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On nodal domains and higher-order Cheeger inequalities of finite reversible Markov processes

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Abstract

Let L be a reversible Markovian generator on a finite set V. Relations between the spectral decomposition of L and subpartitions of the state space V into a given number of components which are optimal with respect to min–max or max–min Dirichlet connectivity criteria are investigated. Links are made with higher-order Cheeger inequalities and with a generic characterization of subpartitions given by the nodal domains of an eigenfunction. These considerations are applied to generators whose positive rates are supported by the edges of a discrete cycle $\mathbb{Z}_N$, to obtain a full description of their spectra and of the shapes of their eigenfunctions, as well as an interpretation of the spectrum through a double-covering construction. Also, we prove that for these generators, higher Cheeger inequalities hold, with a universal constant factor 48.

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1. Introduction and results

Classical Cheeger inequalities for finite reversible Markov processes make a link between the spectral gap and the connectivity constants, which are obtained by minimizing isoperimetric-type quotients over decompositions of the state space into two disjoint parts. There is a significant literature on this subject; see for instance [1,8,9,11,12,18,20–22]. The purpose of this article is to obtain such inequalities between the whole spectrum and decompositions of the state space into *several* parts, when the underlying graph is a cycle. For this we keep on developing the general approach suggested in [19], where the case of trees is treated, which is based on an intermediate Dirichlet connectivity spectrum in a corresponding metric model which is defined through taking into account the first Dirichlet eigenvalues associated with the elements of the decompositions. We also investigate the relations between this intermediate spectrum and the nodal domains of the eigenfunctions of the finite reversible Markov process under consideration. Nevertheless, to deduce the shape of these eigenfunctions when the underlying graph is a cycle, new intermediate quantities will have to be considered.

While we were writing this article, we learned that the Dirichlet connectivity spectrum had already been studied in the continuous context of Laplace–Beltrami operators on Euclidean or Riemannian subdomains with Dirichlet boundary conditions (e.g. see [15] by Helffer et al. and the references therein); however, it seems that the motivations in their context are far from ours, since the regularity and the geometry of the boundaries of the elements of the minimal decompositions are important in their study, while our main motivation is Conjecture 3 described in the sequel, that can also easily be extended to the continuous state space situation.[1] In what follows, we present a thorough study of the minimizing decompositions for the different spectra introduced and we also study their relationships to nodal domains of the corresponding eigenfunctions. To our knowledge, the results on the spectral decomposition of Markov generators on cycles are new and they can be extended to diffusion generators on the cycle which can be written in divergence form (this corresponds to the reversibility assumption).

In the rest of this section we first go through some preliminary definitions and background and then we present an overview of what appears in the forthcoming sections of this article.

1.1. Preliminary definitions and background

In what follows $\mathbb{N}$ and $\mathbb{R}$ are the sets of natural and real numbers, respectively, and for $a, b \in \mathbb{N}$, we define $[\![a, b]\!] := \{a, a+1, \ldots, b\}$ and $[\![b]\!] := [\![1, b]\!]$.

Let V be a finite set of cardinal $N \in \mathbb{N}$, and consider a Markovian generator $L := (L(x, y))_{x,y\in\mathsf{V}}$, i.e. a matrix whose entries are non-negative outside the diagonal and whose row sums are all equal to zero, with $|L| := \max_{x\in\mathsf{V}} |L(x, x)|$. Also, we assume that there exists a positive (nowhere zero) probability measure μ on V such that L is reversible with respect to μ, i.e.

$$\forall\, x \neq y \in \mathsf{V}, \quad \phi(\{x, y\}) := \mu(x)L(x, y) = \mu(y)L(y, x). \tag{1}$$

This assumption implies that L is diagonalizable in $\mathbb{R}$. Let $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ be the eigenvalues with multiplicities of $-L$. The motivation for the study of higher-order

[1] The finite graph context seems more convenient for approaching this problem and if Conjecture 3 is true in its present form, then via usual approximation procedures, it will also be satisfied in a general Markov process framework encompassing diffusion generators.

Cheeger inequalities is to compare these eigenvalues with connectivity related quantities such as isoperimetric numbers (e.g. see [10] and references therein).

As another consequence of the reversibility of L, we have

$$\forall\, x \neq y \in \mathsf{V}, \quad L(x, y) > 0 \Leftrightarrow L(y, x) > 0$$

which leads us to endow V with an undirected and loopless graph structure[2] whose edge set is

$$\mathsf{E} := \{\{x, y\} : L(x, y) > 0\}.$$

Hereafter, this graph is denoted by $\mathsf{G} := (\mathsf{V}, \mathsf{E})$. Let $\mathscr{D}_1(\mathsf{V})$ be the set of all non-empty subsets $\mathsf{A} \subset \mathsf{V}$ and for $k \in [\![2, N]\!]$, consider $\mathscr{D}_k(\mathsf{V})$ the set of k-tuples $\mathcal{A} := (\mathsf{A}_1, \ldots, \mathsf{A}_k)$ of disjoint elements from $\mathscr{D}_1(\mathsf{V})$, called *k-subpartitions*. With any $\mathsf{A} \in \mathscr{D}_1(\mathsf{V})$, we associate its *connectivity* defined by

$$\iota(\mathsf{A}) := \frac{\phi(\partial \mathsf{A})}{\mu(\mathsf{A})}$$

where ϕ is considered as a positive measure defined on E as in Eq. (1), and

$$\partial \mathsf{A} := \{\{x, y\} \in \mathsf{E} : x \in \mathsf{A} \text{ and } y \notin \mathsf{A}\}.$$

Hence, $\phi(\partial \mathsf{A})$ should be interpreted as a measure of the discrete boundary of A. Now, for any $k \in [\![N]\!]$, we introduce the *kth-order isoperimetric constant* as

$$I_k := \min_{(\mathsf{A}_1,\ldots,\mathsf{A}_k)\in\mathscr{D}_k(\mathsf{V})} \max_{j\in[\![k]\!]} \iota(\mathsf{A}_j).$$

The family $(I_k)_{k\in[\![N]\!]}$ is sometimes called the *isoperimetric spectrum* of L. One may naturally ask about the relationship between these numbers and the usual spectrum $(\lambda_k)_{k\in[\![N]\!]}$, and in particular one may ask about the correctness of the following higher-order Cheeger inequalities.

Conjecture 1. *For any $k \in \mathbb{N}$, there exists a universal constant $\chi(k) > 0$ such that for any finite reversible generator L as above, we have*

$$\forall\, k \in [\![N]\!], \quad \chi(k)\frac{I_k^2}{|L|} \leq \lambda_k \leq 2I_k. \quad \square \tag{2}$$

The interested reader is referred to [19,10] for some motivations and background in this regard. It is easy to verify the upper bound $\lambda_k \leq 2I_k$ using the variational formulation of eigenvalues for all $k \in [\![N]\!]$. Also, the case $k = 2$ is well-known and corresponds to the traditional discrete Cheeger inequality with $\chi(2) = 1/2$ (see [18]).[3] It is furthermore easy to verify that the bound $I_N^2/|L| \leq \lambda_N$ is always true. Indeed, up to a change of indices, $\mathscr{D}_N(\mathsf{V})$ is just the family of singleton subsets. But for any $x \in \mathsf{V}$, we have $\iota(\{x\}) = |L(x, x)|$ and it follows that $I_N = |L|$. On the other hand, we have $\lambda_N = \max_{\mathsf{f}\neq 0}(-\mu[\mathsf{f}L[\mathsf{f}]])/\mu[\mathsf{f}^2]$, and consequently, by considering indicator functions of points, we have $\lambda_N \geq |L|$. Moreover, it can be seen that Conjecture 1 is true with the constant $\chi(k) \equiv 1/2$ independent of $k \in \mathbb{N}$ if the graph G is acyclic (see the

[2] Note that this graph structure is not a random walk graph but just a graph that contains the information of positive entries of L. Moreover, note that ϕ as a measure on the set E also defines a flow on this graph.

[3] This bound was first designed in [5] for Riemannian manifolds, and was recently improved to $\chi(2) = 1$ in [20], and it is explained in [8] how to go beyond bounded jump rates on infinite state spaces.

results of Section 2 along with [19] that provide a complete proof. Also, see [10] for a proof in the generic case).

In [19], the spectrum $(\underline{\Lambda}_k)_{k\in[\![N]\!]}$ is introduced, which in some sense is midway between $(\lambda_k)_{k\in[\![N]\!]}$ and $(I_k)_{k\in[\![N]\!]}$. To recall its definition, we first associate a *metric graph* G with the discrete graph G for which $V := \mathsf{V}$ and each edge $\{x, y\} \in \mathsf{E}$ is replaced by a "solid" segment $[x, y]$ of length 1 and we define $G := \cup_{\{x,y\}\in\mathsf{E}}[x, y]$ in which the boundary points of these edge-segments corresponding to a given vertex $x \in \mathsf{V}$ are all identified with a unique point still designated by x (for a similar construction, see [14]). Clearly the edge set of G is defined as $E := \{[x, y] : \{x, y\} \in \mathsf{E}\}$. Also, we refer to any general element of G as a *point*, while the elements of $V = \mathsf{V}$ are referred to as *vertices*. In this setup, the neighborhood of a point $a \in G$ is defined as

$$N(a) := \{x \in \mathsf{V} : \exists\, \{x, y\} \in \mathsf{E},\ a \in [x, y]\}.$$

As a general remark on the notation, we use italic letters (e.g. V, E, f) to refer to objects in the metric model where sans-serif letters (e.g. $\mathsf{V}, \mathsf{E}, \mathsf{f}$) are used to refer to discrete objects. In this setup, by abuse of notation, we use the same symbols for all operators that appear in both metric and discrete models, where the operand will clarify the exact definition of these concepts (e.g. $\mathcal{E}(\mathsf{f})$ (defined below) stands for the energy of a (discrete) function f that is defined on the set of vertices V while $\mathcal{E}(f)$ refers to the energy of a function defined on the metric graph G). Also, if g is a function defined on the metric graph, then g which is defined on the vertex set V of the discrete graph stands for the restriction $g|_{\mathsf{V}}$.

For any $\{x, y\} \in \mathsf{E}$, the segment $[x, y] \in E$ is endowed with the measure $\nu_{x,y} := \phi(\{x, y\})d_{x,y}$, where $d_{x,y}$ is the natural Lebesgue measure on $[x, y]$. We define $d := \sum_{\{x,y\}\in\mathsf{E}} d_{x,y}$ and $\nu := \sum_{\{x,y\}\in\mathsf{E}} \nu_{x,y}$, which are non-negative measures whose total masses are $|\mathsf{E}|$ and $1/2 \sum_{x\in V} \mu(x)\, |L(x, x)|$, respectively. The measure ν enables us to define a Dirichlet form $\mathcal{E}$ on the space $\mathscr{F}(G)$ of absolutely continuous real functions defined on G (i.e. that are absolutely continuous on all edge-segments) via

$$\forall\, f \in \mathscr{F}(G), \quad \mathcal{E}(f) := \int (f')^2\, d\nu \in \mathbb{R}_+ \sqcup \{+\infty\}$$

where f' stands for the weak derivative of f. Analogously, we may define a discrete Dirichlet form on the space $\mathscr{F}(\mathsf{V})$ of real functions defined on V as

$$\forall\, \mathsf{f} \in \mathscr{F}(\mathsf{V}), \quad \mathcal{E}(\mathsf{f}) := \sum_{\{x,y\}\in\mathsf{E}} \left|\mathsf{f}(x) - \mathsf{f}(y)\right|^2 \phi(\{x, y\}) \in \mathbb{R}_+.$$

The probability measure μ is naturally extended to G via the formula $\mu = \sum_{x\in V} \mu(x)\delta_x$ (where we use the same notation for both discrete and metric models), and for a function $f \in \mathscr{F}(G)$ we define

$$\mu[f] = \mu[\mathsf{f}] := \sum_{x\in\mathsf{V}} \mathsf{f}(x)\mu(x).$$

Following our notation, let $\mathscr{D}_1(G)$ (or $\mathscr{D}_1$ for short) be the set of subsets $A \subset G$ which are open and connected, and whose intersection with V is non-empty. For such a subset $A \in \mathscr{D}_1(G)$, let ∂A denote the boundary of A (in the usual topological sense) and also let $\mathscr{F}_0(A)$ be the subspace of $\mathscr{F}(G)$ consisting of functions that vanish on the complementary set $G \setminus A$. Also, for a subset $\mathsf{A} \in \mathscr{D}_1(\mathsf{V})$, $\mathscr{F}_0(\mathsf{A})$ can be defined analogously. Now, we are ready to introduce the (metric and discrete) *principal Dirichlet eigenvalues*, $\lambda_1(A)$ and $\lambda_1(\mathsf{A})$, as follows:

$$\forall\, A \in \mathscr{D}_1(G), \quad \lambda_1(A) := \inf_{f\in\mathscr{F}_0(A):\mu[f^2]\neq 0} \frac{\mathcal{E}(f)}{\mu[f^2]}, \tag{3}$$
$$\forall\, \mathsf{A} \in \mathscr{D}_1(\mathsf{V}), \quad \lambda_1(\mathsf{A}) := \inf_{\mathsf{f}\in\mathscr{F}_0(\mathsf{A}):\mu[\mathsf{f}^2]\neq 0} \frac{\mathcal{E}(\mathsf{f})}{\mu[\mathsf{f}^2]}.$$

These quantities should be interpreted as a measurement of the ease of getting out of A or A for the underlying process (see [19] for a precise description in which one has to introduce instantaneous points to deal with the difference between G and G). Also, if $A \in \mathscr{D}_1$, then $f_A \in \mathscr{F}_0(A)$ stands for the unique minimizing positive function in (3) satisfying $\mu[f_A^2] = 1$ (its positivity and uniqueness come from the connected assumption for A and the Perron–Frobenius theorem; see [19] for the details).

Now, in order to get a better understanding of the minimizers of (3), first, we define $\widetilde{L} : V \times G \to \mathbb{R}$ as an extension of L, in the following way.

- For every $(x, y) \in V \times V$ we have $\widetilde{L}(x, y) = L(x, y)$.
- For any $\{x, y\} \in \mathsf{E}$ and any $z \in G \setminus V$ on the edge-segment $[x, y]$, we define $\widetilde{L}(x, z) := L(x, y)/d([x, z])$, where d is the natural measure on G introduced earlier (throughout the whole paper, d will only be used to measure distances inside edge-segments). For any other $x' \in V \setminus \{x, y\}$, we let $\widetilde{L}(x', z) := 0$.

Moreover, define $\widetilde{\phi}(x, z) := \mu(x)\widetilde{L}(x, z)$ as an extension of ϕ to $V \times G$. For any $A \in \mathscr{D}_1$, let $\mathsf{A} := A \cap V$ and construct the linear operator $\widehat{L}_A = (\widehat{L}_A(x, y))_{x,y\in\mathsf{A}}$ defined on $\mathscr{F}_0(\mathsf{A})$ as

$$\forall\, x, y \in \mathsf{A} \quad \widehat{L}_A(x, y) := \begin{cases} L(x, y) & x \neq y \\ -\displaystyle\sum_{x\neq z\in\mathsf{A}\cup\partial A} \widetilde{L}(x, z) & x = y. \end{cases}$$

For a function $f \in \mathscr{F}(G)$, a subset $A \subseteq G$ is said to be a *nodal domain* of f if it is a connected component of $G \setminus \{x \in V : f(x) = 0\}$. Also, by a nodal domain[4] of a function $\mathsf{f} \in \mathscr{F}(\mathsf{V})$, we mean a nodal domain of the function f, the affine extension of f on edge-segments of G.

Now, let f_A be the restriction to A of the unique minimizing non-negative function f_A in (3) that satisfies $\mu[f_A^2] = 1$. We recall the following result from [19].

Lemma 2. *For any $A \in \mathscr{D}_1$, f_A is the unique positive function defined on $\mathsf{A} := A \cap V$ satisfying $\mu[\mathsf{f}_A^2] = 1$ and $\widehat{L}_A[\mathsf{f}_A] = -\lambda_1(A)\mathsf{f}_A$. Also, we have*

$$\lambda_1(A) = \min_{\mathsf{f}\in\mathscr{F}_0(\mathsf{A})\setminus\{0\}} \frac{-\mu[\mathsf{f}\widehat{L}_A[\mathsf{f}]]}{\mu[\mathsf{f}^2]}$$

for which f_A is a minimizing function whose affine extension is exactly the minimizer of (3), f_A. In addition, if A is a nodal domain of an eigenfunction $\mathsf{g} \neq 0$ associated with an eigenvalue λ of $-L$, then $\lambda_1(A) = \lambda$ and g is proportional to f_A on A.

[4] This is defined by abuse of language in this article and is *different* from the standard definition of the discrete nodal domain (see [3] or [4]).

Now, for any $k \in [\![N]\!]$, consider $\mathscr{D}_k(G)$ (or $\mathscr{D}_k$ for short) as the set of k-tuples $\mathcal{A} := (A_1, \ldots, A_k)$ of disjoint elements from $\mathscr{D}_1(G)$ (such k-tuples will be called *k-subpartitions*). The intermediate quantities mentioned before are defined, for any $k \in [\![N]\!]$, by

$$\underline{\Lambda}_k := \inf_{(A_1,\ldots,A_k)\in\mathscr{D}_k(G)} \left(\max_{j\in[\![k]\!]} \lambda_1(A_j) \right). \tag{4}$$

Hereafter, we refer to the family $(\underline{\Lambda}_k)_{k\in[\![N]\!]}$ as the *Dirichlet connectivity spectrum* of L. Again, via the variational formulation of the eigenvalues, it is not difficult to see that the bounds $\lambda_k \leq \underline{\Lambda}_k$ are always true, for all $k \in [\![N]\!]$. But the following was left as a conjecture (see [19]),

Conjecture 3 (*[19]*)**.** *For any $k \in \mathbb{N}$, there exists a universal constant $\widetilde{\chi}(k) > 0$ such that for any finite generator L as above, we have*

$$\forall\, k \in [\![N]\!], \quad \widetilde{\chi}(k)\underline{\Lambda}_k \leq \lambda_k. \quad \square \tag{5}$$

Clearly by the Perron–Frobenius theorem and Lemma 2, we have $\underline{\Lambda}_2 \leq \lambda_2$ and consequently $\underline{\Lambda}_2 = \lambda_2$. Also, by considering the N-subpartition corresponding to bisection of all edges, we have $\underline{\Lambda}_N \leq 2\,|L| \leq 2\lambda_N$. Furthermore, one of the main results of [19] states that if the graph G is a tree, then we have

$$\forall\, k \in [\![N]\!], \quad \underline{\Lambda}_k = \lambda_k. \tag{6}$$

Remark 4. In [19], the definition of the metric graph, say $\widehat{G}$, associated with $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ is slightly different. There, the segment $[x, y] \in E$ has length $1/\phi(\{x, y\})$ and is endowed with the corresponding Lebesgue measure and these conventions lead as above to a Dirichlet form $\widehat{\mathcal{E}}$ on the space $\mathscr{F}(\widehat{G})$ of absolutely continuous functions on $\widehat{G}$. But let $\psi : G \to \widehat{G}$ be the bijective mapping which, for any edge $\{x, y\} \in \mathsf{E}$, transforms affinely the segment $[x, y] \subset G$ into the corresponding segment $[x, y] \subset \widehat{G}$. The composition mapping

$$f \in \mathscr{F}(\widehat{G}) \mapsto f \circ \psi \in \mathscr{F}(G)$$

is then an isomorphism of vector spaces and we have

$$\forall\, f \in \mathscr{F}(\widehat{G}), \quad \widehat{\mathcal{E}}(f) = \mathcal{E}(f \circ \psi).$$

This relation can be used to translate the results obtained in [19] for $\widehat{G}$ into corresponding ones for G. $\square$

1.2. The organization of forthcoming sections

Here we introduce the sequence of results that we are going to prove in this article, which will appear in the next five sections each devoted to one of the main propositions described below.

It was mentioned in [19], without much hint of a proof, that Conjecture 3 implies Conjecture 1. Our first task will be to provide all the overlooked arguments in Section 2, where we show that to prove Conjecture 1, it is sufficient to prove Conjecture 3 for irreducible generators L (i.e. generators L whose associated graph G is connected). More precisely:

Proposition 5. *Given an integer $k \in [\![N]\!]$ and a generator L, if Inequality (5) is true for L and k, then Inequality (2) is true for L and k with $\chi(k) = \widetilde{\chi}(k)/2$.*

In particular, we recover that Conjecture 1 is true with the constant $\chi(k) \equiv 1/2$ independently of $k \in \mathbb{N}$, if we restrict its assertion to the class of generators whose associated graph is acyclic.

The fact underlying the equality $(\underline{\Lambda}_k)_{k\in[\![N]\!]} = (\lambda_k)_{k\in[\![N]\!]}$ for generators whose associated graph is a tree is that a minimizing subpartition $\mathcal{A} \in \mathscr{D}_k$ in (4) corresponds (at least generically) to the nodal domains of an eigenfunction associated with λ_k. For a better understanding of Conjecture 3, it seems important to be able to verify whether a subpartition $\mathcal{A} \in \mathscr{D}_k(G)$ corresponds to the nodal domains of an eigenfunction. To get a result in this direction, we need to introduce some notions.

A k-subpartition $\mathcal{A} = (A_1, \ldots, A_k) \in \mathscr{D}_k$ is said to be *handy* if for any pair of distinct indices i, j in $[\![k]\!]$ we have $\partial A_i \cap \partial A_j \cap V_3 = \emptyset$, where V_3 is the set of vertices of G whose degrees are at least 3. An eigenfunction of a generator L is said to be *handy* if the collection of its nodal domains constitutes a handy subpartition. Also, the generator L itself is said to be *handy* if any eigenvalue λ of multiplicity m admits m independent handy eigenfunctions. It is instructive to note that if none of the eigenfunctions of a generator L vanishes on V then L is clearly a handy generator (in particular the former property also implies that all eigenvalues have multiplicity 1, since in an eigenspace of dimension 2 it is always possible to construct a function that vanishes on x, for any given $x \in V$).

The *residual set* A_0 of a given subpartition $\mathcal{A} = (A_1, \ldots, A_k) \in \mathscr{D}_k$, with $k \in [\![N]\!]$, is defined as the complementary set of the union of the closures of the A_i, for $i \in [\![k]\!]$. The subpartition $\mathcal{A}$ is said to be a *k-partition* if its residual set A_0 is empty and for any $x \in \partial\mathcal{A}$ (when $\mathcal{A}$ is a subpartition, by convention its boundary is the union of the boundaries of its components), there exist $i \neq j \in [\![k]\!]$ such that $x \in \partial A_i \cap \partial A_j$. We use $\mathscr{P}_k(G)$ (or $\mathscr{P}_k$ for short) to refer to the set of all such k-partitions.

The subpartition $\mathcal{A} = (A_1, \ldots, A_k) \in \mathscr{D}_k$ is said to be *uniform* if for any $i \neq j \in [\![k]\!]$ we have $\lambda_1(A_i) = \lambda_1(A_j)$, where this common value is denoted by $\lambda_1(\mathcal{A})$ (this is also the value of $\lambda_1(A_1 \sqcup A_2 \sqcup \cdots \sqcup A_k)$ if one directly uses the definition as expressed in (3)).

A pair of positive real numbers (r_1, r_2) is said to be *rectifying* for a point $a \in G$ with respect to $(A_1, A_2) \in \mathscr{D}_2$ if $a \in \partial A_1 \cap \partial A_2$, $N(a) \cap A_1 = \{a^-\}$ and $N(a) \cap A_2 = \{a^+\}$, such that

$$r_1 f_{A_1}(a^-)\, \widetilde{\phi}(a^-, a) = r_2 f_{A_2}(a^+)\, \widetilde{\phi}(a^+, a), \tag{7}$$

where f_A is defined as the minimizer of (3).[5]

We say that the pair (r_1, r_2) is *rectifying* for (A_1, A_2) if it is rectifying for all points of the set $\partial A_1 \cap \partial A_2$. Also, a subpartition $\mathcal{A} \in \mathscr{D}_k$, with $k \in [\![N]\!]$, is said to be *rectifiable* if there exists a family $(r_i)_{i\in[\![k]\!]}$ such that for any $i \neq j \in [\![k]\!]$, (r_i, r_j) is rectifying for (A_i, A_j).

Finally, with a subpartition $\mathcal{A} = (A_1, \ldots, A_k) \in \mathscr{D}_k$, we associate the graph $\mathsf{G}_{\mathcal{A}}$ whose vertex set is $\{A_1, \ldots, A_k\}$ and whose edge set consists of the $\{A_i, A_j\}$, with $i \neq j \in [\![k]\!]$, such that $\partial A_i \cap \partial A_j \neq \emptyset$. Then $\mathcal{A}$ is said to be *bipartite* if $\mathsf{G}_{\mathcal{A}}$ is a bipartite graph.

With all of these definitions, we will show:

Proposition 6. *Given $k \in [\![N]\!]$, and a handy partition $\mathcal{A} \in \mathscr{P}_k$, then this partition corresponds to the nodal domains of a handy eigenfunction of $-L$ if and only if it is uniform, rectifiable and bipartite. This eigenfunction is then associated with the eigenvalue $\lambda_1(\mathcal{A})$.*

[5] The concept of rectifiability can be generalized to the case where $\mathrm{card}(N(a) \cap A_1) > 1$ or $\mathrm{card}(N(a) \cap A_2) > 1$. However, this will not be of any use in the sequel, since we will be dealing with just handy subpartitions.

Among the previous conditions, the rectifiability may seem to be the more difficult to check. That is why the Dirichlet connectivity spectrum $(\underline{\Lambda}_k)_{k\in[\![N]\!]}$ is interesting, since its corresponding minimizing subpartitions provide rectifiable subpartitions. But there are other ways to find rectifiable subpartitions and one of them which will be fruitful for the study of Markov processes on cycles is the following. For $k \in [\![N]\!]$ define

$$\overline{\Lambda}_k := \sup_{(A_1,\ldots,A_k)\in\mathscr{P}_k} \left(\min_{j\in[\![k]\!]} \lambda_1(A_j)\right).$$

There is no general comparison between $\underline{\Lambda}_k$ and $\overline{\Lambda}_k$. But some relations can be deduced in special cases (see Proposition 15). Hereafter, we assume that G is connected, and we will prove:

Proposition 7. *For $k \in [\![N]\!]$, let $\mathcal{A} \in \mathscr{D}_k$ be either a minimizing subpartition for $\underline{\Lambda}_k$ or a maximizing partition for $\overline{\Lambda}_k$. If $\mathcal{A}$ is handy, then $\mathcal{A}$ is a uniform and rectifiable partition in $\mathscr{P}_k$.*

But there are some important differences between the optimizing subpartitions and partitions for $\underline{\Lambda}_k$ and $\overline{\Lambda}_k$, respectively. It has been shown in [19] that a minimizer $\mathcal{A} \in \mathscr{D}_k$ for $\underline{\Lambda}_k$ always exists and that it is possible to choose one which is uniform. However, the situation is not so nice for $\overline{\Lambda}_k$, since it may happen that there is no maximizing partition at all. In particular it is known that (see [19]) if a minimizing subpartition for $\underline{\Lambda}_k$ is a handy subpartition then it is actually a uniform partition in $\mathscr{P}_k$, and consequently, we only have to prove the rectifiability condition in the above proposition. Also, by Proposition 6, such a partition has all the required properties to correspond to the nodal domains of a handy eigenfunction of L, except being bipartite. In the continuous framework of Laplace–Beltrami operators in dimension 2, Helffer et al. [15] made a similar observation concerning the bipartiteness condition, under regularity and geometric requirements (in particular the nodal lines must intersect with equal angles).

In this article we focus on the case of handy subpartitions, and in order to be able to generalize our results to an arbitrary kernel we have to make sure that handy kernels are generic. More precisely, let the positive probability μ and the connected graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ be fixed. We denote by $\mathcal{L}(\mu, \mathsf{G})$ the open and convex set of generators which are reversible with respect to μ and whose associated graph is G. We say that a property is *generically* true if it is satisfied for generators belonging to a dense subset of $\mathcal{L}(\mu, \mathsf{G})$ (which is endowed with its natural pointwise topology). For instance we believe in the following assertions.

Conjecture 8. *If G is connected, and $k \in [\![N-1]\!]$, the following statements are generically true.*[6]

(a) *Any minimizing subpartition $\mathcal{A} \in \mathscr{D}_k$ for $\underline{\Lambda}_k$ is handy.*
(b) *There exists a maximizing partition $\mathcal{A} \in \mathscr{P}_k$ for $\overline{\Lambda}_k$ which is handy.*
(c) *Any generator $L \in \mathcal{L}(\mu, \mathsf{G})$ is handy.* □

The results of [19] and Proposition 15 imply that this conjecture is true if G is a tree. Note that the case $k = N$ is not relevant, because, although any partition in $\mathscr{P}_N$ is necessarily handy, we will see in Section 5 that $\overline{\Lambda}_N = +\infty$ if G is a cycle, and consequently, there is no maximizing partition $\mathcal{A} \in \mathscr{P}_N$ for $\overline{\Lambda}_N$ in this case. If the above conjecture was true more generally, it would show that generically, only the bipartiteness condition would be restrictive for a minimizing

[6] Actually, in both parts (a) and (b), we believe that generically all boundary points are in $G \setminus V$.

subpartition of $\underline{\Lambda}_k$ or for a maximizing partition of $\overline{\Lambda}_k$ to correspond to the nodal domains of a handy eigenvector of L.

Let us now come to the case of Markov processes on cycles. For them the graph G is isomorphic to the graph $\mathbb{Z}_N$ endowed with its usual nearest neighbor structure. In this situation, every subpartition is handy and an interlacing property occurs, i.e. for $k \in [\![N-1]\!]$, we have $\underline{\Lambda}_k \le \overline{\Lambda}_k < \underline{\Lambda}_{k+1} \le \overline{\Lambda}_{k+1}$ (see Proposition 15). These features lead to the following properties for the spectral decomposition of L.

Proposition 9. *When* G *is a cycle, we have*

$$\forall\, k \in [\![N]\!], \quad \lambda_k = \begin{cases} 0 = \underline{\Lambda}_1 = \overline{\Lambda}_1, & \text{if } k = 1 \\ \underline{\Lambda}_k, & \text{if } k \text{ is even} \\ \overline{\Lambda}_{k-1}, & \text{if } k \neq 1 \text{ is odd.} \end{cases}$$

Moreover, for $k \in [\![N]\!]$*,* k *even, the eigenfunctions associated with* λ_k *and to* λ_{k+1} *(if* $k+1 \le N$*) have exactly* k *nodal domains, so the equality* $\lambda_k = \lambda_{k+1}$ *is only possible for* k *even and it is equivalent to the equality* $\underline{\Lambda}_k = \overline{\Lambda}_k$*. In particular the multiplicities of the eigenvalues are less than or equal to* 2*.*

As a consequence, the shape of the eigenfunctions associated with generators on cycles can be easily described. But more important for us is the following result which proves Conjecture 3 when the graph G is a cycle, and consequently, one may deduce that all cycles admit a higher-order Cheeger inequality for all of their eigenvalues with a universal constant 48.

Proposition 10. *Assume that* G *is a cycle; then we have*

$$\forall\, k \in [\![N]\!], \quad \underline{\Lambda}_k \le \begin{cases} \lambda_k, & \text{if } k = 1 \text{ or } k \text{ is even} \\ 24\, \lambda_k, & \text{if } k \ge 3 \text{ is odd.} \end{cases}$$

Remark 11. After this paper was submitted for publication, we discovered from an article of Fernandes and Fosseca [13] that the last assertion of Proposition 9 holds more generally for Hermitian matrices whose associated graph is a cycle. About the same time, we learned that variants of the notions of handiness and uniformity of partitions were also introduced (under the names of proper partitions and equipartitions) in a preprint by Band et al. [2], in order to study nodal deficiencies of quantum graphs. Although the operators under study are not the same, they provided a criterion for partitions corresponding to nodal domains of eigenfunctions which shares some similarities with the one presented here, but it is rather based on the critical points of the mapping

$$\mathscr{P}_k \ni \mathcal{A} = (A_1, \ldots, A_k) \mapsto \max_{i \in [\![k]\!]} \lambda_1(A_i)$$

(see also Remark 19). In the two articles the proofs are different. □

2. On higher-order Cheeger inequalities

Our main goal in this section is to prove the fact that to prove Conjecture 1, it is enough to prove Conjecture 3 for irreducible generators. For this, first we prove Proposition 5. The approach is based on a Cheeger-type lower bound on the principal Dirichlet eigenvalue, namely a relation between the latter and a Dirichlet isoperimetric constant, which seems interesting in itself.

More precisely for $\mathsf{A} \subseteq \mathsf{V}$, we define the isoperimetric constant inside A as

$$I(\mathsf{A}) := \min_{\mathsf{B} \subset \mathsf{A} : \mathsf{B} \neq \emptyset} \iota(\mathsf{B}). \tag{8}$$

Corollary 3.2 of Lawler and Sokal [18] gives a Dirichlet analogue to the discrete Neumann-oriented Cheeger bound $\frac{I_2^2}{2|L|} \leq \lambda_2$:

$$\forall\, \mathsf{A} \subseteq \mathsf{V}, \quad \frac{I^2(\mathsf{A})}{2\,|L|} \leq \lambda_1(\mathsf{A}). \tag{9}$$

Proposition 5 is then an immediate consequence of the following corollary.

Corollary 12. *For any reversible generator L given as in the introduction, we have*

$$\forall\, k \in [\![N]\!], \quad \frac{I_k^2}{2\,|L|} \leq \underline{\Lambda}_k.$$

Proof. For any $\mathsf{C} \subseteq \mathsf{V}$, define

$$\overline{\mathsf{C}} := \bigcup_{\substack{\{x,y\} \in \mathsf{E} \\ x \in \mathsf{C}}} [x, y) \;\subseteq G.$$

According to (9), for any $A \in \mathscr{D}_1(G)$, we can find a non-empty set $\mathsf{B} \subset \mathsf{A} := \mathsf{V} \cap A$ such that

$$\frac{\iota^2(\mathsf{B})}{2\,|L|} \leq \lambda_1(\mathsf{A}) = \lambda_1(\overline{\mathsf{A}}) \leq \lambda_1(A).$$

The last inequality follows from the facts that $A \subseteq \overline{\mathsf{A}}$ and λ_1 is decreasing with respect to the set inclusion [19, Lemma 3]. It follows that for any $k \in [\![N]\!]$ and any $\mathcal{A} = (A_1, \ldots, A_k) \in \mathscr{D}_k(G)$, we can find $\mathcal{B} = (\mathsf{B}_1, \ldots, \mathsf{B}_k) \in \mathscr{D}_k(\mathsf{G})$ such that

$$\frac{\left(\max\limits_{i \in [\![k]\!]} \iota(\mathsf{B}_i)\right)^2}{2\,|L|} \leq \max_{i \in [\![k]\!]} \lambda_1(A_i).$$

The announced bound follows at once. □

In the rest of this section we focus on the case of reducible Markov generators and we show that as far as Conjectures 1 and 3 are concerned, one may confine oneself to the case of irreducible kernels whose base graphs are connected. Also, considering the reducibility condition, we will prove some relations between the parameters $\underline{\Lambda}_k$ and $\overline{\Lambda}_k$ in the special cases of trees, forests and cycles. We start with the following basic lemma.

Lemma 13. *Let $(a_i)_{i \in [\![n]\!]}$ and $(b_i)_{i \in [\![m]\!]}$ be two non-decreasing families of numbers with $a_1 = b_1 = 0$. Denote by $(c_i)_{i \in [\![n+m]\!]}$ the non-decreasing ordering (with multiplicities) of the set $\{a_i : i \in [\![n]\!]\} \cup \{b_i : i \in [\![m]\!]\}$ and define, for all $k \in [\![2, n+m]\!]$,*

$$\underline{c}_k := \min_{l \in [\![n]\!] \cap [\![k-m, k-1]\!]} \max(a_l, b_{k-l}), \qquad \overline{c}_k := \max_{l \in [\![n]\!] \cap [\![k-m, k-1]\!]} \min(a_l, b_{k-l})$$

with the convention $\underline{c}_1 = \overline{c}_1 = 0$. Then,

$$\forall\, k \in [\![n+m]\!] \quad \overline{c}_k \leq \underline{c}_k = c_k.$$

Proof. The equality $\underline{c}_1 = c_1 = \min(a_1, b_1) = 0$ is a direct consequence of our choice of $\underline{c}_1$. Fix $k \in [\![2, n+m]\!]$ and without loss of generality assume that $\underline{c}_k = a_j$ with $j \in [\![n]\!] \cap [\![k-m, k-1]\!]$ the smallest such index possible. Then,

$$\underline{c}_k \geq \max(\{a_1, \ldots, a_j\} \cup \{b_1, \ldots, b_{k-j}\}),$$

and consequently, $\underline{c}_k \geq c_k$.

On the other hand, note that $b_{k-j+1} \geq \underline{c}_k$, since otherwise, we would have $\max(a_{j-1}, b_{k-j+1}) \leq \underline{c}_k$ which contradicts our choice of j. Hence,

$$\underline{c}_k \leq \min\{a_{j+1}, \ldots, a_n\} = a_{j+1} \quad \text{and} \quad \underline{c}_k \leq \min\{b_{k-j+1}, \ldots, b_m\} = b_{k-j+1}$$

which shows that $\underline{c}_k \leq c_k$, and we have $\underline{c}_k = c_k$.

For the other inequality, let $\tilde{a}(t)$ and $\tilde{b}(t)$ be the natural piecewise affine extensions of $(a_i)_{i\in[\![n]\!]}$ and $(b_i)_{i\in[\![m]\!]}$ as functions defined on real intervals $[1, n]$ and $[1, m]$, respectively. Then, for a fixed $k \in [\![2, n+m]\!]$ we have the following two cases.

- If there exists a point $u \in [\max(1, k-m), \min(n, k-1)]$ such that $z := \tilde{a}(u) = \tilde{b}(k-u)$ then clearly we have $\overline{c}_k \leq z \leq \underline{c}_k$.
- Otherwise, on the interval $[\max(1, k-m), \min(n, k-1)]$ we either have $\tilde{a}(t) \leq \tilde{b}(t)$ or $\tilde{b}(t) \leq \tilde{a}(t)$ which shows that $\overline{c}_k \leq \underline{c}_k$. $\square$

It should be noted that for all $k \in [\![n+m]\!]$, we have $\overline{c}_k \leq \min(a_n, b_m)$, and consequently, all the elements of the set $\{a_k : k \in [\![n]\!]\} \cup \{b_k : k \in [\![m]\!]\}$ which are larger than $\min(a_n, b_m)$ will not appear in the family $(\overline{c}_k)_{k\in[\![n+m]\!]}$.

Proposition 14. *Conjecture 3 is true if it is true for finite irreducible Markov generators.*

Proof. For $i = 1, 2$, let $L^{(i)}$ be Markov generators on finite sets $\mathsf{V}^{(i)}$, reversible with respect to some positive probability measures $\mu^{(i)}$. We denote by $(\lambda_k^{(i)})_{k\in[\![N^{(i)}]\!]}$ and $(\underline{\Lambda}_k^{(i)})_{k\in[\![N^{(i)}]\!]}$ the corresponding usual and Dirichlet connectivity spectra, respectively. Consider $\mathsf{V} := \mathsf{V}^{(1)} \sqcup \mathsf{V}^{(2)}$ and let L be the generator that acts on $V^{(i)}$ as $L^{(i)}$, for $i = 1, 2$. For $a \in (0, 1)$, let $\mu = a\mu^{(1)} + (1-a)\mu^{(2)}$, and note that L is reversible with respect to μ. Also, define $N := N^{(1)} + N^{(2)}$ and let $(\lambda_k)_{k\in[\![N]\!]}$ and $(\underline{\Lambda}_k)_{k\in[\![N]\!]}$ be the usual and Dirichlet connectivity spectra of L, respectively.

Without loss of generality assume that for a non-increasing sequence $(\widetilde{\chi}(k))_{k\in\mathbb{N}}$, we have

$$\forall\, i \in \{1, 2\}\ \forall\, k \in [\![N^{(i)}]\!], \quad \widetilde{\chi}(k)\underline{\Lambda}_k^{(i)} \leq \lambda_k^{(i)}.$$

Since $V^{(1)}$ and $V^{(2)}$ are not linked by L, the spectrum of L is just the union of the spectra of $L^{(1)}$ and $L^{(2)}$. It follows from Lemma 13 that

$$\forall\, k \in [\![N]\!], \quad \lambda_k = \min_{l\in[\![N^{(1)}]\!]\cap[\![k-N^{(2)},k-1]\!]} \max(\lambda_l^{(1)}, \lambda_{k-l}^{(2)}). \tag{10}$$

For $k \in [\![2, N]\!]$ and $l \in [\![N^{(1)}]\!] \cap [\![k-N^{(2)}, k-1]\!]$, let $\mathcal{A}^{(1)} \in \mathscr{D}_l^{(1)}$ be a minimizer for $\underline{\Lambda}_l^{(1)}$ and also let $\mathcal{A}^{(2)} \in \mathscr{D}_{k-l}^{(2)}$ be a minimizer for $\underline{\Lambda}_{k-l}^{(2)}$. Construct $\mathcal{A} \in \mathscr{D}_k$ as the disjoint union of $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$. It follows that

$$\underline{\Lambda}_k \leq \max(\underline{\Lambda}_l^{(1)}, \underline{\Lambda}_{k-l}^{(2)}),$$

and taking the minimum over $l \in [\![N^{(1)}]\!] \cap [\![k - N^{(2)}, k-1]\!]$, we get

$$\underline{\Lambda}_k \leq \min_{l \in [\![N^{(1)}]\!] \cap [\![k-N^{(2)}, k-1]\!]} \max(\underline{\Lambda}_l^{(1)}, \underline{\Lambda}_{k-l}^{(2)}). \tag{11}$$

This along with (10) and our assumptions shows that

$$\forall\, k \in [\![N]\!], \quad \widetilde{\chi}(k)\underline{\Lambda}_k \leq \lambda_k.$$

On the other hand, if there are more than two irreducible components, then by an iteration of the above argument we come to the same conclusion. □

It is instructive to note that similarly one may show that Conjecture 1 is also true if it is proved for finite irreducible Markov generators.

In what follows we consider some basic inequalities that hold for the Dirichlet connectivity parameters introduced in the special cases of trees, forests and cycles.

Proposition 15. *Let L be reversible generator on the graph G. We have:*

(a) *If G is a cycle, then $0 = \underline{\Lambda}_1 = \overline{\Lambda}_1$, $\underline{\Lambda}_N < \overline{\Lambda}_N = +\infty$ and for any $k \in [\![2, N-1]\!]$,*

$$\overline{\Lambda}_{k-1} < \underline{\Lambda}_k \leq \overline{\Lambda}_k < \underline{\Lambda}_{k+1}.$$

(b) *If G is a tree, then for all $k \in [\![N]\!]$ we have $\lambda_k = \underline{\Lambda}_k = \overline{\Lambda}_k$.*
(c) *If G is a forest, then for all $k \in [\![N]\!]$ we have $\lambda_k = \underline{\Lambda}_k \geq \overline{\Lambda}_k$.*

Proof. Let G be a graph for which either there exists a minimizing uniform partition for $\underline{\Lambda}_k$ or there exists a maximizing uniform partition for $\overline{\Lambda}_k$. Then,

$$\begin{aligned}\underline{\Lambda}_k &= \max_{i \in [\![k]\!]} \lambda_1(A_i) \\ &= \min_{i \in [\![k]\!]} \lambda_1(A_i) \\ &\leq \overline{\Lambda}_k.\end{aligned}$$

For part (a), clearly $0 = \underline{\Lambda}_1 = \overline{\Lambda}_1$. Fix $k \in [\![2, N-1]\!]$ and note that by Proposition 7, handy minimizers of $\underline{\Lambda}_k$ are uniform partitions. Since any subpartition of a cycle is handy, the inequality $\underline{\Lambda}_k \leq \overline{\Lambda}_k$ always holds for generators on cycles.

Also, given uniform partitions $\mathcal{A} = (A_1, \ldots, A_k) \in \mathscr{P}_k$ and $\mathcal{B} = (B_1, \ldots, B_{k+1}) \in \mathscr{P}_{k+1}$, by applying the pigeonhole principle to the boundary points, there exists $i \in [\![k]\!]$ and $j \in [\![k+1]\!]$ such that B_j is strictly included in A_i. Let us recall Lemma 3 of [19] stating that if $A, B \in \mathscr{D}_1$ with B strictly included in A, then $\lambda_1(A) < \lambda_1(B)$. Therefore, by uniformity we get $\lambda_1(\mathcal{A}) < \lambda_1(\mathcal{B})$. Applying this with a minimizer for $\underline{\Lambda}_{k+1}$ and a maximizer for $\overline{\Lambda}_k$ (its existence will be proved in Lemma 20 at the beginning of Section 5), we get $\overline{\Lambda}_k < \underline{\Lambda}_{k+1}$.

To prove $\overline{\Lambda}_N = +\infty$, consider the sequence of N-partitions $(\mathcal{A}^{(n)})_{n \in \mathbb{N}}$, where for any $n \in \mathbb{N}, \partial\mathcal{A}^{(n)}$ consists of the points $(i + (2+n)^{-1})_{i \in \mathbb{Z}_N}$. Then it is easy to verify that $\lim_{n \to +\infty} \lambda_1(\mathcal{A}^{(n)}) = +\infty$.

For part (b), the equality $\lambda_k = \underline{\Lambda}_k$ has already been proved in [19]. On the other hand, assume that there exists a handy minimizer $\mathcal{A} = (A_1, \ldots, A_k) \in \mathscr{D}_k$ for $\underline{\Lambda}_k$. By Proposition 7, $\mathcal{A}$ is a uniform partition and consequently, we have $\underline{\Lambda}_k \leq \overline{\Lambda}_k$ by the argument presented in part (a).

To prove the reverse inequality, let $\mathcal{B} = (B_1, \ldots, B_k)$ be a given k-partition. Due to the fact that the underlying graph is a tree, $\mathsf{G}_{\mathcal{B}}$ is also a tree and it follows that $|\partial\mathcal{B}| \leq k-1$. By the pigeonhole principle, there exists a component A_i, $i \in [\![k]\!]$, of the above minimizer $\mathcal{A}$, that contains no point of $\partial\mathcal{B}$. This means that there exists $j \in [\![k]\!]$ such that $A_i \subset B_j$, and we get

$$\begin{aligned}\min_{l\in[\![k]\!]} \lambda_1(B_l) &\le \lambda_1(B_j)\\ &\le \lambda_1(A_i)\\ &= \lambda_1(\mathcal{A}) = \underline{\Lambda}_k\end{aligned}$$

since $\mathcal{A}$ is uniform. Thus, considering the supremum over $\mathcal{B} \in \mathscr{P}_k$, it follows that $\overline{\Lambda}_k \le \underline{\Lambda}_k$, and consequently equality (b) holds if there exists a handy minimizer for $\underline{\Lambda}_k$.

But by Proposition 22 of [19], minimizers of $\underline{\Lambda}_k$ are generically handy for the tree G. Also, by an argument similar to the one presented in Theorem 25 of [19] one may prove that $\underline{\Lambda}_k$ and $\overline{\Lambda}_k$ as real functions on the set $\mathcal{L}(\mu, \mathsf{G})$ are continuous (i.e. with respect to the entries of the Markov generator). These two facts prove that equality (b) holds for all generators on trees.

For part (c), it is not difficult to see that a true equality always holds in (11), and in a similar way, we get

$$\overline{\Lambda}_k = \max_{l\in[\![N^{(1)}]\!]\cap[\![k-N^{(2)},k-1]\!]} \min(\overline{\Lambda}^{(1)}_l, \overline{\Lambda}^{(2)}_{k-l}).$$

Hence, for any forest by Lemma 13 and part (b) we have

$$\forall\, k \in [\![N]\!], \quad \overline{\Lambda}_k \le \underline{\Lambda}_k = \lambda_k. \quad \square$$

Remark 16. In what follows we discuss cases for which the inequalities of Proposition 15 are strict. To see this, using the situation and notation of the proof of Proposition 14 for a graph with two connected components, we have $\overline{\Lambda}_k \le \min\{\overline{\Lambda}^{(1)}_{N^{(1)}}, \overline{\Lambda}^{(2)}_{N^{(2)}}\}$, but by Lemma 13,

$$\{\underline{\Lambda}_k : k \in [\![N]\!]\} = \{\underline{\Lambda}^{(1)}_k : k \in [\![N^{(1)}]\!]\} \cup \{\underline{\Lambda}^{(2)}_k : k \in [\![N^{(2)}]\!]\}.$$

This shows that even the inequality $\max(\underline{\Lambda}_k, \overline{\Lambda}_k) \le \min(\underline{\Lambda}_{k+1}, \overline{\Lambda}_{k+1})$ cannot be true in general. For instance, assume that one of the components is an empty graph with just one vertex and also let the other component be a tree, and note that for all $k \in [\![N]\!]$ we have $\overline{\Lambda}_k = 0$ while the parameter $\underline{\Lambda}_k$ is nonzero for $k \ge 3$. Moreover, the same example shows that the inequality $\overline{\Lambda}_k \le \underline{\Lambda}_k$ can be strict for forests.

On the other hand, it is instructive to mention that if Conjecture 8 is true then by the first paragraph of the previous proof and the continuity of functions $\underline{\Lambda}_k$ and $\overline{\Lambda}_k$ on $\mathcal{L}(\mu, \mathsf{G})$, one may deduce that the inequality $\underline{\Lambda}_k \le \overline{\Lambda}_k$ is true for all irreducible generators.

Moreover, we would like to mention that the functions $\underline{\Lambda}_k$ and $\overline{\Lambda}_k$ are not necessarily continuous on the whole set of generators on a set V. To see this, consider the situation where $N^{(1)} = 1$ and the graph of $L^{(2)}$ is a tree. Let $V^{(1)} = \{x\}$ and choose some vertex y in $V^{(2)}$. For $\epsilon \ge 0$, let $L(\epsilon)$ be deduced from L by imposing that $L(\epsilon)(x, y) = L(\epsilon)(y, x) = \epsilon$ and by leaving the other entries unchanged (except $L(\epsilon)(x, x) = L(x, x) - \epsilon$ and $L(\epsilon)(y, y) = L(y, y) - \epsilon$). Classical perturbation results (see for instance the book of Kato [17]) imply that the usual spectrum $(\lambda_k(\epsilon))_{k\in[\![N]\!]}$ is continuous at $\epsilon = 0$. Note that for $\epsilon > 0$, the graph associated with $L(\epsilon)$ is a tree and we have $(\underline{\Lambda}_k(\epsilon))_{k\in[\![N]\!]} = (\overline{\Lambda}_k(\epsilon))_{k\in[\![N]\!]} = (\lambda_k(\epsilon))_{k\in[\![N]\!]}$. But due to the above considerations, $\overline{\Lambda}_k = 0$ for all $k \in [\![N]\!]$. Thus, $(\overline{\Lambda}_k(\epsilon))_{k\in[\![N]\!]}$ is not continuous at $\epsilon = 0$ for $k \ge 3$. $\square$

3. Characterization of handy spectral nodal domains

In this section we provide a proof of Proposition 6, as well as a criterion for the rectifiability of a given subpartition that will be useful for deducing Proposition 7.

Let $\mathsf{g} \neq 0$ be an eigenfunction associated with an eigenvalue λ of $-L$, and let g be its affine extension to G with nodal domains $A_1, \ldots, A_k$. Assume that $\mathcal{A} := (A_1, \ldots, A_k)$ is a handy partition in $\mathscr{P}_k$. We show that it is uniform, rectifiable and bipartite.

First, note that Lemma 2 implies that for all $i \in [\![k]\!]$, $\lambda_1(A_i) = \lambda$, and consequently, $\mathcal{A}$ is uniform. To prove rectifiability and bipartiteness, for $i \in [\![k]\!]$, define $\sigma_i \in \{-1, +1\}$ and choose $r_i > 0$ such that $g = \sigma_i r_i f_{A_i}$ on A_i. Let $\{A_i, A_j\}$ be an edge in the graph $\mathsf{G}_{\mathcal{A}}$ defined before Proposition 6 and let a be a point in $\partial A_i \cap \partial A_j$. By definition of the nodal domains, we have $g(a) = 0$ and $\sigma_i \sigma_j = -1$. It also follows that the couple (r_i, r_j) is rectifying for a with respect to (A_i, A_j) and since this is independent of the chosen $a \in \partial A_i \cap \partial A_j$, (r_i, r_j) is rectifying for (A_i, A_j), and consequently, $\mathcal{A}$ is rectifiable by the family $(r_i)_{i \in [\![k]\!]}$. It is also bipartite, because the edges of $\mathsf{G}_{\mathcal{A}}$ are only between the two independent sets $\{A_i : i \in [\![k]\!] \text{ with } \sigma_i = 1\}$ and $\{A_i : i \in [\![k]\!] \text{ with } \sigma_i = -1\}$. This completes the proof of the direct implication in Proposition 6.

For the converse, fix $k \in [\![N]\!]$ and assume that $\mathcal{A} := (A_1, \ldots, A_k) \in \mathscr{P}_k$ is a handy, uniform, rectifiable and bipartite partition, and let $(r_i)_{i \in [\![k]\!]}$ be the corresponding rectifying family. Also, let $(\sigma_i)_{i \in [\![k]\!]}$ be a $\{-1, 1\}$-valued family such that the graph $\mathsf{G}_{\mathcal{A}}$ is bipartite between the two parts $\{A_i : i \in [\![k]\!] \text{ with } \sigma_i = 1\}$ and $\{A_i : i \in [\![k]\!] \text{ with } \sigma_i = -1\}$. We verify that $\mathsf{g} := \sum_{i \in [\![k]\!]} \sigma_i r_i \mathsf{f}_{A_i}$ is an eigenvector of L associated with the eigenvalue $-\lambda_1(\mathcal{A})$. This will complete the proof of Proposition 6, since by bipartition, it is clear that up to a permutation of indices, $\mathcal{A}$ corresponds to the nodal domains of g.

Indeed, Lemma 2 implies that for any $i \in [\![k]\!]$, by uniformity, on $\mathsf{A}_i := A_i \cap V$ we have

$$\begin{aligned} \widehat{L}_{A_i}[\mathsf{g}] &= \sigma_i r_i \widehat{L}_{A_i}[\mathsf{f}_{A_i}] \\ &= -\lambda_1(\mathcal{A}) \sigma_i r_i \mathsf{f}_{A_i} \\ &= -\lambda_1(\mathcal{A}) \mathsf{g}. \end{aligned}$$

Let $a \in \partial A_i \cap \partial A_j$ and first, assume that $a \notin V$, which implies that $[a^-, a^+]$ is the unique edge in E containing a, where $a^- \in A_i$ and $a^+ \in A_j$. Then, rectifiability and bipartiteness imply that

$$\widetilde{L}(a^-, a)\mathsf{g}(a^-) = L(a^-, a^+)(\mathsf{g}(a^-) - \mathsf{g}(a^+)).$$

This shows that $\widehat{L}_{A_i}[\mathsf{g}]$ coincides with $L[\mathsf{g}]$ on A_i. On the other hand, assume that $a \in V$, and note that by handiness $[a^-, a]$ and $[a^+, a]$ are the unique edges in E included in the closures of A_i and A_j, respectively. Thus, by rectifiability, bipartiteness and the fact that $\mathsf{g}(a) = 0$, we have

$$L[\mathsf{g}](a) = L(a, a^-)\mathsf{g}(a^-) + L(a, a^+)\mathsf{g}(a^+) = 0 = -\lambda_1(\mathcal{A})\mathsf{g}(a).$$

Now, since $\mathcal{A} \in \mathscr{P}_k$, we conclude that $L[\mathsf{g}] = -\lambda_1(\mathcal{A})\mathsf{g}$, as announced. This completes the proof of Proposition 6.

In the rest of this section we are going to present a convenient criterion for the rectifiability of a given subpartition $\mathcal{A} := (A_1, \ldots, A_k) \in \mathscr{D}_k$, for some $k \in [\![N]\!]$. A finite family of points from the metric graph G, $(a_j)_{j \in [\![l]\!]}$, with $l \in [\![k]\!]$, is said to be a *necklace* (with respect to $\mathcal{A}$) if there exists a family $(n_j)_{j \in [\![l]\!]}$ of distinct indices from $[\![k]\!]$ such that for all $j \in [\![l]\!]$, $a_j \in (\partial A_{n_j} \cap \partial A_{n_{j+1}})$ (with the convention $A_{n_{l+1}} := A_{n_1}$). The necklace $(a_j)_{j \in [\![l]\!]}$ is said to be rectifiable if we can find a family $(r_j)_{j \in [\![l]\!]}$ such that for all $j \in [\![l]\!]$, (r_j, r_{j+1}) is rectifying for a_j with respect to $(A_{n_j}, A_{n_{j+1}})$ (with the convention $r_{l+1} := r_1$). The subpartition

$\mathcal{A}$ itself is said to be *necklace-rectifiable* if all of its necklaces are rectifiable. Of course if $\mathcal{A}$ is rectifiable, then it is clearly necklace-rectifiable. The following lemma shows that the converse is also true.

Lemma 17. *A subpartition $\mathcal{A} \in \mathscr{D}_k$ is rectifiable if and only if it is necklace-rectifiable.*

Proof. Fix $k \in [\![N]\!]$ and let $\mathcal{A} := (A_1, \ldots, A_k) \in \mathscr{D}_k$ be a necklace-rectifiable subpartition. We verify that this subpartition is also rectifiable. Let $i \neq j \in [\![k]\!]$ be such that $\{A_i, A_j\}$ is an edge of $\mathsf{G}_{\mathcal{A}}$ and fix $x \in \partial A_i \cap \partial A_j$. Let (r_i, r_j) be a rectifying pair for x with respect to (A_i, A_j) and consider x' which is another point in $\partial A_i \cap \partial A_j$. Then (x, x') is a necklace for $\mathcal{A}$ (associated with the cycle $A_i \to A_j \to A_i$ of $\mathsf{G}_{\mathcal{A}}$) and the fact that it is rectifiable is equivalent to the statement that (r_i, r_j) is also rectifying for x' with respect to (A_i, A_j), because all rectifying pairs are proportional. It follows that (r_i, r_j) is rectifying for (A_i, A_j).

Assume temporarily that $\mathsf{G}_{\mathcal{A}}$ is connected. We define a family $(r_i)_{i\in[\![k]\!]}$ in the following way. We arbitrarily choose $r_1 = 1$. Then, for $i \in [\![2, k]\!]$, let $(n_j)_{j\in[\![l]\!]}$ be a path going from 1 to i, i.e. $n_1 = 1, n_l = i$, all of its elements are distinct and for all $j \in [\![l-1]\!]$, $\{A_{n_j}, A_{n_{j+1}}\}$ is an edge of $\mathsf{G}_{\mathcal{A}}$. Recursively, on $j \in [\![2, l]\!]$, we define r_{n_j} uniquely by imposing that $(r_{n_{j-1}}, r_{n_j})$ is rectifying for $(A_{n_{j-1}}, A_{n_j})$ (this can be done, according to the first part of this proof). We end up with a definition for r_j and it is not difficult to check that the value obtained is not dependent on the choice of the path going from 1 to j, due to the fact that $\mathcal{A}$ is necklace-rectifiable (consider all the cycles induced by the concatenation of two paths going from 1 to j, the second one in reverse order). Let us verify that for any edge $\{A_i, A_j\}$ of $\mathsf{G}_{\mathcal{A}}$, (r_i, r_j) is rectifying for (A_i, A_j). Indeed, this is a consequence of the above construction and of the fact that there exists a path $(n_m)_{m\in[\![l]\!]}$ starting from 1 and satisfying either $n_{l-1} = i, n_l = j$ or $n_{l-1} = j, n_l = i$.

Note that if $\mathsf{G}_{\mathcal{A}}$ is not connected, it is sufficient to proceed as before separately on each of its connected components. □

Now, we quantify the rectifiability of a necklace. Hence, let $(a_j)_{j\in[\![l]\!]}$, with $l \in [\![k]\!]$, be a given necklace with respect to $\mathcal{A} \in \mathscr{D}_k$. Up to a change of indices, we assume that $a_j \in \partial A_j \cap \partial A_{j+1}$ for $j \in [\![l]\!]$. We associate with this necklace the quantity

$$C := \prod_{j\in[\![l]\!]} \frac{f_{A_j}(a_j^-)\, \widetilde{\phi}(a_j^-, a_j)}{f_{A_{j+1}}(a_j^+)\, \widetilde{\phi}(a_j^+, a_j)},$$

where the meaning of the expressions $\widetilde{\phi}$ and f_{A_j} is recalled from the paragraph preceding Lemma 2 and a_j^-, a_j^+ are defined to be the unique vertices in $A_j \cap N(a_j)$ and $A_{j+1} \cap N(a_j)$, respectively (the indices are taken modulo l). Then we have:

Lemma 18. *The necklace $(a_j)_{j\in[\![l]\!]}$ is rectifiable if and only if $C = 1$.*

Proof. Assume that $r_j > 0$ is given, for some $j \in [\![l-1]\!]$. The requirement that (r_j, r_{j+1}) is rectifying for a_j with respect to (A_j, A_{j+1}) uniquely determines r_{j+1}. Indeed by definition it amounts to the equality

$$r_j f_{A_j}(a_j^-)\, \widetilde{\phi}(a_j^-, a_j) = r_{j+1} f_{A_{j+1}}(a_j^+)\, \widetilde{\phi}(a_j^+, a_j).$$

Starting from r_1, we construct iteratively $r_1, \ldots, r_l$. The only obstruction to $(r_j)_{j\in[\![l]\!]}$ being rectifying for the necklace $(a_j)_{j\in[\![l]\!]}$ is that (r_l, r_1) must be rectifying for a_l with respect to (A_l, A_1). Let $r > 0$ be such that (r_l, r) is rectifying for a_l with respect to (A_l, A_1), according to the above computation we have $r = Cr_1$. Hence, the equality $r = r_1$ is equivalent to $C = 1$. □

4. Rectifiability of optimizers for $\underline{\Lambda}_k$ and $\overline{\Lambda}_k$

Our goal here is to prove Proposition 7, which indicates two ways for producing rectifiable subpartitions.

We begin with the case of a minimizing handy subpartition $\mathcal{A} = (A_1, \dots, A_k) \in \mathscr{D}_k$ for $\underline{\Lambda}_k$, for some $k \in [\![N]\!]$. Since G is assumed to be connected, by results of [19] and the hypothesis we know that $\mathcal{A}$ is a uniform k-partition. Indeed, the argument behind this fact is simple, because if $\mathcal{A}$ is not uniform, then it is possible (due to the fact that all boundary vertices have degree 2) to enlarge infinitesimally such A_i such that $\lambda_1(A_i) = \max_{j\in[\![k]\!]} \lambda_1(A_j)$ and to move the other ones appropriately to produce a better partition, which contradicts the minimality of $\mathcal{A}$ (see [19]). The same idea can be applied to prove that any handy maximizer partition $\mathcal{A} \in \mathscr{P}_k$ for $\overline{\Lambda}_k$ is uniform.

Therefore, it remains to prove the rectifiability of handy optimizers. We will use Lemmas 17 and 18 to prove that any handy minimizer for $\underline{\Lambda}_k$ is rectifiable. The same arguments can prove the assertion for handy maximizers of $\overline{\Lambda}_k$.

Thus, let $(a_j)_{j\in[\![l]\!]}$, with $l \in [\![k]\!]$, be a necklace. We adopt again the notation of the previous section, in particular that the necklace induces the cycle $A_1 \to A_2 \to \cdots \to A_l \to A_1$ in $\mathsf{G}_{\mathcal{A}}$ and $a_j \in \partial A_j \cap \partial A_{j+1}$, for $j \in [\![l]\!]$. We want to show that

$$C = 1. \tag{12}$$

The basic idea of the proof is to perturb infinitesimally the positions of the boundary points $(a_j)_{j\in[\![l]\!]}$ and to take advantage of the minimizing property of $\mathcal{A}$. For this let $(t_j)_{j\in[\![l]\!]}$ be a family of positive real numbers and a small enough number $\epsilon > 0$. For $j \in [\![l]\!]$, we consider $a_j(\epsilon) \in [a_j, a_j^+]$ such that $d([a_j(\epsilon), a_j^+]) = d([a_j, a_j^+]) - \epsilon t_j$. Hence, for ϵ small enough, one may define the subpartition $\mathcal{A}^{(\epsilon)} = (A_1^{(\epsilon)}, \dots, A_k^{(\epsilon)}) \in \mathscr{D}_k$ which is similar to $\mathcal{A}$, except that the boundary points $(a_j)_{j\in[\![l]\!]}$ have been moved to the points $(a_j(\epsilon))_{j\in[\![l]\!]}$ (in particular we have $A_j^{(\epsilon)} = A_j$, for $j \in [\![l+1, k]\!]$).

Now, we want to evaluate the infinitesimal influence of this perturbation on the principal Dirichlet eigenvalues. For this, fix the arbitrary family of positive real numbers $(r_j)_{j\in[\![l]\!]}$ and define the function $f_j^{(\epsilon)} \in \mathscr{F}_0(A_j^{(\epsilon)})$ for each $j \in [\![l]\!]$ such that it coincides with f_{A_j} on the vertices in $A_j \cap V$, $f_j^{(\epsilon)}(a_j) := r_j\epsilon$, vanishes outside of $A_j^{(\epsilon)}$ and is affinely extended on $A_j^{(\epsilon)}$. Then, for all $j \in [\![l]\!]$ we have (indices are taken modulo l)

$$\begin{aligned}\mathcal{E}(f_j^{(\epsilon)}) &= \mathcal{E}(f_{A_j}) + f_{A_j}^2(a_{j-1}^+)\left(\frac{\epsilon t_{j-1}}{d([a_{j-1}^+, a_{j-1}]) - \epsilon t_{j-1}}\right)\widetilde{\phi}(a_{j-1}^+, a_{j-1})\\ &\quad + \left(f_{A_j}(a_j^-) - r_j\epsilon\right)^2 \widetilde{\phi}(a_j^-, a_j) + \left(r_j\epsilon\right)^2\left(\frac{d([a_j^+, a_j])}{\epsilon t_j}\right)\widetilde{\phi}(a_j^+, a_j)\\ &\quad - \left(f_{A_j}(a_j^-)\right)^2\widetilde{\phi}(a_j^-, a_j).\end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial}{\partial\epsilon}\left(\mathcal{E}(f_j^{(\epsilon)})\right)\Big|_{\epsilon=0} &= t_{j-1} f_{A_j}^2(a_{j-1}^+)\,\frac{\widetilde{\phi}(a_{j-1}^+, a_{j-1})}{d([a_{j-1}^+, a_{j-1}])} - 2r_j f_{A_j}(a_j^-)\,\widetilde{\phi}(a_j^-, a_j)\\ &\quad + \frac{r_j^2}{t_j}\,\widetilde{\phi}(a_j^+, a_j) d([a_j^+, a_j]).\end{aligned}$$

Now, for the arbitrary family of positive real numbers $(s_j)_{j\in[\![l]\!]}$ let

$$t_j := s_j\, d([a_j^+, a_j])\, \widetilde{\phi}(a_j^+, a_j), \qquad r_j := s_j\, f_{A_j}(a_j^-)\, \widetilde{\phi}(a_j^-, a_j).$$

Hence,

$$\frac{\partial}{\partial\epsilon}\left(\mathcal{E}(f_j^{(\epsilon)})\right)\bigg|_{\epsilon=0} = s_{j-1}\, f_{A_j}^2(a_{j-1}^+)\, \widetilde{\phi}^2(a_{j-1}^+, a_{j-1}) - s_j\, f_{A_j}^2(a_j^-)\, \widetilde{\phi}^2(a_j^-, a_j).$$

If $C \neq 1$, we can assume that $C > 1$ (otherwise consider the reverse necklace $(a_{l+1-j})_{j\in[\![l]\!]}$). Take $s_1 := 1$ and define iteratively, for $j \in [\![2, l]\!]$,

$$s_j := s_{j-1}\left(\frac{f_{A_j}(a_{j-1}^+)\widetilde{\phi}(a_{j-1}^+, a_{j-1})}{f_{A_j}(a_j^-)\widetilde{\phi}(a_j^-, a_j)}\right)^2 C^{\frac{2}{l}}.$$

These are compatible modulo l, since

$$s_{l+1} = s_1 \prod_{j\in[\![l]\!]}\left(\frac{f_{A_{j+1}}(a_j^+)\widetilde{\phi}(a_j^+, a_j)}{f_{A_j}(a_j^-)\widetilde{\phi}(a_j^-, a_j)}\right)^2 C^2 = s_1,$$

and consequently we end up with

$$\forall\, j \in [\![l]\!], \quad \frac{\partial}{\partial\epsilon}\left(\frac{\mathcal{E}(f_j^{(\epsilon)})}{\mu[(f_j^{(\epsilon)})^2]}\right)\bigg|_{\epsilon=0} = \frac{\partial}{\partial\epsilon}\left(\mathcal{E}(f_j^{(\epsilon)})\right)\bigg|_{\epsilon=0}\frac{1}{\mu[f_{A_j}^2]} < 0.$$

Thus, for $\epsilon > 0$ small enough, we get

$$\max_{j\in[\![l]\!]} \lambda_1(A_j^{(\epsilon)}) < \max_{j\in[\![l]\!]} \lambda_1(A_j).$$

Since $\lambda_1(A_j^{(\epsilon)}) = \lambda_1(A_j)$ for $j \in [\![l+1, k]\!]$, it follows that the subpartition $\mathcal{A}^{(\epsilon)}$ is a minimizer for $\underline{\Lambda}_k$ (in particular to avoid an immediate contradiction, we must have $l+1 \leq k$). However, note that this minimizing subpartition is still handy, and by Proposition 7 of [19], it must be uniform. This is not the case and we conclude that, indeed, $C = 1$.

Thus we have shown that if the handy subpartition $\mathcal{A}$ is minimizing for $\underline{\Lambda}_k$, then its necklaces are rectifiable according to Lemma 18. Now, Lemma 17 implies that $\mathcal{A}$ is rectifiable. This ends the proof of Proposition 7.

Remark 19. If Conjecture 8 is true, it appears that generically, for any $k \in [\![N]\!]$, there are two ways of generating a handy, uniform and rectifiable k-partition. One is by looking for a minimizer of the mapping

$$\mathscr{P}_k \ni \mathcal{A} = (A_1, \ldots, A_k) \mapsto \max_{i\in[\![k]\!]} \lambda_1(A_i)$$

and the other one is by looking for a maximizer of the mapping

$$\mathscr{P}_k \ni \mathcal{A} = (A_1, \ldots, A_k) \mapsto \min_{i\in[\![k]\!]} \lambda_1(A_i).$$

It would had been more satisfying if we could have played with only one mapping. More precisely, endow $\mathscr{P}_k$ with a differential structure (which is the gluing of several differential manifolds with boundaries, whose dimensions can be computed from the acceptable infinitesimal increasing and decreasing simultaneous moves of the boundaries of the elements of the partition at hand). We are wondering whether there exists a "natural" mapping from $\mathscr{P}_k$ to $\mathbb{R}$ whose critical points are (generically) handy, uniform and rectifiable partitions. The first candidates which come to mind are the mappings

$$\mathcal{L}_{p,k} : \mathscr{P}_k \ni \mathcal{A} = (A_1, \ldots, A_k) \mapsto \frac{1}{k} \sum_{i \in [\![k]\!]} \lambda_1^p(A_i)$$

where $k \in [\![N]\!]$ and $p \in [1, +\infty)$. A careful look at the above arguments shows that if a partition $\mathcal{A}$ is handy and is also a critical point for one of the above mappings $\mathcal{L}_{p,k}$ (even with $p \in \mathbb{R} \setminus \{0\}$), then $\mathcal{A}$ is uniform and rectifiable (the proof of Proposition 7 of [19] is also valid for such critical partitions).

For the case of the usual two-dimensional Laplacian on some rectangles or on the equilateral triangle, Helffer and Hoffmann-Ostenhof [16] have shown that the minimizers of the mapping $\mathcal{L}_{1,k}$ (for $k = 3$ and $k = 2$) are not uniform. This is not really contradictory with the above assertions, because the symmetries of their examples suggest that they are typically not working with what we would call a generic situation.

Despite its obvious interest, the investigation of the properties of the mappings $\mathcal{L}_{p,k}$ is beyond the scope of the present paper. $\square$

5. Spectral decomposition of cycles

As announced before, in this section we restrict our attention to reversible generators whose underlying graph is a cycle. Our main goal is to prove Proposition 9.

For this let L be a given generator, reversible with respect to a positive probability μ and whose underlying graph G is a cycle, that in what follows is identified with the usual nearest neighbor graph structure of $\mathbb{Z}_N$. The metric graph G can be seen as the cycle of length N obtained by identifying the points 0 and N of the segment $[0, N]$.

The first step toward establishing Proposition 9 is to check that a maximizer does exist for the quantity $\overline{\Lambda}_k$, for all $k \in [\![N-1]\!]$. Note that for the cycle $\mathbb{Z}_N$, by Proposition 15(a), we have $\overline{\Lambda}_N = +\infty$. Also, note that the equality $\overline{\Lambda}_N = +\infty$ does not hold in general. For instance, $\overline{\Lambda}_N$ is finite for generators whose graph is a tree due to Proposition 15(b).

Lemma 20. *For any $k \in [\![N-1]\!]$, there exists $\mathcal{A} = (A_1, \ldots, A_k) \in \mathscr{P}_k$ such that*

$$\overline{\Lambda}_k = \min_{i \in [\![k]\!]} \lambda_1(A_i).$$

Proof. First note that due to Proposition 15(a), $\overline{\Lambda}_k$ is finite for $k \in [\![N-1]\!]$. Let $(\mathcal{A}^{(n)})_{n \in \mathbb{N}}$ be a sequence of elements of $\mathscr{P}_k$ which is maximizing for $\overline{\Lambda}_k$, i.e.

$$\lim_{n \to \infty} \min_{i \in [\![k]\!]} \lambda_1(A_i^{(n)}) = \overline{\Lambda}_k.$$

Taking into account the compactness of the Hausdorff topology on the set of compact subsets of $\mathbb{R}/(N\mathbb{Z})$, we can extract a subsequence of $(\partial\mathcal{A}^{(n)})_{n \in \mathbb{N}}$ (still denoted as $(\partial\mathcal{A}^{(n)})_{n \in \mathbb{N}}$ for notational convenience) and a compact subset $B \subset G$ such that

$$\lim_{n \to \infty} \partial\mathcal{A}^{(n)} = B.$$

Necessarily, we have $l := \mathrm{card}(B) \leq k$ (and $l \geq k/2$), and we let $A_1, \ldots, A_l$ be the connected components of $G \setminus B$. The family $\mathcal{A} := (A_1, \ldots, A_l)$ is covering, in the sense that G is the union of the closures of the A_i, but $\mathcal{A}$ may not be a partition of $\mathscr{P}_l$, because some of its components may not intersect $\mathbb{Z}_N$. Indeed, consider $A'_1, \ldots, A'_{l'}$, the subsets A_i, with $i \in [\![l]\!]$, which satisfy $A_i \cap \mathbb{Z}_N \neq \emptyset$. Defining $\mathcal{A}' = (A'_1, \ldots, A'_{l'})$, we have that $\mathcal{A}'$ is an l'-subpartition and

$$\min_{i \in [\![l']\!]} \lambda_1(A'_i) = \overline{\Lambda}_k.$$

This is a consequence of the fact that the mapping

$$G^2 \ni (x, y) \mapsto \lambda_1((x, y)) \in \mathbb{R}_+ \sqcup \{+\infty\}$$

is easily seen to be continuous (see for instance [19, Lemma 6]), where in the rhs (x, y) is the interval of G obtained by going from x to y "counter-clockwise". By convention if $(x, y) \cap \mathbb{Z}_N = \emptyset$, we let $\lambda_1((x, y)) = +\infty$.

Note that $l' \neq 0$. This comes from the assumption that $k \leq N-1$, which implies that $\mathbb{Z}_N \not\subset B$ and thus there exist $i \in \mathbb{Z}_N$ and $j \in [\![l']\!]$ such that $i \in A'_j$. If $\mathcal{A}' \notin \mathscr{P}_{l'}$, let us show that we can find $\mathcal{A}'' \in \mathscr{P}_l$ with

$$\min_{i \in [\![l]\!]} \lambda_1(A''_i) \geq \min_{i \in [\![l']\!]} \lambda_1(A'_i). \tag{13}$$

Indeed, if $\mathcal{A}' \notin \mathscr{P}_{l'}$, necessarily $l' < l$ and the A_i, with $i \in [\![l]\!]$, which have been removed from $\mathcal{A}$ to obtain $\mathcal{A}'$ are of the form $(x, x+1)$ with $x \in \mathbb{Z}_N$. For $\epsilon \in (0, 1)$ small enough, we have, for all such x,

$$\lambda_1((x, x+1+\epsilon)) \geq \min_{i \in [\![l']\!]} \lambda_1(A'_i),$$

and because the lhs goes to infinity when ϵ goes to zero, this is a consequence of the above continuity property. Still for such x, consider the set $(x+\epsilon, x+1+\epsilon)$ if $x \notin \partial\mathcal{A}'$ (remark that $\lambda_1((x+\epsilon, x+1+\epsilon)) > \lambda_1((x, x+1+\epsilon)))$ and $(x, x+1+\epsilon)$ if $x \in \partial\mathcal{A}'$, and call these sets $A''_{l'+1}, \ldots, A''_l$. Now, diminish a little the A'_i, with $i \in [\![l']\!]$, so that they do not overlap with the above sets, to get the A''_i, for $i \in [\![l']\!]$ (having noticed that since $A''_i \subset A'_i$, we have $\lambda_1(A''_i) \geq \lambda_1(A'_i)$). The l-partition $\mathcal{A}'' := (A''_1, \ldots, A''_l)$ satisfies (13). If $\mathcal{A}'$ was to belong to $\mathscr{P}_{l'}$, we would have in fact $\mathcal{A}' = \mathcal{A}$ and $l' = l$. In this case, just take $\mathcal{A}'' := \mathcal{A}$.

Now, we will modify $\mathcal{A}''$ to obtain $\mathcal{A}''' \in \mathscr{P}_k$ such that

$$\min_{i \in [\![k]\!]} \lambda_1(A'''_i) \geq \min_{i \in [\![l]\!]} \lambda_1(A''_i). \tag{14}$$

It will follow that

$$\min_{i \in [\![k]\!]} \lambda_1(A'''_i) \geq \overline{\Lambda}_k$$

and thus $\mathcal{A}'''$ will be the wanted maximizer.

If $l = k$, there is nothing to do; we just take $\mathcal{A}''' := \mathcal{A}''$. Otherwise, namely if $l < k$, first consider the case where $\partial\mathcal{A}'' \cap \mathbb{Z}_N \neq \emptyset$ and choose $x \in \partial\mathcal{A}'' \cap \mathbb{Z}_N$. We create a new component $(x-\epsilon, x+\epsilon)$, with $\epsilon \in (0, 1)$ small enough that

$$\lambda_1((x-\epsilon, x+\epsilon)) > \min_{i \in [\![l]\!]} \lambda_1(A''_i).$$

At the same time we diminish a little the A''_i which admit x as a boundary point. We get an $(l+1)$-partition whose minimal principal Dirichlet eigenvalue is not less than $\min_{i \in [\![l]\!]} \lambda_1(A''_i)$.

If $l+1<k$ and if this $(l+1)$-partition still admits a boundary point in $\mathbb{Z}_N$, we start again the above procedure. Repeating such transformations, we end up with an m-partition $\widetilde{\mathcal{A}}$ with

$$\min_{i\in[\![m]\!]} \lambda_1(\widetilde{A}_i) \geq \min_{i\in[\![l]\!]} \lambda_1(A''_i),$$

such that either $m=k$ or $\partial\widetilde{\mathcal{A}}\cap\mathbb{Z}_N=\emptyset$. In the first case, we just take $\mathcal{A}''' := \widetilde{\mathcal{A}}$. Otherwise there exists $i \in [\![m]\!]$ such that $\widetilde{A}_i$ contains two consecutive points of $\mathbb{Z}_N$ (because $m<k\leq N-1$). Then we cut $\widetilde{A}_i$ at the middle of the edge separating the consecutive points, to get two elements of $\mathscr{D}_1$ whose principal Dirichlet eigenvalues are larger than $\lambda_1(\widetilde{A}_i)$. Replacing $\widetilde{A}_i$ by these two subsets, we get an $(m+1)$-partition whose minimal principal Dirichlet eigenvalue is not less than $\min_{i\in[\![m]\!]}\lambda_1(\widetilde{A}_i)$. By repeating this procedure, we end up with a k-partition $\mathcal{A}'''$ satisfying (14). □

Remark 21. The existence of a maximizer for $\overline{\Lambda}_k$, for $k\in[\![N-1]\!]$, does not hold true in general. Consider the generator on $[\![4]\!]$ whose matrix is given by

$$\begin{pmatrix} -4 & 1 & 1 & 2 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 2 & 0 & 0 & -2 \end{pmatrix}.$$

A maximizer for $\overline{\Lambda}_2$ would be $([2,1),[1,3]\cup[1,4])$ (also $([3,1),[1,2]\cup[1,4])$ and the partition $([4,1),[2,1]\cup[1,3])$ would provide a non-uniform maximizer); unfortunately the second component is not open, contrary to our requirements. A maximizing sequence $(A^{(n)})_{n\in\mathbb{N}}$ can be constructed, by imposing that for all $n\in\mathbb{N}$, $\partial\mathcal{A}^{(n)}=\{x_n\}$, where $(x_n)_{n\in\mathbb{N}}$ is a sequence of points from $(2,1)$ (or from $(3,1)$) converging to 1. Note that $([2,1),[3,1))$ is a minimizing subpartition for $\underline{\Lambda}_2$ and we recover that $\underline{\Lambda}_2 = \overline{\Lambda}_2$ (=1 here) for trees, as was proved in Proposition 15. Heuristically speaking, this example is typically not "generic", due to the identity $L(1,2)=L(1,3)$. □

The next step is to prove that actually the parameters $\underline{\Lambda}_k$, $k\in[\![N]\!]$ and $\overline{\Lambda}_k$, $k\in[\![N-1]\!]$ are eigenvalues of $-L$ when k is even.

Proposition 22. *Let $k\in[\![N]\!]$ be an even integer. Then, $\underline{\Lambda}_k$ is an eigenvalue of $-L$ and also if $k\neq N$, $\overline{\Lambda}_k$ is also an eigenvalue of $-L$.*

Proof. Due to the fact that the degree of the vertices of $\mathbb{Z}_N$ is 2, all subpartitions are handy. Thus, for $k\in[\![N]\!]$, Proposition 7 implies that the minimizer $\mathcal{A}$ for $\underline{\Lambda}_k$ as well as the maximizer $\mathcal{B}$ for $\overline{\Lambda}_k$ (provided $k\neq N$) are necessarily uniform and rectifiable k-partitions. If furthermore k is even, it appears that $\mathcal{A}$ and $\mathcal{B}$ are also bipartite. Then, Proposition 6 yields that $\underline{\Lambda}_k=\lambda_1(\mathcal{A})$ and $\overline{\Lambda}_k=\lambda_1(\mathcal{B})$ are eigenvalues of $-L$. □

Proposition 22 shows that the spectrum of $-L$ is given by $\underline{\Lambda}_1=0$, $(\underline{\Lambda}_k)_{k\in[\![2,N]\!],k \text{ even}}$ and $(\overline{\Lambda}_k)_{k\in[\![2,N-1]\!],k\text{ even}}$, if all of these values are distinct. Indeed, if N is even, we get $1+N/2+N/2-1=N$ values, while if N is odd, we obtain $1+(N-1)/2+(N-1)/2$ values. So if for all $k\in[\![2,N-1]\!]$, k even, $\underline{\Lambda}_k\neq\overline{\Lambda}_k$, by Propositions 15(a) and 22 we deduce that the spectrum of $-L$ is as described in Proposition 9. The general case will be an immediate consequence of the following result.

Proposition 23. *Let $k \in [\![2, N-1]\!]$, k even, be given and assume that $\underline{\Lambda}_k = \overline{\Lambda}_k$. Then the eigenvalue $\underline{\Lambda}_k$ of $-L$ has multiplicity 2 and the associated eigenvectors have exactly k nodal domains.*

Proof. Let $\mathcal{A} = (A_1, \ldots, A_k) \in \mathscr{P}_k$ be a minimizer for $\underline{\Lambda}_k$, with k as in the proposition. We adopt the convention that $A_1 = (b_1, b_2)$, $A_2 = (b_2, b_3), \ldots, A_k = (b_k, b_1)$, where the boundary points satisfy $0 \leq b_1 < b_2 < \cdots < b_k < N$ in $G = \mathbb{R}/(N\mathbb{Z})$ identified with $[0, N)$. First, assume that one of the A_i, with $i \in [\![k]\!]$, contains two elements of $\mathbb{Z}_N$. Up to a shift of indices, we can assume that $1, 2 \in A_1$. Let $y_1 \in (1, 2)$ be given and consider the mapping

$$F : (0, N) \ni s \mapsto \lambda_1((y_1, y_1 + s)) \in \mathbb{R}_+ \sqcup \{+\infty\}$$

(where $y + s$ has to be interpreted as a point of $G = \mathbb{R}/(N\mathbb{Z})$). We have $F(s) = +\infty$ for $s \in (0, 2 - y_1]$, but on the interval $[2 - y_1, N)$, F is (strictly) decreasing. Since $y_1 \in A_1$, we have $(y_1, b_2) \subset A_1$ and $A_2 \subset (y_1, b_3)$ and these inclusions are strict. Thus there exists a unique point $y_2 \in A_2$ such that $\lambda_1(y_1, y_2) = \lambda_1(A)$. Iterating this procedure, we can find $y_3 \in A_3$ such that $\lambda_1(y_2, y_3) = \lambda_1(A)$ etc. We end up constructing $y_k \in A_1$ such that $\lambda_1(y_{k-1}, y_k) = \lambda_1(A)$.

A priori there are three possibilities for this last point: $y_k < y_1$, or $y_k = y_1$, or $y_k > y_1$. Let us show that the assumption $\underline{\Lambda}_k = \overline{\Lambda}_k$ implies that $y_k = y_1$.

First, the case $y_k < y_1$ is always impossible. Indeed, if we define $B_i := (y_i, y_{i+1})$ for $i \in [\![k]\!]$, then $\mathcal{B} := (B_1, \ldots, B_k)$ is a minimizing subpartition for $\underline{\Lambda}_k$. According to Proposition 7, it should be a partition, but this is not the case, so we end up with a contradiction which implies that $y_k \geq y_1$.

Now, assume that $y_k > y_1$. It is then possible to transform the points $y_i, i \in [\![k]\!]$, into new points $y_i', i \in [\![k]\!]$, with $y_k' = y_1 = y_1'$, in such a way that if $\mathcal{B} := (B_1, \ldots, B_k)$ is the partition for which $\partial\mathcal{B} = \{y_i' : i \in [\![k]\!]\}$, we get

$$\forall\, i \in [\![k]\!], \quad \lambda_1(B_i) > \lambda_1((y_i, y_{i+1})).$$

Indeed, in the covering $\mathbb{R}$ of G, this can be achieved by diminishing continuously the points $y_i,\ i \in [\![2, k]\!]$, in such a manner that the intervals between consecutive points have the same principal Dirichlet eigenvalue with respect to the periodization of the generator. When the points starting from y_k reach y_1, we get the points $y_i',\ i \in [\![k]\!]$. We deduce that

$$\overline{\Lambda}_k \geq \min_{i \in [\![k]\!]} \lambda_1(B_i) > \min_{i \in [\![k]\!]} \lambda_1((y_i, y_{i+1})) = \lambda_1(\mathcal{A}) = \underline{\Lambda}_k$$

which is in contradiction with our main assumption.

Thus, as announced, $y_k = y_1$ and the partition $\mathcal{B}$, defined by $B_i := (y_i, y_{i+1})$ for $i \in [\![k]\!]$, is minimizing for $\underline{\Lambda}_k$. According to Proposition 6, there exists an eigenfunction f associated with the eigenvalue $-\underline{\Lambda}_k = -\overline{\Lambda}_k$ of L whose nodal domains are given by the elements of $\mathcal{B}$. Similarly, there exists an eigenfunction g associated with the eigenvalue $-\overline{\Lambda}_k$ whose nodal domains are given by the elements of $\mathcal{A}$. Let us check that f and g are not proportional. This comes from the choice of y_1, which insures that g has the same sign on the vertices 1 and 2, while f has opposite sign on these points. Thus, the equality $\underline{\Lambda}_k = \overline{\Lambda}_k$ implies that the dimension of the eigenspace of $-L$ associated with $\underline{\Lambda}_k$ is at least 2. It cannot be strictly larger than 2, or otherwise we would end up with too many independent eigenfunctions. Indeed, here is the counting, taking into account Proposition 15(a) to insure that we are not counting the same eigenvalue twice. First, $\underline{\Lambda}_1 = 0$ is an eigenvalue of multiplicity 1 and for any $l \in [\![2, N]\!]$, l even, if $\underline{\Lambda}_l \neq \overline{\Lambda}_l$, we get two different eigenvalues of multiplicity 1 (except if N is even and $l = N$; then only $\underline{\Lambda}_N$ is an eigenvalue, of multiplicity 1) and if $\underline{\Lambda}_l = \overline{\Lambda}_l$, we get one eigenvalue of multiplicity 2. Whatever N is (odd or even), this produces N eigenvalues (with multiplicities).

It remains to check the assertion about the nodal domains of any arbitrary eigenfunction f associated with $\underline{\Lambda}_k$, when $\underline{\Lambda}_k = \overline{\Lambda}_k$ with k even. Let l be the number of nodal domains of f and let $\mathcal{B}$ be the l-subpartition consisting of these domains. First, we show that f cannot vanish on two consecutive vertices. Let $x \in \mathbb{Z}_N$ be such that $\mathsf{f}(x) = 0$; then the equality $0 = \underline{\Lambda}_k f(x) = -L[f](x)$ implies that $\mathsf{f}(x-1)$ and $\mathsf{f}(x+1)$ must have different signs and in particular neither of them can vanish and they cannot both vanish. But in the latter case, the relation $0 = \underline{\Lambda}_k f(x+1) = -L[f](x+1)$ would lead to $f(x+2) = 0$ and by iteration we would end up with the contradiction $f = 0$. As a consequence, $\mathcal{B}$ is a partition and l is even. Note also that $\mathcal{B}$ is uniform and $\lambda_1(\mathcal{B}) = \underline{\Lambda}_k$. Necessarily $l = k$, because otherwise, via the pigeonhole principle, we would end up with the conclusion that one element of $\mathcal{B}$ is strictly included in one element of $\mathcal{A}$ (case $l > k$) or one element of $\mathcal{A}$ is strictly included in one element of $\mathcal{B}$ (case $l < k$), facts in contradiction with $\lambda_1(\mathcal{B}) = \lambda_1(\mathcal{A})$. It appears also that the construction given at the beginning of this proof gives all the eigenvectors associated with $\underline{\Lambda}_k$, up to a factor, by letting y_1 wander inside $[b_1, b_2)$. □

Let us give some precision regarding the sentence following Proposition 9. We have seen in [19] that the first Dirichlet eigenvector $f_A \geq 0$ associated with a segment A (path), with absorption only at the ends of the segment, has the following shape. Starting from zero at the lhs absorbing point, it increases until it reaches the highest value of f_A; maybe it stays at the same value at the next vertex, but after that it decreases until it reaches zero at the rhs absorbing point. The construction considered in the above proof then enables us to see that any eigenvector f associated with $\underline{\Lambda}_k$ or $\overline{\Lambda}_k$, with $k \in [\![2, N]\!]$ even, has the following shape. It is a succession of increasing phases and decreasing phases and between each two of them, f attains its local minima or maxima at one vertex or two consecutive vertices. There are exactly $k/2$ increasing or decreasing, respectively, phases and during each of them, f crosses zero (either at a true vertex of V or at a "virtual" point of G).

This behavior was well-known in the homogeneous case (namely for the generator L whose diagonal entries are either $1/2$ or 0, depending on whether they correspond to nearest neighbors or not, on the discrete cycle $\mathbb{Z}_N$), because then the eigenvectors are described as follows:

- the eigenvector $\mathbb{1}$ (always taking the value 1) is associated with the eigenvalue 0,
- the eigenspace associated with the eigenvalue $1 - \cos(2\pi k/N)$, for $k \in [\![\lfloor (N-1)/2 \rfloor]\!]$, is generated by the mappings $\mathbb{Z}_N \ni x \mapsto \cos(2\pi kx/N)$ and $\mathbb{Z}_N \ni x \mapsto \sin(2\pi kx/N)$,
- if N is even, $\mathbb{Z}_N \ni x \mapsto \cos(\pi x)$ is also an eigenvector, associated with the eigenvalue -1.

Thus there is a universal qualitative behavior for the eigenvectors of generators whose associated graph is a cycle, whether or not there are double eigenvalues (as in the homogeneous case).

6. Proof of Conjecture 3 for cycles

In this section we prove Proposition 10 which implies that Conjecture 3 and, thus, Conjecture 1 (a generalized Cheeger inequality) are valid with a universal constant for all $k \geq 1$ and all generators on cycles.

Fix $k \in [\![N]\!]$ odd (if k is even, there is nothing to prove, due to the equality $\underline{\Lambda}_k = \lambda_k$ of Proposition 9). Let $\mathcal{A} = (A_1, \ldots, A_k) \in \mathscr{P}_k$ be a minimizing partition for $\underline{\Lambda}_k$ which is uniform and rectifiable. For such a partition, we adopt the same convention as before:

that $A_1 = (b_1, b_2), A_2 = (b_2, b_3), \ldots, A_k = (b_k, b_1)$, where the boundary points satisfy $0 \le b_1 < b_2 < \cdots < b_k < N$, with $G = \mathbb{R}/(N\mathbb{Z})$ identified with $[0, N)$. It is also convenient to use periodic notation: $b_{k+1} = b_1$, $A_0 = A_k$, $A_{k+1} = A_1$ etc. The main step will consist in finding $i \in [\![k]\!]$ such that

$$\lambda_1(A_i \sqcup \{b_{i+1}\} \sqcup A_{i+1}) \ge (1/24)\, \lambda_1(\mathcal{A}). \tag{15}$$

Indeed, assuming that this is true and, to simplify the notation, that $i = k$, consider the partition $\mathcal{A}' := (A'_1, \ldots, A'_{k-1}) \in \mathscr{P}_{k-1}$ given by

$$\forall\, l \in [\![k-1]\!], \quad A'_l := \begin{cases} A_k \sqcup \{b_1\} \sqcup A_1, & \text{if } l = 1 \\ A_l, & \text{if } l \in [\![2, k-1]\!]. \end{cases}$$

Taking into account Proposition 9, we get

$$\begin{aligned} (1/24)\, \underline{\Lambda}_k &= (1/24)\, \lambda_1(\mathcal{A}) \\ &\le \lambda_1(A'_1) \\ &= \min_{l \in [\![k-1]\!]} \lambda_1(A'_l) \\ &\le \overline{\Lambda}_{k-1} \\ &= \lambda_k. \end{aligned}$$

To find the index $i \in [\![k]\!]$ such that (15) is satisfied, we need some preliminaries. First, we recall the following lemma from [7] concerning approximating the first Dirichlet eigenvalue of generators on paths (birth–death processes) with Dirichlet conditions at both ends of the path. For more on this and its relationships to Poincaré and weighted Hardy inequalities see [6,7].

Lemma 24 *([7, Corollary 7.8]).* *Let (L, μ) be a reversible generator on a path $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ with state space $\mathsf{V} = \{x_0, \ldots, x_{N+1}\}$ and edge set $\mathsf{E} = \{\{x_i, x_{i+1}\} : 0 \le i \le N\}$. Also, let $\lambda_1 := \lambda_1(\{x_1, \ldots, x_N\})$; then we have*

$$\kappa^{-1}/4 \le \lambda_1 \le \kappa^{-1},$$

where

$$\kappa^{-1} := \min_{1 \le n \le m \le N} \left[\left(\sum_{i=0}^{n-1} \frac{1}{\phi_i} \right)^{-1} + \left(\sum_{i=m}^{N} \frac{1}{\phi_i} \right)^{-1} \right] \left(\sum_{j=n}^{m} \mu(x_j) \right)^{-1},$$

in which $\phi_i := \phi(\{x_i, x_{i+1}\}) = \mu(x_i)\, L(x_i, x_{i+1})$, for $0 \le i \le N$.

Let G be a metric path and as in Section 1.1, define two measures μ and ρ on G as follows:

$$\mu := \sum_{x \in V} \mu(x)\delta_x, \qquad \rho := \sum_{\{x,y\} \in \mathsf{E}} \rho_{x,y}, \tag{16}$$

where for each edge $\{x, y\} \in \mathsf{E}$, $\rho_{x,y} := d_{x,y}/\phi(\{x, y\})$, in which $d_{x,y}$ is the natural Lebesgue measure on $[x, y]$. Note that one may think of ρ as a measure of resistance on the edges of G. Now, it is straightforward to generalize Lemma 24 to the metric model as follows.

Lemma 25. *Let G be a metric path endowed with the reversible generator (L, μ). Then, for every interval $A := (a, b) \subset G$, we have*

$$(1/4)\, \kappa^{-1}(A) \le \lambda_1(A) \le \kappa^{-1}(A),$$

where

$$\kappa^{-1}(A) := \inf_{a<x\le y<b} \left(\frac{1}{\rho((a,x))} + \frac{1}{\rho((y,b))} \right) \frac{1}{\mu([x,y])}, \tag{17}$$

where the measures μ, ρ are defined as in (16).

Proof. Assume that the infimum in (17) occurs for the points $x_0, y_0 \in G$. To prove the lemma, it is enough to show that $\{x_0, y_0\} \subseteq V(G)$ and apply Lemmas 2 and 24. By contradiction and without loss of generality assume that $x_0 \in G \setminus V(G)$. Then choose x_0' next to x_0 such that $\rho(a, x_0') > \rho(a, x_0)$ and (x_0, x_0') contains no vertices of $V(G)$. Then $\mu([x_0', y_0]) = \mu([x_0, y_0])$ and hence

$$\left(\frac{1}{\rho((a,x_0'))} + \frac{1}{\rho((y_0,b))} \right) \frac{1}{\mu([x_0',y_0])} < \left(\frac{1}{\rho((a,x_0))} + \frac{1}{\rho((y_0,b))} \right) \frac{1}{\mu([x_0,y_0])},$$

which is impossible. $\square$

Now, let us come back to the minimizing partition $\mathcal{A} = (A_1, \dots, A_k) \in \mathscr{P}_k$ for $\underline{\Lambda}_k$ considered at the beginning of this section. For all $i \in [\![k]\!]$, we denote by c_i the unique middle point of A_i, in the sense that $\rho((b_i, c_i)) = \rho((c_i, b_{i+1})) = \rho(A_i)/2$. Consider the k-partition $B := (B_1, \dots, B_k)$ such that

$$\forall\, i \in [\![k]\!], \quad B_i := (c_i, c_{i+1})$$

(with the convention $b_{k+1} = b_1$). There exists at least one index $i \in [\![k]\!]$ such that $\lambda_1(B_i) \ge \underline{\Lambda}_k = \lambda_1(\mathcal{A})$. Indeed, otherwise we would get

$$\max_{i\in[\![k]\!]} \lambda_1(B_i) < \underline{\Lambda}_k$$

which would be in contradiction with the definition of $\underline{\Lambda}_k$. The next result shows that this index i is the one that we were looking for.

Proposition 26. *The bound (15) is satisfied if $i \in [\![k]\!]$ is such that $\lambda_1(B_i) \ge \lambda_1(\mathcal{A})$.*

Proof. Without loss of generality assume that $i = k$, define $A_1' := A_k \sqcup \{b_1\} \sqcup A_1$ and consider the same notation as before: $A_1 = (b_1, b_2)$, $A_k = (b_k, b_1)$, $B_k = (c_k, c_1)$. Let $x_0, y_0 \in A_1'$ be two points that minimize (17) achieving $\kappa^{-1}(A_1')$. Thus

$$\kappa^{-1}(A_1') = \left(\frac{1}{\rho((b_k,x_0))} + \frac{1}{\rho((y_0,b_2))} \right) \frac{1}{\mu([x_0,y_0])}.$$

We prove the bound (15) through the following three cases.

Case 1. $\rho((b_k, x_0)) \ge (3/5)\,\rho(A_k)$ and $\rho((y_0, b_2)) \ge (3/5)\,\rho(A_1)$.

Then $c_k < x_0 < y_0 < c_1$ and we compute that

$$\begin{aligned} \rho((c_k,x_0)) &= \rho((b_k,x_0)) - \frac{\rho(A_k)}{2} \\ &\ge \frac{\rho((b_k,x_0))}{6}. \end{aligned}$$

Similarly $\rho((y_0, b_2)) \le 6\rho((y_0, c_1))$ and we deduce via Lemma 25 that

$$\lambda_1(A_1') \ge \frac{1}{4}\,\kappa^{-1}(A_1') = \frac{1}{4} \left(\frac{1}{\rho((b_k,x_0))} + \frac{1}{\rho((y_0,b_2))} \right) \frac{1}{\mu([x_0,y_0])}$$

$$\begin{aligned}&\geq \frac{1}{4}\left(\frac{1}{6\,\rho((c_k,x_0))}+\frac{1}{6\,\rho((y_0,c_1))}\right)\,\frac{1}{\mu([x_0,y_0])}\\&\geq \frac{1}{24}\,\kappa^{-1}(B_k)\geq \frac{1}{24}\,\lambda_1(\mathcal{A}).\end{aligned}$$

Case 2. $\rho((b_k,x_0)) < (3/5)\,\rho(A_k)$ and $\rho((y_0,b_2)) < (3/5)\,\rho(A_1)$.

Choose the points $d_1, d_2 \in A'_1$ as the middle points of the intervals (c_k, b_1) and (b_1, c_1), respectively, i.e.

$$\begin{aligned}&\rho((c_k,d_1))=\rho((d_1,b_1))=(1/4)\,\rho(A_k),\\&\rho((b_1,d_2))=\rho((d_2,c_1))=(1/4)\,\rho(A_k).\end{aligned}$$

Then, we have $x_0 < d_1 < d_2 < y_0$ and by Lemma 25 we have

$$\begin{aligned}&\lambda_1(\mathcal{A})\leq\kappa^{-1}(A_k)\leq\left(\frac{1}{\rho((b_k,x_0))}+\frac{1}{\rho((d_1,b_1))}\right)\,\frac{1}{\mu([x_0,d_1])},\\&\lambda_1(\mathcal{A})\leq\kappa^{-1}(B_k)\leq\left(\frac{1}{\rho((c_k,d_1))}+\frac{1}{\rho((d_2,c_1))}\right)\,\frac{1}{\mu([d_1,d_2])},\\&\lambda_1(\mathcal{A})\leq\kappa^{-1}(A_1)\leq\left(\frac{1}{\rho((b_1,d_2))}+\frac{1}{\rho((y_0,b_2))}\right)\,\frac{1}{\mu([d_2,y_0])}.\end{aligned}$$

Hence,

$$\begin{aligned}&\mu([x_0,d_1])\,\lambda_1(\mathcal{A})\leq\left(\frac{1}{\rho((b_k,x_0))}+\frac{12}{5\,\rho((b_k,x_0))}\right),\\&\mu([d_1,d_2])\,\lambda_1(\mathcal{A})\leq\left(\frac{12}{5\,\rho((b_k,x_0))}+\frac{12}{5\,\rho((y_0,b_2))}\right),\\&\mu([d_2,y_0])\,\lambda_1(\mathcal{A})\leq\left(\frac{12}{5\,\rho((y_0,b_2))}+\frac{1}{\rho((y_0,b_2))}\right),\end{aligned}$$

and consequently,

$$\mu([x_0,y_0])\,\lambda_1(\mathcal{A})\leq 6\,\left(\frac{1}{\rho((b_k,x_0))}+\frac{1}{\rho((y_0,b_2))}\right).$$

This, along with Lemma 25, implies that

$$\lambda_1(\mathcal{A})\leq 6\,\kappa^{-1}(A'_1)\leq 24\,\lambda_1(A'_1).$$

Case 3. $\rho((b_k,x_0)) < (3/5)\,\rho(A_k)$ and $\rho((y_0,b_2)) \geq (3/5)\,\rho(A_1)$.

Again, let d_1 be the middle point of the interval (c_k, b_1) and like in Case 2, we have

$$\begin{aligned}\mu([x_0,d_1])\,\lambda_1(\mathcal{A})&\leq\left(\frac{1}{\rho((b_k,x_0))}+\frac{1}{\rho((d_1,b_1))}\right)\\&\leq\left(\frac{1}{\rho((b_k,x_0))}+\frac{12}{5\,\rho((b_k,x_0))}\right),\\\mu([d_1,y_0])\,\lambda_1(\mathcal{A})&\leq\mu([d_1,y_0])\,\lambda_1(B_k)\\&\leq\left(\frac{1}{\rho((c_k,d_1))}+\frac{1}{\rho((y_0,c_1))}\right)\\&\leq\left(\frac{12}{5\,\rho((b_k,x_0))}+\frac{6}{\rho((y_0,b_2))}\right).\end{aligned}$$

Thus,

$$\mu([x_0, y_0])\, \lambda_1(\mathcal{A}) \le 6\left(\frac{1}{\rho((b_k, x_0))} + \frac{1}{\rho((y_0, b_2))}\right),$$

and the result follows. $\square$

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Appendix. Double coverings

In this appendix we present a spectral interpretation of parameters $(\underline{\Lambda}_k)_{k\in[\![N]\!]}$ and $(\overline{\Lambda}_k)_{k\in[\![N]\!]}$ via a double-covering construction that helps to lift partitions to bipartite partitions on a covering space. In this regard, a generator $L^{(2)}$ on a set $\mathsf{V}^{(2)}$ is said to be a double covering of L if there exists a mapping $\pi : \mathsf{V}^{(2)} \to \mathsf{V}$ such that:

- any $x \in \mathsf{V}$ admits two pre-images by π,
- for any $x \in \mathsf{V}^{(2)}$, the restriction of π on $N^{(2)}(x) := \{y \in \mathsf{V}^{(2)} : L^{(2)}(x, y) > 0\}$ is one to one from $N^{(2)}(x)$ onto $N(\pi(x)) := \{y \in \mathsf{V} : L(\pi(x), y) > 0\}$,
- for any $x, y \in \mathsf{V}^{(2)}$, either $L^{(2)}(x, y) = L(\pi(x), \pi(y))$ or $L^{(2)}(x, y) = 0$.

The first two conditions show that the graph $\mathsf{G}^{(2)}$ associated with $L^{(2)}$ is a double covering of G in the usual graph theory sense and the third one (in conjunction with the second one) says that $L^{(2)} \circ \pi = \pi \circ L$, where π is seen as acting on functions $f \in \mathscr{F}(\mathsf{V})$ through $\pi[f] = f \circ \pi$. It should be noted that if L is reversible with respect to the invariant measure μ, then $L^{(2)}$ is also reversible with respect to the invariant measure $\mu^{(2)}$ on $V^{(2)}$ defined as $\mu^{(2)}(x) = \mu(\pi(x))$. In what follows we prove:

Proposition 27. *Let $\mathcal{A}$ be a handy rectifiable and uniform partition. Then there exists a double covering $L^{(2)}$ of L such that $-\lambda_1(\mathcal{A})$ is an eigenvalue of $L^{(2)}$.*

Proof. For this fix $k \in [\![N]\!]$ and let $\mathcal{A} = (A_1, \ldots, A_k) \in \mathcal{P}_k$ be a handy, rectifiable and uniform partition. Now, we construct a double covering $L^{(2)}$ of the underlying generator L such that $\lambda_1(\mathcal{A})$ is an eigenvalue of $L^{(2)}$. We define

$$\mathsf{V}^{(2)} := \mathsf{V} \times \{-1, +1\}$$

and let $\pi : \mathsf{V}^{(2)} \to \mathsf{V}$ and $\sigma : \mathsf{V}^{(2)} \to \{-1, +1\}$ be the associated canonical projections. Also, for each $x \in \mathsf{V}$ define the index $I(x)$ as follows:

$$\forall\, x \in \mathsf{V}, \quad I(x) := \begin{cases} i & \text{if } x \in A_i, \\ \min\{i, j\} & \text{if } x \in \partial A_i \cap \partial A_j. \end{cases}$$

Now, define the generator $L^{(2)}$ on $\mathsf{V}^{(2)}$ as

$$\forall\, x, y \in \mathsf{V}^{(2)},$$

$$L^{(2)}(x, y) := \begin{cases} L(\pi(x), \pi(y)) & \text{if } I(\pi(x)) = I(\pi(y)) \text{ and } \sigma(x) = \sigma(y), \\ L(\pi(x), \pi(y)) & \text{if } I(\pi(x)) \ne I(\pi(y)) \text{ and } \sigma(x) \ne \sigma(y), \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

It can be verified that $L^{(2)}$ as defined in (18) is actually a double covering of L. At the level of processes, the intertwining relation $L^{(2)} \circ \pi = \pi \circ L$ implies that if $(X_t^{(2)})_{t\geq 0}$ is a Markov process whose generator is $L^{(2)}$, then $(\pi(X_t^{(2)}))_{t\geq 0}$ is a Markov process admitting L as its generator. The above relation also implies that the space of "even" functions (namely the functions on $\mathsf{V}^{(2)}$ which can be written in the form $f \circ \pi$, where f is a function defined on V) is left stable under $L^{(2)}$ and that its restriction to this space can be put in conjugacy with L. In particular, the spectrum of L is included in the spectrum of $L^{(2)}$.

Let $G^{(2)}$ be the metric graph associated with the generator $L^{(2)}$ as in the introduction. The mapping $\pi : \mathsf{V}^{(2)} \to \mathsf{V}$ admits a natural extension from $G^{(2)}$ to G, still denoted by π. It is straightforward to see that for any $i \in [\![k]\!]$, $\pi^{-1}(A_i) \subset G^{(2)}$ consists of two connected components: one, say B_{2i-1} is such that

$$B_{2i-1} \cap \mathsf{V}^{(2)} \subset \{x \in \mathsf{V}^{(2)} : \sigma(x) = -1\}$$

and the other one, say B_{2i}, satisfies

$$B_{2i} \cap \mathsf{V}^{(2)} \subset \{x \in \mathsf{V}^{(2)} : \sigma(x) = +1\}.$$

The sets B_i, for $i \in [\![2k]\!]$, are clearly disjoint, open and connected, and by the definition of $L^{(2)}$, $\mathcal{B} := (B_i)_{i\in[\![2k]\!]}$ is a handy $2k$-partition of $G^{(2)}$. In what follows, we prove that $\mathcal{B}$ is also bipartite, uniform and rectifiable, and consequently, the proof follows from Proposition 6.

As was remarked in the definition of $\mathcal{B}$, for any $i \in [\![k]\!]$, the restriction of the mapping σ to $B_i \cap \mathsf{V}^{(2)}$ is constant (with value $(-1)^i$). This leads us to define the mapping Σ on the vertex set $\mathsf{V}(\mathsf{G}_{\mathcal{B}})$ as $\Sigma(B_i) = (-1)^i$ which shows that $\mathsf{G}_{\mathcal{B}}$ is bipartite, with two parts $\{B \in \mathsf{V}(\mathsf{G}_{\mathcal{B}}) : \Sigma(B) = -1\}$ and $\{B \in \mathsf{V}(\mathsf{G}_{\mathcal{B}}) : \Sigma(B) = +1\}$.

Moreover, for $i \in [\![2k]\!]$, the operator $\widehat{L}^{(2)}_{B_i}$ defined on $B_i \cap \mathsf{V}^{(2)}$ (as defined preceding Lemma 2) is just the operator $\widehat{L}_{A_{\lfloor i/2\rfloor}}$ defined on $A_{\lfloor i/2\rfloor} \cap \mathsf{V}$ if functions on $A_{\lfloor i/2\rfloor} \cap \mathsf{V}$ are identified with functions on $B_i \cap \mathsf{V}^{(2)}$ via the operator π. We deduce that $\lambda_1(B_i) = \lambda_1(A_{\lfloor i/2\rfloor})$ and that $\mathcal{B}$ is uniform with $\lambda_1(\mathcal{B}) = \lambda_1(\mathcal{A})$.

For rectifiability, let $(r_i)_{i\in[\![k]\!]}$ be a rectifying family for $\mathcal{A}$ and consider the family $(r_i^{(2)})_{i\in[\![k]\!]}$ given by

$$\forall\, i \in [\![2k]\!], \quad r_i^{(2)} := r_{\lfloor (i+1)/2\rfloor}.$$

Then it is easy to verify that Equality (7) naturally lifts to the covering space $G^{(2)}$ and consequently, $(r_i^{(2)})_{i\in[\![k]\!]}$ is a rectifying family for $\mathcal{B}$. □

Remark 28. The above construction is valid for a general partition to give rise to a bipartite partition relatively to a double covering of the initial generator. □

If Conjecture 8 was true, we could conclude that generically, for any $k \in [\![N]\!]$ there exists a double covering $L^{(2)}$ of L such that $-\underline{\Lambda}_k$ is an eigenvalue of $L^{(2)}$ (and there exists another double covering of L which admits $\overline{\Lambda}_k$ as an eigenvalue). If these double coverings exist, they depend on the value of $k \in [\![N]\!]$, as is the case for cycles. But in the following proposition, we will see that for the generator L on the cycle $\mathbb{Z}_N$, all parameters $(-\underline{\Lambda}_k)_{k\in[\![N]\!]}$ and $(-\overline{\Lambda}_k)_{k\in[\![2,N-1]\!]}$ appear in the spectrum of a double covering of L.

Proposition 29. *Let L be a generator on the cycle $\mathbb{Z}_N$. Let $\mathcal{L}^{(2)}$ be the unique irreducible double covering of L. Denote as $(\underline{\Lambda}_k^{(2)})$ and $(\overline{\Lambda}_k^{(2)})$ its Dirichlet connectivity spectra. The ordinary* *spectrum of* $L^{(2)}$ *is given by*

$$0 = \underline{\Lambda}_1 < \underline{\Lambda}_2^{(2)} \le \overline{\Lambda}_2^{(2)} < \underline{\Lambda}_2 \le \overline{\Lambda}_2 < \underline{\Lambda}_3 \le \overline{\Lambda}_3 < \cdots < \underline{\Lambda}_{N-1} \le \overline{\Lambda}_{N-1} < \underline{\Lambda}_N.$$

Moreover, we have

$$\forall\, k \in [\![2, N]\!], \qquad \begin{cases} \underline{\Lambda}_{2k}^{(2)} = \underline{\Lambda}_k \\ \overline{\Lambda}_{2k}^{(2)} = \overline{\Lambda}_k. \end{cases}$$

Proof. Let $k \in [\![N]\!]$ be an integer and $\mathcal{A} = (A_1, \ldots, A_k) \in \mathcal{P}_k$ be a minimizing partition for $\underline{\Lambda}_k$ and consider the double covering $L^{(2)}$ defined as in (18). Note that in general this construction strongly depends on the partition $\mathcal{A}$; however, in the case of a cycle, up to an isomorphism, the generator $L^{(2)}$ only depends on the parity of k, i.e. if k is odd, $L^{(2)}$ is $\mathcal{L}^{(2)}$, the periodic doubling of L on $\mathbb{Z}_{2N}$ introduced in the statement of the above proposition, and if k is even, $L^{(2)}$ just acts as L on two disconnected copies of $\mathsf{V} = \mathbb{Z}_N$.

Now, for $k \in [\![3, N]\!]$ an odd integer, Propositions 7 and 27 imply that $-\underline{\Lambda}_k$ is an eigenvalue of $L^{(2)}$. Also, since the double covering for an odd k is independent of k, the spectrum of $\mathcal{L}^{(2)}$ contains $-\underline{\Lambda}_k$ and $-\overline{\Lambda}_k$ for every odd $k \in [\![3, N]\!]$. Furthermore, since the spectrum of L is included in the spectrum of $L^{(2)}$, the parameters $-\underline{\Lambda}_1 = 0$ and $-\underline{\Lambda}_k$ and $-\overline{\Lambda}_k$ for even $k \in [\![2, N]\!]$ (except $-\overline{\Lambda}_N$) also appear in the spectrum of $L^{(2)}$. On the other hand, the same arguments as are used in the proof of Proposition 23 give rise to the fact that if $-\underline{\Lambda}_k = -\overline{\Lambda}_k$, this eigenvalue has multiplicity at least 2 for $L^{(2)}$. Hence, the spectrum of $L^{(2)}$ contains (with multiplicity) both Dirichlet connectivity spectra $(\underline{\Lambda}_k)_{k \in [\![N]\!]}$ and $(\overline{\Lambda}_k)_{k \in [\![2, N-1]\!]}$. Only two eigenvalues are missing in this description, $\underline{\Lambda}_2^{(2)}$ and $\overline{\Lambda}_2^{(2)}$, namely those admitting eigenfunctions with two nodal domains. Therefore, the first statement follows from Proposition 15. Moreover, the last statement follows from Proposition 9 applied to $\mathcal{L}^{(2)}$. $\square$

Clearly to get a spectral interpretation of $\underline{\Lambda}_k^{(2)}$ and $\overline{\Lambda}_k^{(2)}$ with k odd, one should consider the irreducible double covering of $\mathcal{L}^{(2)}$ (i.e. the irreducible 4-covering of L) whose graph is $\mathbb{Z}_{4N}$.

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