Syllabus – Advanced course
Qualitative studies of PDEs: a dynamical systems approach

- **Main goal:** The aim of this advanced course is to provide an overview of some techniques used in the study of partial differential equations from a dynamical systems point of view.

- **Contents:** We will focus on studying coherent structures, such as periodic patterns and traveling waves in spatially extended systems. The lectures will be illustrated and motivated by a variety of simple model problems, such as the Allen-Cahn or Nagumo equation, reaction-diffusion systems and pattern-formation models such as the Swift-Hohenberg equation. We will learn about techniques to study existence and stability of such coherent structures. More specifically, there will be four different parts in these lectures.

  Part 1: we present the main ideas in the finite dimensional case. More precisely, we study the nonlinear asymptotic stability of equilibrium points of autonomous ordinary differential equations in $\mathbb{R}^n$. We first recall well known results about linear (in)stability implies nonlinear (in)stability in the case of hyperbolic equilibrium points. Then, in a second step, we present new results in the case of non hyperbolic equilibrium points and develop the concepts of center manifolds.

  Part 2: we give a theoretical framework to study the spectrum of closed linear operators. Closed linear operators naturally appear when linearizing a partial differential equation around a special solution (a traveling wave for example).

  Part 3: this is a direct continuation of Part 2. The idea is to identify the relationship between the spectrum of a given closed operator $\mathcal{L}$ and the dynamics of the linear equation $\partial_t u = \mathcal{L}u$ generated by this operator. As a key application, we demonstrate the nonlinear asymptotic stability of traveling fronts solutions for scalar bistable reaction-diffusion equations.

  Part 4: we present center manifold theorems in infinite dimensions. This is the natural generalization of the results presented in Part 1 in the finite dimensional case for ODEs.

- **Prerequisite:** There is no real prerequisite for this advanced course except some basic knowledge on ODEs. However, we (strongly) encourage students to take the course: *Elliptic and parabolic PDEs*.

- **References:** We suggest two references:

  M. Haragus and G. Iooss, *Local Bifurcations, Center Manifolds, and Normal Forms in Infinite-Dimensional Dynamical Systems*, EDP Sciences, Springer, 2011.

  T. Kapitula and K. Promislow, *Spectral and Dynamical Stability of Nonlinear Waves*, Applied Mathematical Sciences, Springer, 2013.

- **Key-words:** PDEs, dynamical systems, center manifolds, periodic patterns, traveling waves.

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