Course B3 : Theoretical and numerical analysis of dispersive partial differential equations

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Abstract: We will present on a model case some of the recent developments in the theoretical and numerical analysis of dispersive partial differential equations

The analysis of dispersive partial differential equations (PDE) has known tremendous developments in the last thirty years, both on its theoretical and numerical aspects. It has benefited from the introduction of techniques coming from various areas of mathematics such as harmonic analysis, dynamical systems or the calculus of variations. The goal of this series of lectures will be to present some of the recent developments on a prototype case, the time dependent Nonlinear Schrödinger Equation (NLS).

The Schrödinger equation is one of the famous model of quantum theory but also arises in various fields of physics, e.g. in nonlinear optics for laser beam propagation or in cold atom physics to describe Bose Einstein condensation. It presents remarkable properties, e.g. the preservation in time of several quantities and the existence of soliton solutions (waves which travel at a constant speed in time, keep the same spatial profile along the evolution in time and do not scatter). The numerical approximation of time dependent Schrödinger equations requires specific care to be able to preserve their theoretical properties and to compute soliton solutions other long time.

In the theoretical part of this series of lectures, we will cover the Cauchy theory of NLS, the existence and classification of soliton profiles by variational method, the stability of ground state solitons, and Merle's classification of the blow-up dynamics at minimal mass.

In the numerical part of this series of lectures, we will introduce and analyse some numerical schemes that turn out to be well adapted to the study of the Schrödinger equation. We will split the analysis between time discretization and space approximation. The time discretization is at the heart of the strategy to preserve conserved quantities in the numerical approximation, whereas space approximation will be concerned with the boundary conditions. Many numerical tests will be performed during real time experiments.

References :

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