Geometric Modal Logic

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Alethic modalities pertain to truths as being not only factual.

Still, the range of what is possible (the content of the collection of all possible worlds, in modern modal semantics) is referred to as a kind of super-fact.

Modal logic should go beyond it: what the actual range of the possible turns out to be should be understood as a range *among others*.

Hence, a (second-order) range of all ranges for the possible.

But the actual range of all possible ranges for the possible should itself be conceived of as a second-order range *among others*, and so on.

That consideration of different levels of possibility becomes critical in the case of modal iteration.

Modal iteration = superposition of modal clauses, as when you say, for example, that some truth is necessarily necessary, or necessarily possibly necessary, etc.

The intuition of a range of possibles ranges of possible worlds (or even of ranges of possibles ranges . . . of possible ranges of possible worlds) is at the core of modal reasoning.

The usual semantics for modal logic does *not* genuinely implement that intuition.

Wanted: An alternative semantical framework that interprets iterated modalities in a better way.

Abstract features of that framework: possible worlds of variable levels are introduced and, for each modal iteration, a set of possible worlds is considered *relatively to* some already given possible world.

Two main opposite conceptions of necessity ought to be brought out and distinguished:

- the conception of necessity as a *modality*, whose duplication is senseless (R. Carnap);
- the conception of necessity as an *operator*, whose iterability is the very rationale (C. I. Lewis).

There are advantages and drawbacks in each of both options.

The first one precludes one from introducing a genuine modal *logic*. There is nothing to calculate, the concept of logical necessity is conceived of in terms of a logical space. Modal iteration is meaningless. *Tractatus*, 5.525:

> *The certainty, possibility, or impossibility of a situation is not expressed by a proposition, but by an expression's being a tautology, a proposition with sense, or a contradiction.*

The second option (which champions necessity as an operator) gives rise to an algebra of modalities which has nothing to do with any *modal* logic and loses sight of the specificity of logical modalities (conflation of modal and temporal logics).

Modal logic = versatile formalism to resort to in the field of applied logic.

Kripke semantics ([Kripke, 1963]), based on 'accessibility relations' between possible worlds, despite driving the operator approach to prevail, carries out a kind of conciliation ot both options.

Syntax

Given propositional variables p, q, r, ..., the set $\mathcal{F}$ of *formulas* of propositional modal logic is defined inductively:

- any propositional variable belongs to $\mathcal{F}$;
- if $\phi \in \mathcal{F}$, then $\neg\phi \in \mathcal{F}$;
- if $\phi, \psi \in \mathcal{F}$, then $(\phi \wedge \psi)$ and $(\phi \vee \psi) \in \mathcal{F}$;
- if $\phi \in \mathcal{F}$, then $\Box\phi \in \mathcal{F}$ ($\Box\phi$ = "Necessarily, ϕ").
- if $\phi \in \mathcal{F}$, then $\Diamond\phi \in \mathcal{F}$ ($\Diamond\phi$ = "Possibly, ϕ").

($\Box$ and $\Diamond$ are interdefinable, so only either one has to be mentioned in the language.)

Modal degree of a formula: obvious.

Proofs

Axioms of a "normal" system:

- all the tautologies of nonmodal propositional logic: $(p \vee \neg p)$, $((p \to q) \to (\neg q \to \neg p))$, etc.
- axiom **K**: $\Box(p \to q) \to (\Box p \to \Box q)$
- axiom **T**: $\Box p \to p$

Inference rules: modus ponens, 'necessitation' rule $(\phi/\Box\phi)$ and rule of uniform substitution $(\chi(p)/\chi[\phi/p])$.

The ensuing system is system T. Further axioms: axiom 4, $\Box p \to \Box\Box p$, which defines S4 $=$ T $+$ **4**, or axiom **5**, $\Diamond p \to \Box\Diamond p$, which defines S5.

Semantics

A *Kripke model* for propositional modal logic consists in a quadruple $\mathcal{M} = \langle W, R, w_0, V \rangle$, where:

- W is a set whose elements are called "possible worlds",
- Among which, a given arbitrary world w_0 is distinguished as being "the actual world",
- R is a binary relation R on W,
- V assigns to each atomic proposition p a subset $V(p)$ of W.

Inductive definition of $\mathcal{M}, w \vDash \phi$:

- $\mathcal{M}, w \vDash p$ iff $w \in V(p)$
- $\mathcal{M}, w \vDash \neg\phi$ iff $\mathcal{M}, w \nvDash \phi$
- $\mathcal{M}, w \vDash (\phi \wedge \psi)$ iff $\mathcal{M}, w \vDash \phi$ and $\mathcal{M}, w \vDash \psi$
- $\mathcal{M}, w \vDash \Box\phi$ iff $\forall w'$such that wRw', $\mathcal{M}, w' \vDash \phi$.

A formula ϕ is said to be *valid* in $\mathcal{M}$ iff $\mathcal{M}, w \vDash \phi$ for every $w \in W$.

Completeness results: for example, validity of **4** exactly corresponds to the transitivity of the relation R.

Modal iteration in Kripkean semantics becomes ramification in the graph of R:

$\mathcal{M}, w \vDash \Box\Box p$ iff $\forall v$ s.t. $wRv, \forall u$ s.t. $vRu, \mathcal{M}, u \vDash p$

The higher the modal degree, the longer the branch of the tree to consider.

I would like to suggest a new form of conciliation of the two basic options about necessity: let's try to grant the possibility to iterate a modal clause, while acceding to the remark that such an iteration should commit us to far more than is usually acknowledged.

Besides, let's do this within the framework of possible worlds.

Truth of a given proposition ϕ = satisfaction of ϕ in the actual world,

Necessary truth of ϕ = satisfaction, not only in the actual world w_0, but in any possible world $w \in W$

Now, what should we expect to account for the fact that ϕ is necessarily necessarily true?

We shifted from a given distinguished world w_0 to a set of worlds W, with $w_0 \in W$.

So, in an analogous way, we should shift now from the current set of possible worlds W to a set W^2 *of sets* of possible worlds, with $W \in W^2$.

Within the standard semantics itself, the first introduction of $\Box$ results in a widening of the place for the evaluation of the truth value of ϕ. Let's call such a widening a semantical *change of scale*. The same type of change should occur when the necessity operator is iterated.

The assessment of the truth value of ϕ required only to consider a single world w_0. The assessment of the truth value of $\Box\phi$ requires to consider a new type of entities, namely a set of worlds W. Consequently, the assessment of the truth value of $\Box\Box\phi$ should require to consider a further type of entities, namely a set of sets of worlds, which would be to W as W is to w_0. And so on, in case of any further modal iteration.

Such a progression is downright foreign to Kripke modal semantics, where the stock of possible worlds is *fixed* once for all.

Admittedly, we have the progression w_0, $E_{w_0} = \{w : w_0 R w\}$, $(E_w)_{w \in E_{w_0}}, \ldots$.

Still, any set (of sets) of worlds is contained in advance in W, and nothing relates two different subsets E_w and $E_{w'}$ as making up some higher-order world. There are no worlds of different levels, and a set of possible worlds is not itself a second-order possible world.

Modal iteration is not really accounted for.

Saying that a proposition is necessarily necessary claims *incomparably more* than saying that this proposition is simply necessary.

Speaking of something as 'possibly possible', we implicitly let the variation system itself vary, we shift from a given system of possibility into a frame inside which this system is only one among others, and we say that *respectively to* some *other* system, such or such state of affairs becomes possibly true.

To take modal iteration seriously, we should allow for higher-order nested systems of possibility, where each higher-order possible world is relative to some lower-order possible world.

The necessity of some necessary truth may itself rely on contingent factors.

$\Diamond\Diamond\phi = \phi$ *could have been* possible.

The actual array of possible worlds, after all, is only one among others equally possible: it could have been different. Hence a change of scale suggests itself.

A semantical change of scale should match each syntactical modal iteration.

Endeavours to 'gentzenize' modal logic: see [Masini, 1992], which introduces '2-sequents', that is, vertical lists of ordinary sequents.

A 2-sequent may be seen as a setting of different validity levels, each occurrence of a formula A being convertible into a modalization $\Box A$ at the immediately upper level.

Other previous developments:

- "Hypermodal logics" (Dov Gabbay)
- "Structural theory of sets" (Alexandru Baltag)
- Analysis of higher-order vagueness (Kit Fine)

Wanted: a stratified structure within which each possible world w gives rise to a set of possible worlds relative to w (w = index for a system of possibility).

Such a structure would lead from a set of possible worlds to a set of sets of possible worlds, and so on.

Candidate: the transition from a given manifold M to its "tangent bundle" TM.

To each point x of M is attached the tangent space to M at x, T_xM. $T_xM =$ fiber above x.

$TM =$ disjoint union of all the spaces T_xM, $x \in M$.

Natural projection $p : TM \to M$.

TM is itself a manifold. Hence TTM. And so on.

Scheme

0-level possible worlds = points of a manifold M. Evaluation of p at M.

T_xM = set of all 1-level possible worlds *relative to* x. Evaluation of $\Box p$ at TM.

$T_{(x,v_x)}TM$ = set of all 2-level possible worlds relative to (x, v_x). Evaluation of $\Box\Box\phi$ at TTM. And so on.

Each time, TM is twice as dimensional as M, which fits the intuitive idea of a frame widening.

Project: representing modal iteration (as a change of scale) through a geometric fibration.

Accessibility is taken litterally (path on a manifold). Whereas Kripke-style possible worlds remain without any connection to each other, possible worlds are given the texture of a geometric universe.

First step: any propositional variable p gets interpreted by a set of curves $\gamma_p : \mathbb{R} \to M$ on M, and p is true at x (in M) if one of the curves γ_p passes through x.

Four problems

$\neg p$, $(p \wedge q)$, etc., need to be interpreted as sets of curves as well, starting from the interpretation of p.

Modal lift: the interpretation of $\Box p$ has to be set in some reference to TM. For example, if we have a distribution on M, i.e., a family of subspaces $H_x \subset T_x M$, for $x \in M$: $\Box p$ is true at x if $x = p(t_0)$ for a certain t_0, <u>with $p'(t_0) \in H_x$</u>.

Nonmodal lift: if $\Box$ is assigned to a family of curves on TM and q is interpreted by a family of curves on M, how to interpret $(\Box p \to q)$? The interpretation of q has to be "hoisted" onto TM.

Problems: (P1) is the definition of the modal lift of any formula, leading from a family of curves interpreting a formula ϕ to a restricted family of curves interpreting $\Box\phi$ (or to an enlarged family of curves interpreting $\Diamond\phi$, depending on the choice of the primitive operator).

(P2) pertains to the introduction of a distribution not only on M, but also on TM, on TTM, ...

(P3) is the definition of the nonmodal lift of any formula ϕ, so as to face differences in modal degree.

(P4) is as to whether the normal axioms of modal propositional logic hold.

Last thing: to know if $\Box(\Box p \vee q)$ (degree 2) is true at $x \in M$, it is necessary to ask whether the corresponding curve on TTM passes through a certain point of TTM linked (in a way to be specified) to x.

This supposes to define for each $x \in M$ a matching point in TM above x, and so finally for each $x \in M$ a sequence $x^0 = x, x^1 \in TM, x^2 \in TTM$, etc. of counterparts of x.

A *geometric modal frame* consists in

(i) a sequence of differential fibrations $M = M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \ldots$ making it possible to naturally define the nonmodal lift $L(\gamma)$ onto M_{i+1} of any curve γ on M_i;

(ii) a distribution on every manifold M_i allowing to define naturally the modal lift $\lambda(\gamma)$ onto a submanifold of M_{i+1} of any curve γ on M_i;

(iii) a sequence $(x, x^1 \in M_1, x^2 \in M_2, \ldots)$ above any $x \in M$.

A *valuation* on such a frame is the assignment, to every propositional variable p, of a family of curves on M.

A frame endowed with a valuation is a *geometric modal model*.

FIRST TRACK: simpler case, $\Box$ as primitive

Each variable p is interpreted by a set I_p of curves on a manifold M. Hypothesis: the evaluation of $\Box p$ at x takes place in T_xM. A link is needed between $\Box p$ and the set of all derivatives γ'_p of the curves in I_p, namely:

$M, x \vDash \Box p$ iff there are γ_p and t s.t. $x = \gamma_p(t)$ and $\gamma'_p(t)$ satisfies a certain condition within T_xM, that is, belongs to some $H_x \subset T_xM$.

The latter clause corresponding to $\Box$ is existential, but in fact its meaning is that the shift from p to $\Box p$ amounts to a restriction, through the further condition of the conformity to $(H_x)_{x \in M}$.

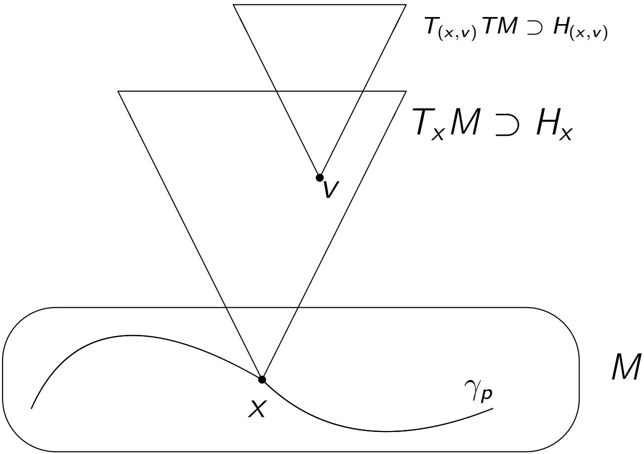
The subspace H_x should be conceived of as a set of constraints which locally restrict the range of possibilities. The distribution $(H_x)_{x \in M}$ determines the array of admissible trajectories, as for the movement of a physical body: physical law = "foliation" on a configuration space.

In the same way (higher modal degrees), a distinguished subspace $H_{(x,v)}$ of $T_{(x,v)}TM$ is needed for each $(x \in M, v \in H_x)$:

$$M, x \vDash \Box\Box p \text{ iff } \exists v \in H_x \;\; TM, v \vDash \Box p$$

iff

$$\exists v \in H_x \; \exists \gamma_p \in I_p \; \exists t \; (v = \gamma_p'(t) \text{ and } \gamma_p''(t) \in H_{(x,v)}).$$



x

v

Such a semantics establishes a connection between $\Box p$ and $\gamma'_p(t)$, between $\Box\Box p$ and $\gamma''_p(t)$, and so on: parallel between necessitation and derivation.

Serious problem: the determination of distinguished subspaces H_x, $H_{(x,v)}$, and so on.

First solution: within **S5** any modal formula is deductively equivalent to a formula of degree ≤ 1.

Consequence: a geometric model for **S5** will need only, on top of M, a field of tangent subspaces – in particular, a non-zero vector field.

Vector field on a manifold M = map X which assigns to each point x of M a vector X_x of T_xM.

Let's write $\overline{\gamma}$ for $\{\gamma(t) : t \in \mathbb{R}\}$. Then let's state:

$I_{\neg\phi} = \{\gamma \text{ curve on } M : \overline{\gamma} \cap \overline{I_\phi} = \emptyset\}$

$I_{(\phi\wedge\psi)} = I_\phi \cap I_\psi$

$I_{(\phi\vee\psi)} = I_\phi \cup I_\psi$

$I_{\Box\phi} = \{\gamma \in I_\phi : \ \gamma \text{ is an integral curve of } X\}$

$M, x \vDash \chi$ iff $x \in \overline{I_\chi}$

Axiom T is valid by construction.

Axiom K and the axiom $\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$ follow from the Cauchy-Lipschitz theorem which states the existence and unicity of the maximal integral curve passing through a given point (in the case of a non-zero vector field).

Tautologies and modus ponens turn out to be valid.

The necessitation rule and the uniform substitution rule don't hold any more, but given our perspective, that is in order.

A *modalized topos* is a topos E (whose subobject classifier is noted Ω) with an arrow $j : \Omega \to \Omega$ in E, called a "modal operator", such that the following diagrams commute:

$$\begin{tikzcd} & 1 \arrow[dr, "\mathrm{true}"] & \\ \Omega \arrow[ur, "\mathrm{true}"] \arrow[rr, "j"] & & \Omega \end{tikzcd} \quad \begin{tikzcd} & \Omega \arrow[dl, "j"'] \arrow[dr, "j"] & \\ \Omega \arrow[rr, "j"] & & \Omega \end{tikzcd} \quad \begin{tikzcd} \Omega \times \Omega \arrow[r, "j \times j"] \arrow[d, "\wedge"'] & \Omega \times \Omega \arrow[d, "\wedge"] \\ \Omega \arrow[r, "j"] & \Omega. \end{tikzcd}$$

Topological modal logic: cf.
[Kremer and Mints, 2005] and [Venema, 1999].

Modality in a topos : generalization of topological closure.

Problem: the axiom **4** is forced to hold.

SECOND TRACK: general case, $\diamond$ as primitive

Riemannian manifold = manifold M endowed with a "metrics", so that it becomes possible to measure the speed of any movement on M and thus the distance between any two points of M. It also allows one to define *geodesic curves*.

Any metrics on M gives rise to the corresponding "Levi-Civita connection", ∇. Intuitively, ∇ assigns two vector fields X and Y both defined around x_0 to the infinitesimal deviation of Y from $X(x_0)$. A vector field X ((defined along a curve on M) is *parallel* if $\nabla(X, X) = 0$.

A vector field X is in fact a curve on TM.

If X is parallel, then the tangent vector $X'(0) \in T_{X(0)}TM$ is said to be *horizontal*.

Then, a curve $\gamma : \mathbb{R} \to TM$ on TM is said to be *horizontal* if $\gamma'(t)$ is horizontal for any $t \in \mathbb{R}$.

Example: if γ is a geodesic, γ' is horizontal.

Fact: For any curve γ on M and any vector $v \in T_{\gamma(0)}M$, there is a unique horizontal curve $\tilde{\gamma}^v$ on TM passing through v and lying above γ (i.e., $p \circ \tilde{\gamma}^v = \gamma$, where $p : TM \to M$).

$\tilde{\gamma}^v$ is called "the horizontal lift of γ through v at $t = 0$".

Fact: If γ is a geodesic of M, the horizontal lift of γ passing through $\gamma'(0)$ is γ', which is itself a geodesic of TM.

Any metrics g on M gives rise to a natural metrics (the "Sasaki metrics") g_T on TM.

Iteration: starting from a Riemannian manifold (M, g),

$M_0 = M, g_0 = g$, $M_{n+1} = TM_n, g_{n+1} = (g_n)_T$ and $p_{n+1} : (M_{n+1}, g_{n+1}) \to (M_n, g_n)$.

Finally, the connection ∇ allows one to define the *parallel transport* $J^{\gamma}_{t,t'}$ along a curve γ of any vector $v \in T_{\gamma(t)}M$ to some vector in $T_{\gamma(t')}M$.

This is essential to connect possible worlds of the same level but lying above different points (= belonging to different systems of possibility).

(M, g) is not enough to get a modal frame. We must add a family $\mathrm{Ac}(M) = \{\gamma_i : i \in I\}$ of "accessibility curves" on M.

For $x \in M$, $A^1(x) := \bigcup_{x \in \overline{\gamma_i}} \overline{\gamma_i}$ = set of accessible worlds from x.

<u>Problem</u>: recall that possible worlds *relative to x* should be elements of TM, not of M.

Indeed, points of M = possible worlds of order 0, points of TM = worlds of order 1, ...

Solution: there is a way of coding the points of M accessible from x with elements of TM, using the "exponential map".

For $v \in T_xM$, let c_v be the geodesic of M such that $c_v(0) = x$ and $c'(0) = v$.

$P^1(x) := \{v \in T_xM : c_v(1) \in A^1(x)\}$ is the representant in T_xM of $A^1(x)$.

$\pi^1(x) := \{c'_v(t) : v \in P^1(x), t \in \mathbb{R}\}$ is then the extension to TM (beyond T_xM) of all worlds accessible from x. (The map $(v, t) \mapsto c_v(t)$ is the "geodesic flow" of M.)

$\pi^1(x)$ = set of all possible worlds in TM which are relative to $x \in M$.

$P^1(x)$ = set of isolated 1-worlds relative to x.

$\pi^1(x)$ = set of curves of 1-worlds relative to x.

The construction of π^1 can be iterated:
$\pi^2 : TM \to \{\text{curves of } TTM\}$, π^3, ...

Indeed, (M, g) is replaced by (TM, g^T), accessibility curves γ_i on M are replaced by 2-accessibility curves $J_w^{\gamma_i} : t \mapsto J_{t_w,t}^{\gamma_i}(w)$ (for all $w \in \pi^1(x)$) on TM, and so on.

Last task: the definition of the modal and the nonmodal lifts.

To that end, one introduces, for each n, a selection $\mathrm{Ad}_n(M)$ of *admissible curves* on M_n: the interpretation of each formula of modal degree n is required to be a subset of $\mathrm{Ad}_n(M)$.

One only asks:

- if $\delta \in \mathrm{Ad}_{n+1}(M)$, then $p_{n+1}(\delta) \in \mathrm{Ad}_n(M)$
- for any $\gamma \in \mathrm{Ad}_n(M)$, there exists $\delta \in \mathrm{Ad}_{n+1}(M)$ that lifts γ (i.e., $p_{n+1}(\delta) = \gamma$).

Then, any curve γ on M_n gives rise to two sets of curves on M_{n+1}:

- $\lambda(\gamma) = \{\tilde{\gamma}^{v_0} \in \mathrm{Ad}_{n+1}(M) : v_0 \in T_{\gamma(0)} M_n\}$
- $L(\gamma) = \{\delta \in \mathrm{Ad}_{n+1}(M) : p_{n+1} \circ \delta = \gamma\}$

Since the difference in modal degree between two subformulas of a same formula can be > 1, one defines iterated nonmodal lifting: $L_n^n = \mathrm{id}$, $L_n^{n+m} = L_{n+m-1}^{n+m} \circ \ldots \circ L_n^{n+1}$.

We are done: a *metrical modal frame* is a structure $\underline{M} = \langle M, g, \mathrm{Ac}(M), (\mathrm{Ad}_n(M))_{n\geq 0}\rangle$.

A valuation V on $\underline{M}$ is the assignment, to any propositional variable p, of the set of all the continuous portions (defined up to reparametrization) of certain members of $\mathrm{Ad}_0(M)$.

(So $\overline{V(p) \cap V(q)} = \overline{V(p)} \cap \overline{V(q)}$

and $\overline{\mathrm{Ad}_0(M) \setminus V(p)} = \overline{\mathrm{Ad}_0(M)} \setminus \overline{V(p)}$.)

A metrical modal frame endowed with a valuation is a *metrical modal model*.

$\overline{\gamma}$ = set of all points of γ

For any set Γ of curves: $\overline{\Gamma} = \bigcup_{\gamma \in \Gamma} \overline{\gamma}$,
$\lambda(\Gamma) = \bigcup_{\gamma \in \Gamma} \lambda(\gamma)$ and $L(\Gamma) = \bigcup_{\gamma \in \Gamma} L(\gamma)$.

The valuation V is extended to all formulas by induction:

- $V(\neg\phi) = \mathrm{Ad}_n(M) \setminus V(\phi)$ (for $\deg(\phi) = n$)
- $V(\phi \wedge \psi) = V(\phi) \cap L^n_m(V(\psi))$ (for $\deg(\phi) = n$ and $\deg(\psi) = m < n$)
- $V(\phi \vee \psi) = V(\phi) \cup L^n_m(V(\psi))$ (same hypothesis)
- $V(\Diamond\phi) = \lambda(V(\phi))$

Finally, for any formula ϕ of degree n and for any $x \in M$, one defines:

$$\langle \underline{M}, V \rangle, x \vDash \phi \ \text{ iff } \ \pi^n(x) \cap \overline{V(\phi)} \neq \emptyset$$

A semantics faithful to modal widening is now completely set up.

Remark: if $y \in \pi^n(x)$ is in $\overline{V(\phi)}$, then there exists $z \in \pi^{n+1}(x)$ in $\overline{L(V(\phi))}$, so if a formula is true at x, all its nonmodal lifts will be true at all higher levels. Nonmodal lifting does not cause us to lose anything of the lifted formula.

Axioms **T** and **K** turn out to be valid.

Axiom **4** ($\Diamond\Diamond p \to \Diamond p$) is not valid, which is in order.

($M, x \models \Diamond p$ requires that $\lambda(V(p))$ contains at least one point of $\pi^1(x)$. But $M, x \models \Diamond\Diamond p$ requires only that $\lambda(\lambda(V(p)))$ contains an element of $\pi^2(w)$ for some $w \in \pi^1(x)$, and w can very well be other than x.)

- If $\langle M, g \rangle$ is simple (any two points are linked by a single geodesic),
- if $\mathrm{Ac}(M)$ is composed of geodesics only,
- if $\mathrm{Ac}(M) \subseteq \mathrm{Ad}_0(M)$,
- if $\mathrm{Ad}_0(M)$ constitutes a partition of M
- and, for $n \geq 2$, $\mathrm{Ad}_n(M) =$ set of all horizontal lifts, passing through an horizontal vector, of all members of $\mathrm{Ad}_{n-1}(M)$,

then $\underline{M}$ will be called *minimal*.

Proposition

Axiom **4** *is valid in any minimal frame.*

S4* := S4 minus the necessitation rule and with the rule of uniform substitution restricted to the case where the substituens and the substituendum have the same modal degree.

Conjecture

S4 is complete w.r.t. the class of all minimal frames.*

At any rate, axiom **4** cannot hold but only in very special (sparse) frames.

Conclusion

Other frameworks could be considered, in particular "contact geometry" (contact manifolds instead of Riemannian manifolds).

No easy completeness results. But the point was mainly philosophical: one can make sense of modalization as inducing a "change of scale".

Kremer, P. and Mints, G. (2005).
Dynamic topological logic.
Annals of Pure and Applied Logic, 131:133–158.

Kripke, S. (1963).
Semantical considerations on modal logic.
Acta Philosophica Fennica, 16:83–94.

Masini, A. (1992).
2-sequent calculus : a proof-theory of modalities.
Annals of Pure and Applied Logic, 58.

Venema, Y. (1999).
Points, lines and diamonds: a two-sorted modal logic for projective planes.
Journal of Logic and Computation, 9(5):601–621.