz) = \frac{p^b}{\Gamma(b)} z^{b-1} e^{-pz} \mathbf{1}_{(0,\infty)}(z)dz$$

   is independent of $Y \sim K^{(2)}(a,b,p)$ then $Z+Y \sim K^{(2)}(a+b,-b,p)$ and an immediate proof is given by computing the Laplace transform of $X+Y$ and by using (*).
3. Because

   if $H \sim \gamma_{a+b,p}(dz)$ is independent of $Y \sim K^{(2)}(a,b,p)$ then $\frac{H}{1+Y} \sim K^{(2)}(a+b,-b,p)$ and an immediate proof is given by computing the Mellin transform of $\frac{H}{1+Y}$ and by using (*).

Dyson has discovered (3) in the particular case $b=0$ and $a \in \mathbb{N}^*$.

Stationary distribution of a Markov chain.

Principle.* Let E be metric separable space with its Borel sigma field, let C be the space of continuous functions from E to E (with the smallest sigma field such that $f \mapsto f(x)$ is measurable for all $x \in E$) and let $F_1, \dots, F_n, \dots$ be independent iid rv on C with distribution ν. Consider

$$Z_n(x) = F_1 \circ F_2 \circ \dots F_n(x)$$

and

$$Y_n(x) = F_n \circ F_{n-1} \circ \dots F_1(x).$$

Assume that $\lim_{n \to} Z_n(x) = Z$ exists almost surely and does not depend on x. Then the distribution μ of Z is the unique stationary distribution of the Markov chain $(Y_n(x))_{n \geq 1}$. In particular when $(Y, F) \in E \times C$ are independent with a $F \sim \nu$ then $F(Y) \sim Y$ if and only if $Y \sim \mu$.

*G.L. (1986) 'A contraction principle for certain Markov chains and its applications.' *Contemp. Math.* **50**, 263-273. ; James Propp and David Wilson (1996) 'Exact Sampling with Coupled Markov Chains and Applications to Statistical Mechanics' *Random Structures and Algorithms* **9** 223-252.

Random continued fractions and products of (2,2) random matrices.

if $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible denote

$$h_M(x) = \frac{ax+b}{cx+d}.$$

Clearly $h_M \circ h_{M_1} = h_{MM_1}$.

Example :

Consider the random Moebius transformations $F_n(x) = \frac{H_n}{1+x}$ when the H_n are iid and > 0. if Z is

$$Z = \lim_{n\to\infty} F_1 \circ F_2 \circ F_3 \ldots F_n(x) = \cfrac{H_1}{1 + \cfrac{H_2}{1+\cfrac{H_3}{1+\cfrac{H_4}{1+\ldots}}}}$$

the distribution of Z is also the stationary distribution on $(0, \infty)$ of the Markov chain

$$Y_n = F_n \circ F_{n-1} \circ \ldots \circ F_1(x_0)$$

or $Y_n = F_n(Y_{n-1})$. We skip the fact that Z does exist almost surely.

Dyson case We now apply this principle in a first particular case $H_n \sim \gamma_{a,p}$ for all n with $H_1, \ldots, H_n, \ldots$ independent. In this case we know that if $H \sim \gamma_{a,p}$ and Y are independent then $Y \sim \frac{H}{1+Y}$ if and only $Y \sim K^{(2)}(a, 0, p)$. Now to stick to the aims of the Dyson's paper we observe that $H = x\Lambda$ has distribution $\gamma_{a,1/x}$ if $\Lambda \sim \gamma_{a,1} = \gamma_a$. Therefore with the notations introduced before, if $\nu = \gamma_a$ the measure μ_x is $K^{(2)}(a, 0, 1/x)$. From this Dyson is able to get μ explicitely in the particular case where $a \in \mathbb{N}^*$

Generalisation to more general Kummer distributions Let $a, a+b>0, p>0$ Now we assume $H_{2n-1}\sim \gamma_{a,p}$ and $H_{2n}\sim\gamma_{a+b,p}$ with again $H_1,\ldots,H_n,\ldots$ independent. If $G_n(x)=\frac{H_n}{1+x}$ we apply the above principle to $F_n=G_{2n-1}\circ G_{2n}$ which are iid Moebius transformations. Suppose that $Y\sim\sim K^{(2)}(a,b,p)$ is independent of H_1 and H_2. Thus $G_2(Y)\sim K^{(2)}(a+b,-b,p)$ and $F_1(Y)\sim Y$. Therefore we can apply the principle and we find that the distribution of the continued fraction

$$Z=\lim_{n\to\infty} Z_n(x)=F_1\circ F_2\circ\ldots F_n(x)$$

is $K^{(2)}(a,b,p)$. Here again, to come back to the Dyson's motivations we find that $\mu_x=K^{(2)}(a,b,1/x)$. I have not yet undertaken the calculation of μ...

A little challenge. In a recent and very rich paper*, Angelo Koudou and Pierre Vallois have shown in particular the following result :

Theorem K-V : Let X and Y be independent positive non Dirac random variables and consider

$$U = \frac{XY}{1+X+Y}, \; V = X - U = \frac{X(1+X)}{1+X+Y}$$

Then U and V are independent if and only if there exist $a, b, p > 0$ such that $X \sim K^{(2)}(a+b, -b, p)$ and such that $Y \sim \beta^{(2)}(a, b)$. Under these circumstances $U \sim K^{(2)}(a, b, p)$ and $V \sim \gamma_{b,p}$.

Actually their proof of the characterisation adds an extra hypothesis of existence of C^2 densities for all variables. Another point is that their direct result gives another proof- without the extraordinary formula- of our point (3) which has been so useful.

*(2009) 'Some independence properties of the type Matsumoto-Yor' *Preprint.*

Here are the details. Exchanging the roles of (a, b) and $(a+b, -b)$ point (3) says : if $X_1 \sim \gamma(a, p)$ and $Y_1 \sim K^{(2)}(a+b, -b, p)$ and are independent then $\frac{X_1}{1+Y_1} \sim K^{(2)}(a, b, p)$. In the other hand a consequence of Theorem K-V is this : let $X_1 \sim \gamma(a, p)$ $X_1' \sim \gamma(b, p)$ and $X \sim K^{(2)}(a+b, -b, p)$ be independent and define $Y = X_1/X_1'$. Trivially $Y \sim \beta^{(2)}(a, b)$. Now taking U and V as in Theorem K-V, we get

$$U \times \frac{1}{V} = \frac{Y}{1+X} = \frac{X_1}{1+X} \times \frac{1}{X_1'}$$

Now since from Theorem K-V $V \sim X_1'$, since U and V are independent since $\frac{X_1}{1+X}$ and X_1' are independent we can claim that $U \sim \frac{X_1}{1+X}$ (a way to prove the claim is to take the Mellin transforms of $\frac{U}{V}$ and $\frac{X_1}{1+X} \times \frac{1}{X_1'}$: the fact that the Mellin transform of $V \sim X_1'$ is analytic gives us permission to simplify). Finally, since Theorem K-V says that $U \sim K^{(2)}(a, b, p)$ we get a new proof of the point (3).

Questions

1. Can we use the principle through the distribution of the generalised Dyson random continued fractions to prove completely the Koudou Vallois characterization ?

2. We have mentioned that the knowledge of the distribution μ_x of the continued fraction $H_j(x)$ gives the knowledge of asymptotic distribution μ of the eigenvalues of the random matrix iA where A an anti-symmetric Jacobi matrix with iid entries. Dyson performs this calculation when the Λ_j are γ_n distributed where n is an integer. The answer involves polynomials of degree n. Therefore replacing n by the positive number $a > 0$ is probably not trivial.

3. More generally, we have seen that the case $\Lambda_{2n-1} \sim \gamma_a$ and $\Lambda_{2n} \sim \gamma_b$ leads to explicit distribution μ_x of the Kummer$^{(2)}$ type. Finding the corresponding μ is even a more general interesting problem.

4. If $W_1, \ldots, W_n, \ldots$ are independent such that $W_{2n-1} \sim \gamma_{a,p}$ and $W_{2n} \sim \gamma_{b,p}$ it is known * that the random continued fraction

$$\frac{1}{W_1 + \frac{1}{W_2 + \frac{1}{W_3 \cdots}}}$$

as a generalized inverse Gaussian distribution (GIG). The quoted paper by Marklov *et al.* is devoted to the case where the W_j are replaced by $W_j e^{i\alpha}$ and the very interesting distribution obtained there which is spread in a cone of the complex plane gets probabilistic interpretations in Comtet and Tourigny. The point I want to make is the following : writing $1/W_j = x\Lambda_j$ with $p = x$ shows that the last random continued fraction is of the Dyson type, and that μ_x is also known here and is a GIG distribution. Therefore the explicit calculation of μ from the knowledge of μ_x is a problem which has to be solved.

*G.L. and V. Seshadri, (1983) 'A characterization of the generalized inverse Gaussian distribution by continued fractions.' *Z. Wahrsheinlichkeitstheorie und Verv. Geb.* **62**, 485-489.

5. The case *where Λ_j takes two values 0 or 1 has been investigated : in this case μ_x is a Denjoy distribution, a quite singular distribution. Dyson methods for recovering μ from the knowledge of Ω_μ are postulating that μ has a density. This is probably not the case anymore and these methods have to be adapted for finding μ there.

*Chassaing, Ph., Letac, G. and Mora, M. (1984) 'Brocot sequences and random walks on $SL(2,\mathbb{C})$.' Probability on Groups. *Lecture notes in mathematics, Springer*. **1034**, 37-50.

Three non Dysonian random continued fractions

Dysonians random continued fractions can be seen as the infinite iterations of Moebius transforms of the form $h_M(z) = \frac{x\Lambda}{1+x\Lambda z}$ where

$$M = \begin{bmatrix} x\Lambda & 0 \\ x\Lambda & 1 \end{bmatrix}$$

Since $h_{\lambda M} = h_M$ note that $M = \begin{bmatrix} xA & 0 \\ xA & B \end{bmatrix}$ is Dysonian as well.

1. I should mentioned here that the paper by Koudou and Vallois contains also other results leading to interesting continued fractions. They fix a constant $\delta > 0$ and consider the Moebius random functions generated by

$$M_j = \begin{bmatrix} 0 & \delta Y_j \\ 1 + Y_j & 1 \end{bmatrix}$$

where the random 'inputs' Y_n have beta distributions of type two, namely

$$\beta^{(2)}_{\alpha,p}(dx) = \frac{1}{B(\alpha,p)} \frac{x^{\alpha-1}}{(1+x)^{\alpha+p}} \mathbf{1}_{(0,\infty)}(x)dx$$

and the continued fractions have distributions of the form

$$Cx^{a-1}(1+x)^b(\delta+x)^c\mathbf{1}_{(0,\infty)}(x)dx$$

2. The paper by Asci *et al.* * considers continued fractions generated by

$$M_j = \begin{bmatrix} 1 & 0 \\ W_j & 1 \end{bmatrix}$$

The random inputs W_j have beta distributions of type two and the continued fractions have distributions of the form

$$\frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} \ {}_2F_1(p,q;r;x)\mathbf{1}_{(0,1)}(x)dx$$

where ${}_2F_1(p,q;r;x)$ is the Gauss hypergeometric function. The proof of this uses a fascinating formula about ${}_3F_2(a,b,c;d,e;1)$ discovered by Thomae in 1879.

*Asci, C., Letac, G. and Piccioni, M. (2008) 'Beta-hypergeometric distributions and random continued fractions.' *Statist. Probab. Lett.*, **78**, issue **13**, 1711-1721.

3. The paper by Marklof *et al* observes the following : if $w = a + ib$ with $b > 0$, consider the Cauchy distribution $C_w(dx) = \frac{1}{\pi}\frac{bdx}{b^2+(x-a)^2}$. Now let $X_1, \ldots, X_n, \ldots$ be independent random variables such that $X_n \sim C_{w_n}$. Define

$$X = X_1 - \frac{1}{X_2 - \frac{1}{X_3 - \ldots}}, \quad w = w_1 - \frac{1}{w_2 - \frac{1}{w_3 - \ldots}}.$$

Then $X \sim C_w$. This is a non Dysonian example since

$$M_j = \begin{bmatrix} X_j & -1 \\ 1 & 0 \end{bmatrix}.$$

Furthermore X is not positive.