Elliptic PDEs and Calculus of Variations

P. Bousquet, R. Ignat

The aim of this course is to present elementary methods used in the study of elliptic PDEs, with particular emphasis on the problems of existence, uniqueness and regularity of solutions, including the connections with the Calculus of Variations.

1. Introduction:

Elliptic divergence form operators, Weak solutions.

2. Laplace's Equation:

Regularity, Quantitative properties of harmonic functions (Caccioploi's inequality, H^k estimates, Compactness....)

3. Existence of weak solutions:

Lax-Milgram's Theorem, Fredholm alternative, Schauder's method, Duality method (Stampacchia's Theorem).

4. Regularity of weak solutions:

DeGiorgi's theorem ($C^{0,\alpha}$ regularity), Moser's iteration technique, Schauder's estimates.

5. Maximum principle:

Weak Maximum principle, Hopf's lemma, Strong Maximum principle, Harnack's inequality, Moving plane method, Symmetry of positive solutions.

6. Solutions of nonlinear elliptic PDEs:

Local invertibility, Nemitskii operators, Global invertibility (Continuity method), Nonexistence results (Pohozaev's identity).

7. Calculus of variations and critical points:

The Direct Method, Constrained Minimization, Euler-Lagrange equations, Morse's Deformation Theorem, Palais-Smale sequences, Mountain Pass lemma, Applications to semi-linear elliptic PDEs.

8. Compactness:

Compensated compactness (div-curl lemma), Concentrated Compactness.

References:

1. Han, Q. and Lin, F. *Elliptic partial differential equations.* Courant Lecture Notes in Mathematics, 2011.

2. Evans, L. *Partial differential equations.* Graduate Studies in Mathematics, AMS, Providence, RI, 2010.

3. Gilbarg, D. and Trudinger, N. *Elliptic partial differential equations of second order.* Springer-Verlag, Berlin, 2001.

4. Giaquinta, M. *Introduction to regularity theory for nonlinear elliptic systems.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1993.