Controllability of parabolic PDEs : old and new

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The goal of this course is to introduce the audience to the controllability problems for evolution partial differential equations and more particularly for parabolic equations. Let us consider the following controlled evolution system

$$\partial_t y + Ay = Bv, \quad y(0) = y_0, \tag{$\star$}$$

where $y(t)$ is the state at time t, y_0 the initial data, A is an elliptic operator defined on a domain Ω (think of it as the negative Laplace operator with homogeneous Dirichlet boundary condition for instance), $v(t)$ is the control at time t and B the control operator (that is the way the control acts on the system).

The main question we will study is the one of the approximate (resp. null) controllability : for a given time T and initial data y_0, does it exist a control map $t \in (0, T) \mapsto v(t)$ such that the corresponding solution $y(T)$ of ($\star$) at time T is small (resp. zero) ?

The main example we shall investigate is the one of the heat equation ($A = -\Delta$) with distributed control ($B = 1_\omega$, $\omega \subset \Omega$) or boundary control ($B =$"the boundary Dirichlet data operator").

Tentative schedule

1. **Controllability questions for autonomous finite dimensional linear systems of ODEs :**
   - Kalman rank condition.
   - Hautus test.
2. **Controllability of infinite dimensional dynamical systems :**
   - Functional framework
   - Equivalence between controllability and observability
   - Fattorini-Hautus test
   - HUM functional and applications
3. **The heat equation :**

   We will study in detail this equation and introduce three methods that we will compare one with each other:
   - The moment method in 1D.
   - The Lebeau-Robbiano approach.
   - The Fursikov-Imanuvilov approach.

   The main tools underlying those methods are fine spectral properties for the operator A on the one hand, and/or Carleman estimates in another hand.
4. **Systems of coupled parabolic equations :**

   We will consider here parabolic systems (with more than 1 component in the solution) that we wish to control by as few controls as possible. A typical example will be

$$\begin{cases} \partial_t y_1 - \Delta y_1 = a_{11}(x) y_1 + a_{12}(x) y_2 + 1_\omega v \\ \partial_t y_2 - \Delta y_2 = a_{21}(x) y_1 + a_{22}(x) y_2. \end{cases} \tag{$\star\star$}$$

Observe that the control v only acts in the first equation. Therefore, it is clear that the particular form of the coupling terms between the equations (for instance the coefficient a_{21} in the example ($\star\star$)) will play a crucial role in the analysis. This topic has recently received a very careful attention in the literature and we will describe many new phenomenon (some of them unexpected) that may occur in this setting.

5. **Numerical analysis :**

   If there is enough time, I will also discuss issues coming from the numerical analysis of such control problems.

References

[1] Farid Ammar-Khodja, Assia Benabdallah, Manuel González-Burgos, and Luz Teresa. Recent results on the controllability of linear coupled parabolic problems: A survey. *Mathematical Control and Related Fields*, 1(3):267–306, September 2011.

[2] Franck Boyer. On the penalised HUM approach and its applications to the numerical approximation of null-controls for parabolic problems. *ESAIM: Proceedings*, 41:15–58, December 2013.

[3] Jean-Michel Coron. *Control and nonlinearity*, volume 136 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.