Introduction to metric and Riemannian geometry

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Abstract:

The goal of this course is to introduce the audience to both metric and Riemannian geometry. Since the pioneeering work of Gromov in the early eighties, a tremendous amount of work has been devoted to Metric Measure Spaces. The leading idea is to encode the geometry of the space into a distance and a measure (modeling the volume of the space and of its subsets) through their interactions. On the other hand, the classical tools from Riemannian geometry remain relevant for fine studies of the underlying space and will also be addressed. Finally, I will discuss the central notion of curvature through a metric characterization of spaces with curvature bounded -from above and/or below- by a number k. This notion, originally introduced by Alexandrov, can be described rather simply in metric terms and will be useful for the advanced course on geometric group theory and 3 manifolds.

More specifically the topics adressed will be the following:

- Length spaces
  - Length structures
  - Examples
  - Length structures induced by metrics
  - Shortest paths
- Smooth length structures/ Riemannian metric
  - Riemannian metric
  - Examples. Nash's isometric embedding theorem.
  - Levi-Civita connection
  - Geodesics from an analytical point of view
  - Normal coordinates
- Densities and Volume
  - Densities on a Riemannian manifold
  - Volume estimates

- Space forms
  - The sphere
  - The hyperbolic space
- Variation formula(s)
  - Jacobi fields
  - Gauss lemma
  - Conjugate points
- Metric spaces with bounded curvature
  - Definitions
  - Angles. Analysis of distance functions
  - Examples
  - First properties
  - A survey of more adavanced facts (globalization theorem).

References:

- D. Burago, Y. Burago, and S. Ivanov , *A course in Metric Geometry*, AMS, 2001.
- S. Gallot, D. Hulin, J. Lafontaine, *Riemannian Geometry*, Springer, third edition, 2004.