An example of application of discrete Hardy's inequalities

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Summary : After having given a short proof of weighted Hardy's inequalities on $\mathbb{N}$, adapted from the continuous case, we will show how to use them to evaluate, up to an universal factor, spectral gaps and logarithmic Sobolev constants of birth and death processes on $\mathbb{Z}$. We will illustrate this method by calculations on an example.

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1 Introduction

Weighted discrete Hardy's inequalities are a simple and very efficient way to evaluate spectral gaps on trees, since theoretically this method enables one to get an estimation up to a factor 32 (or 8 if, as below, the tree is the usual one on $\mathbb{Z}$). The purpose of this note is to present this approach and to give an example of such a calculation. The birth and death Markov kernels under study were considered by C. Meise, who wanted to know the order of their spectral gap, to evaluate the speed of convergence to equilibrium of an other Markov chain on the n-uples, $n \in \mathbb{N}^* \setminus \{1\}$, of elements of the additive group $\mathbb{Z}_3$, whose union set is generating (for more details about this problem, see [8], where Meise gets its needed estimation by using a method due to Zeifman [13, 5]).

In fact, there is already a lot of methods to evaluate the spectral gap of a birth and death process, for instance: nonnegative matrix approach ([12]), Zeifman's method ([13, 5]), optimal path considerations ([6]), Chen's coupling and distance arguments ([7]), variants of Cheeger's inequalities ([2]) ... But in each of these approachs, there is a "good" choice to be made: one has to find a function in [12] or [13, 5], a length function in [6], a distance in [7], a renormalisation in [2], and there is no universal way to make such a suitable guess, one has to try and see on the specific examples at hand.

The main advantage of the Hardy's inequalities is that they do not depend on yours prediction skills, they gives you a (relatively) explicit estimation of the spectral gap in terms of yours data (birth and death rates and the associated reversible probability). Sure, this evaluation is only good up to an universal factor 8, so in particular examples the previous methods can be more efficient, nevertheless in practice this factor is not a crucial point (see for instance the consequences of the estimation needed by Meise), and the robustess aspect of the Hardy's inequalities seems more important.

Furthermore, this approach is also valid for the estimation of the Sobolev-logarithmic constant (see the third section below), and for the latter, there is no other general method. Indeed, the Hardy's inequalities give criteria for the existence of positive spectral gap and Sobolev-logarithmic constant for birth and death processes, in the spirit of the work of Bobkov and Götze for diffusions on $\mathbb{R}$ (see [1]).

But the serious drawback of the discrete Hardy's inequalities is that they can only be applied to Markov processes whose underlying graph is a tree (nevertheless subtrees can be useful for general finite Markov processes, see for instance [9]).

Let us first recall the weighted discrete Hardy's inequalities on $\mathbb{N}$, with a short proof of these bounds, directly adapted from the one given by Muckenhoupt [10] for the similar continuous result on $\mathbb{R}_+$. For the general case of weighted discrete Hardy's inequalities on rooted trees, see the appendice of [9], where these inequalities are deduced from those given by Evans, Harris and Pick in [4] for continuous trees (whose proof was itself inspired by the approach considered by Sawyer in [11] for $\mathbb{R}_+$).

So let μ and ν be two positive functions on $\mathbb{N}^*$, which will be seen as measures, we are interested in the smallest constant $A \in \mathbb{R}_+^* \sqcup \{+\infty\}$ such that the following inequalities are satisfied for all functions $f : \mathbb{N}^* \to \mathbb{R}$,

$$\sum_{x\in\mathbb{N}^*} \mu(x)\left(\sum_{0<y\leq x} f(y)\right)^2 \leq A \sum_{x\in\mathbb{N}^*} \nu(x) f^2(x)$$

To estimate A, let us introduce the constant B defined by

$$B \;=\; \sup_{x>0}\left(\sum_{y=1}^{x}\frac{1}{\nu(y)}\right)\sum_{y\geq x}\mu(y)$$

Then the weighted (by μ and ν) discrete Hardy's inequalities on $\mathbb{N}$ just say

Proposition 1

The constant B is a good approximation of A, in the sense that we are always assured of

$$B \leq A \leq 4B$$

Proof:

We begin by proving that $A \leq 4B$: let f be any function of $\mathbb{L}^2(\mathbb{N}^*,\nu)$, we have to see that

$$\sum_{x\in\mathbb{N}^*}\left(\sum_{0<y\leq x}f(y)\right)^2\mu(x) \;\leq\; 4B\sum_{x\in\mathbb{N}^*}f^2(x)\,\nu(x) \tag{1}$$

For $x\in\mathbb{N}^*$, denote

$$N(x) \;=\; \sum_{y=1}^{x}\frac{1}{\nu(y)}$$

We start from the left hand side of (1), for which we use a Cauchy-Schwarz inequality, to obtain the following upper bound

$$\sum_{x\in\mathbb{N}^*}\mu(x)\sum_{0<y\leq x}f^2(y)\nu(y)N^{\frac{1}{2}}(y)\sum_{0<z\leq x}\frac{1}{\nu(z)}\frac{1}{N^{\frac{1}{2}}(z)}$$

To deal with the last sum, we use the concavity inequality saying that for all $a,b>0$,

$$b^{\frac{1}{2}}-a^{\frac{1}{2}} \geq \frac{1}{2}\frac{1}{b^{\frac{1}{2}}}(b-a)$$

so we are assured that for all $z>0$ (with the convention that $N(0)=0$),

$$\frac{1}{\nu(z)}\frac{1}{N^{\frac{1}{2}}(z)} \;\leq\; 2(N^{\frac{1}{2}}(z)-N^{\frac{1}{2}}(z-1))$$

Thus we get

$$\begin{aligned}\sum_{0<z\leq x}\frac{1}{\nu(z)}\frac{1}{N^{\frac{1}{2}}(z)} \;&\leq\; 2N^{\frac{1}{2}}(x)\\ &\leq\; 2\left(\frac{B}{\mu([x,+\infty[)}\right)^{\frac{1}{2}}\end{aligned}$$

We now have to consider

$$2\sqrt{B}\sum_{y\in\mathbb{N}^*}f^2(y)\nu(y)N^{\frac{1}{2}}(y)\sum_{x\geq y}\mu(x)\frac{1}{\mu^{\frac{1}{2}}([x,+\infty[)}$$

and to do so we use again the previous convexity inequality, to obtain

$$\begin{aligned}\sum_{x\geq y}\mu(x)\frac{1}{\mu^{\frac{1}{2}}([x,+\infty[)} &\leq 2\sum_{x\geq y}\mu^{\frac{1}{2}}([x,+\infty[)-\mu^{\frac{1}{2}}([x+1,+\infty[)\\ &= 2\mu^{\frac{1}{2}}([y,+\infty[)\\ &\leq 2\left(\frac{B}{N(y)}\right)^{\frac{1}{2}}\end{aligned}$$

Putting together all these calculations, we end up with (1).

To prove the simpler lower bound $A\geq B$, it is enough to apply the inequality

$$\sum_{x\in\mathbb{N}^*}\left(\sum_{0<y\leq x}f(y)\right)^2\mu(x) \;\leq\; A\sum_{x\in\mathbb{N}^*}f^2(x)\,\nu(x)$$

with well chosen functions f.

More precisely, for $x_0\in\mathbb{N}^*$ fixed, let f be defined by

$$\forall\ y\in\mathbb{N}^*,\qquad f(y)=\begin{cases}1/\nu(y) & \text{, if } 0<y\leq x_0\\ 0 & \text{, otherwise}\end{cases}$$

we get therefore

$$\begin{aligned}A\sum_{0<y\leq x_0}\frac{1}{\nu(y)} &\geq \sum_{x\in\mathbb{N}^*}\left(\sum_{0<y\leq x}f(y)\right)^2\mu(x)\\ &\geq \sum_{x\geq x_0}\left(\sum_{0<y\leq x_0}\frac{1}{\nu(y)}\right)^2\mu(x)\end{aligned}$$

from where we deduce that $A\geq\mu([x_0,+\infty[)\sum_{0<y\leq x_0}1/\nu(y)$. To get the result we want, it remains to take the supremum in $x_0\in\mathbb{N}^*$.

□

Now let us see how this result can be applied to get an evaluation of the spectral gap for birth and death processes on $\mathbb{Z}$: so let $(a(i))_{i\in\mathbb{Z}}$ and $(b(i))_{i\in\mathbb{Z}}$ be two sequences of positive numbers, which will respectively stand for the birth and death rates. Up to a multiplicative constant, there is an unique associated reversible measure μ, i.e. satisfying for all $i\in\mathbb{Z}$, $\mu(i)a(i)=\mu(i+1)b(i+1)$. Let us assume that μ is in fact of finite weight, so from now on we can restrict ourselves to the case where it is a probability.

The spectral gap of this birth and death process is the ergodic constant defined by

$$\lambda \;=\; \inf_{f\in\mathbb{L}^2(\mu)\setminus\mathrm{Vect}(\mathbf{1})}\frac{\mathcal{E}(f,f)}{\mu((f-\mu(f))^2)}$$

where the Dirichlet form $\mathcal{E}$ is given for all functions $f\in\mathbb{L}^2(\mu)$ by

$$\mathcal{E}(f,f) \;=\; \sum_{i\in\mathbb{Z}}(f(i+1)-f(i))^2\mu(i)a(i)$$

For $i\in\mathbb{Z}$, let us define

$$\begin{aligned}B_+(i) &= \sup_{x>i}\left(\sum_{y=i+1}^{x}\frac{1}{\mu(y)b(y)}\right)\sum_{y\geq x}\mu(y)\\ B_-(i) &= \sup_{x<i}\left(\sum_{y=x}^{i-1}\frac{1}{\mu(y)a(y)}\right)\sum_{y\leq x}\mu(y)\end{aligned}$$

Let m denote a median of μ (i.e. an integer $m \in \mathbb{Z}$ such that $\mu(]-\infty, m[) \leq 1/2$ and $\mu(]m, +\infty[) \leq 1/2$), and let

$$B \;=\; B_+(m) \vee B_-(m)$$

Its interest appears in the following estimation:

Proposition 2

With the previous notations, we have

$$\frac{1}{4B} \leq \lambda \leq \frac{2}{B}$$

Proof:

We first prove the lower bound on the spectral gap: let $f \in \mathbb{L}^2(\mu) \setminus \mathrm{Vect}(\mathbb{1})$ be given, we consider the two functions defined by

$\forall\, x \in \mathbb{Z}$,

$$\begin{aligned} f_-(x) &= (f(x) - f(m))\mathbb{1}_{]-\infty,m]}(x) \\ f_+(x) &= (f(x) - f(m))\mathbb{1}_{[m,+\infty[}(x) \end{aligned}$$

We have for them

$$\begin{aligned} \mathcal{E}(f,f) &= \mathcal{E}(f - f(m)\mathbb{1}, f - f(m)\mathbb{1}) \\ &= \mathcal{E}(f_-, f_-) + \mathcal{E}(f_+, f_+) \end{aligned}$$

and

$$\begin{aligned} \mu((f - \mu(f))^2) &\leq \mu((f - f(m)\mathbb{1})^2) \\ &= \mu(f_-^2) + \mu(f_+^2) \end{aligned}$$

so

$$\begin{aligned} \frac{\mathcal{E}(f,f)}{\mu((f-\mu(f))^2)} &\geq \min\left(\frac{\mathcal{E}(f_-,f_-)}{\mu(f_-^2)}; \frac{\mathcal{E}(f_+,f_+)}{\mu(f_+^2)}\right) \\ &\geq (A_+ \vee A_-)^{-1} \end{aligned}$$

where A_+ (respectively A_-) is the best constant such that the following inequalities are satisfied for all functions $g : \mathbb{N}^* \to \mathbb{R}$,

$$\begin{aligned} \sum_{x \in \mathbb{N}^*} \mu(m+x) \left(\sum_{0<y\leq x} g(y)\right)^2 &\leq A_+ \sum_{x\in\mathbb{N}^*} \mu(m-1+x) a(m-1+x) g^2(x) \\ &= A_+ \sum_{x\in\mathbb{N}^*} \mu(m+x) b(m+x) g^2(x) \end{aligned}$$

(resp. $\sum_{x\in\mathbb{N}^*} \mu(m-x)\left(\sum_{0<y\leq x} g(y)\right)^2 \leq A_- \sum_{x\in\mathbb{N}^*} \mu(m-x)a(m-x)g^2(x)$).

This fact shows that the bound $\lambda \geq 1/(4B)$ follows from the Hardy's inequalities.

For the reversed inequality $\lambda \leq 2/B$, by symmetry of the problem we can assume for instance that $B = B_+(m)$, and we begin by treating the case where $B < +\infty$. Let $0 < \epsilon < B_+(m)$ be given and consider a function $g : \mathbb{N}^* \to \mathbb{R}$ (without any restriction, we could assume that g is taking values in $\mathbb{R}_+$) such that

$$\frac{\sum_{x\in\mathbb{N}^*}\mu(m+x)\left(\sum_{0<y\leq x} g(y)\right)^2}{\sum_{x\in\mathbb{N}^*}\mu(m+x)b(m+x)g^2(x)} \;\geq\; A_+-\epsilon \tag{2}$$

where the numerator can be supposed to be finite and non-zero.

We define a non-constant function $f : \mathbb{Z} \to \mathbb{R}$ by setting

$$\forall\, x\in\mathbb{Z},\qquad f(x) = \begin{cases} 0 & ,\text{ if } x\leq m \\ g(1)+\cdots+g(x-m) & ,\text{ if } x>m \end{cases}$$

As $\mu(\{f=0\}) \geq \mu(]-\infty,m]) \geq 1/2$, we get by a Cauchy-Schwarz inequality that $\mu(f)^2 \leq \mu(\{f>0\})\mu(f^2) \leq \mu(f^2)/2$, so

$$\begin{aligned}\mu((f-\mu(f))^2) &= \mu(f^2)-\mu(f)^2 \\ &\geq \mu(f^2)/2 \\ &\geq (A_+-\epsilon)\mathcal{E}(f,f)/2 \\ &\geq (B_+-\epsilon)\mathcal{E}(f,f)/2 \\ &\geq (B_+-\epsilon)\lambda\mu((f-\mu(f))^2)/2\end{aligned}$$

from which we get $\lambda \leq 2(B_+-\epsilon)^{-1}$. Letting ϵ go to 0_+, we obtain the expected result. If $B=+\infty$, we proceed in same manner, but we have to consider functions g such that the left hand side of (2) is finite but very large.

□

In practice, it can be difficult to find exactly a median, and sometimes it is better to consider the constant

$$B' \;=\; \inf_{i\in\mathbb{Z}}(B_+(i)\vee B_-(i))$$

because we are still assured of the same bounds (which also show that B and B' are of the same order).

Proposition 3

As before, we can approximate λ by $1/B'$:

$$\frac{1}{4B'} \leq \lambda \leq \frac{2}{B'}$$

Proof:

The first part of the proof of the proposition 2 shows in fact that for all $i\in\mathbb{Z}$ (cut the function $f-f(i)\mathbb{1}$ in i instead of cutting $f-f(m)\mathbb{1}$ in m), we are assured of $\lambda \leq (4(B_+(i)\vee B_-(i)))^{-1}$. On the other hand, the bound $\lambda \leq 2/B'$ follows from the trivial inequality $B'\leq B$. □

This result shows that if one wants to get a "good" lower bound of the spectral gap (which is often the critical point), one only need to guess an "adequate choice" of x and to apply the estimate

$$\lambda \;\geq\; \frac{1}{4(B_+(x)\vee B_-(x))}$$

Of course, the previous considerations are still valid if $\mathbb{Z}$ is replaced by a infinite or finite discrete interval (this situation has to be considered for instance if the $a(i)$ or the $b(i)$, $i \in \mathbb{Z}$, are only assumed to be non-negative), as it will be the case below. In fact, when the interval is not the whole $\mathbb{Z}$, but the rates of birth and death are positive on this interval (except on the frontier for one of them), we have furthermore unicity, up to a multiplicative constant, for the invariant measure μ, which is then reversible. This was not necessarily true for $\mathbb{Z}$.

2 The example

It is now time to introduce the example of Meise: it is a familly of birth and death processes indexed by $n \in \mathbb{N}^* \setminus \{1\}$, taking values in $\{1, \cdots, n\}$. More precisely, for all $n \in \mathbb{N}^* \setminus \{1\}$, the birth and death rates are respectively given by

$$\begin{aligned} \forall\, 1 \leq i < n, \qquad a_n(i) &= \frac{i(n-i)}{n(n-1)} \\ \forall\, 1 < i \leq n, \qquad b_n(i) &= \frac{i(i-1)}{pn(n-1)} \end{aligned}$$

where $p > 0$ is a fixed parameter.

Meise was interested in showing that the spectral gaps λ_n of these chains are of order $1/n$: there exists two constants $0 < c_1 < c_2 < +\infty$ such that for all $n \geq 2$, we have the bounds

$$\frac{c_1}{n} \leq \lambda_n \leq \frac{c_2}{n} \tag{3}$$

(in fact Meise has got a more precise result, as he can show that $\lim_{n\to\infty} n\lambda_n = 1$, note that such equivalences are out of reach by using only Hardy's bounds, but the goal of the following calculations is only to illustrate this method because they have some interesting features, maybe should we have chosen a more terrible example !).

In order to prove them, let us denote by μ_n the associated reversible probability on $\{1, \cdots, n\}$. It is quite immediate to calculate that it is given by

$$\forall\, 1 \leq i \leq n, \qquad \mu_n(i) \;=\; ((1+p)^n - 1)^{-1} p^i \binom{n}{i} = \frac{p^i n!}{((1+p)^n - 1)i!(n-i)!}$$

For $n > 1 + \lfloor 1/p \rfloor$, let us define m_n as $\lfloor z_n \rfloor + 1$, where $z_n = (pn-1)/(1+p)$ (and $\lfloor \cdot \rfloor$ denote the integer part), it is an interesting point in $\{1, \cdots, n\}$ because μ_n is non-decreasing (respectively decreasing) on $\{1, \cdots, m_n\}$ (resp. $\{m_n, \cdots, n\}$). More precisely, as we have for all $1 \leq i < n$,

$$\frac{\mu_n(i+1)}{\mu_n(i)} \;=\; \frac{a_n(i)}{b_n(i+1)} = \frac{p(n-i)}{i+1}$$

this ratio is decreasing on $\{1, \cdots, n\}$ and take the value 1 only for the real number $i = z_n$.

Using this remark, making the convention that $\mu_n(x) = 0$ for $x \notin \{1, \cdots, n\}$, and defining $l_n = 2\lfloor \sqrt{n} \rfloor + 3$, we see that for all $n > 1 + \lfloor 1/p \rfloor$,

$$\begin{aligned} \frac{\mu_n(m_n + l_n)}{\mu_n(m_n)} &\leq \left(\frac{p(n - z_n - \sqrt{n})}{z_n + 1 - \sqrt{n}} \right)^{\sqrt{n}} \\ &= \left(\frac{1 - p/\sqrt{n}}{1 + 1/\sqrt{n}} \right)^{\sqrt{n}} \\ &\leq \rho_1 \end{aligned}$$

where $\rho_1 = \exp(-p - 1/2) < 1$ (we have made use of the concavity inequalities $x/(x+1) \leq \ln(1+x) \leq x$, valid for all $x > -1$). With a similar calculation, we see that

$$\forall\, n > 1 + \lfloor 1/p \rfloor, \qquad \frac{\mu_n(m_n - l_n)}{\mu_n(m_n)} \;\leq\; \rho_2$$

with $\rho_2 = \exp(-1 - p/(1+p)) < 1$ (in fact, it could be possible to be numerically more precise here, and via Stirling formula, one can explicit the limit values, as n tends to infinity, of the previous ratios).

We deduce from these bounds that we have for $n > 1 + \lfloor 1/p \rfloor$,

$$\forall\, m_n \leq x \leq n, \qquad \frac{\mu_n(x + l_n)}{\mu_n(x)} \leq \rho_1$$

$$\forall\, 1 \leq x \leq m_n, \qquad \frac{\mu_n(x - l_n)}{\mu_n(x)} \leq \rho_2$$

But these estimations are only good near m_n (this is related to the fact that if by an affine transformation, we repace $[1, n]$ by $[-m_n/\sqrt{n}, (n - 1 - m_n)/\sqrt{n}]$, then the image of μ_n "looks like" a centered Gaussian distribution with variance $p/(1+p)^2$), and as we are away from this point with a distance of order n, it is easy to get better estimates: for instance, as

$$\frac{\mu_n(k_n - 1)}{\mu_n(k_n)} \;\leq\; \frac{1}{2}$$

with $k_n = \lfloor p(n+1)/(2+p) \rfloor$, we have

$$\forall\, 1 \leq x \leq k_n, \qquad \frac{\mu_n(x-1)}{\mu_n(x)} \leq \frac{1}{2}$$

In fact these informations are mainly the only ones needed to get a lower bound on λ_n by using the Hardy's inequalities from the point m_n. The heuristic reason is the following one: on $\{m_n, \cdots, n\}$ and on $\{k_n, \cdots, m_n\}$ (in reverse order), in a space scale $\sqrt{n}$, μ_n decreases somewhat faster than power laws (whose parameters ρ_1 and ρ_2 do not depend on n) and the rates of transition in the direction of m_n are bounded below by a constant which does not depend on n, and this contributes to an estimate of the spectral gap of order $1/n$. On the other hand, on the interval $\{1, \cdots, k_n\}$ (also looked at in reverse order), in a space scale 1, μ_n decreases in some sense faster than a power law (of parameter $1/2$), but the rates of transition toward k_n are at least $1/n$ (which reduce the speed of escape from 1 by the same factor), so this also gives an estimation of the spectral gap of order $1/n$. Here are the more detailed calculations: we begin by evaluating

$$B_{n,+} \;=\; \max_{x > m_n} \left(\sum_{y = m_n + 1}^{x} \frac{1}{\mu_n(y) b_n(y)} \right) \mu_n([x, n])$$

It is easy to explicit a $n_0 \geq 1 + \lfloor 1/p \rfloor$ large enough such that for all $n \geq n_0$ and all $m_n < y \leq n$, $b(y) \geq p/(2(p+1)^2)$. Let x be fixed in $\{m_n, \cdots, n\}$, then we have

$$\begin{aligned} \frac{p}{2(p+1)^2} \sum_{y = m_n + 1}^{x} \frac{1}{\mu_n(y) b_n(y)} \;&\leq\; \sum_{y = m_n + 1}^{x} \frac{1}{\mu_n(y)} \\ &=\; \sum_{i \in \mathbb{N}} \; \sum_{m_n \vee (x - (i+1) l_n) < y \leq m_n \vee (x - i l_n)} \frac{1}{\mu_n(y)} \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{i\in\mathbb{N}\,:\,x-il_n\ge m_n} l_n \frac{1}{\mu_n(x-il_n)} \\
&\le \sum_{i\in\mathbb{N}\,:\,x-il_n\ge m_n} l_n \rho_1^i \frac{1}{\mu_n(x)} \\
&\le \frac{l_n}{1-\rho_1}\frac{1}{\mu_n(x)}
\end{aligned}
$$

while

$$
\begin{aligned}
\mu_n([x,n]) &\le \sum_{i\in\mathbb{N}} \sum_{n\wedge(x+il_n)\le y< n\wedge(x+(i+1)l_n)} \mu_n(y) \\
&\le \sum_{i\in\mathbb{N}\,:\,x+il_n\le n} l_n \mu_n(x+il_n) \\
&\le \sum_{i\in\mathbb{N}\,:\,x-il_n\le n} l_n \rho_1^i \mu_n(x) \\
&\le \frac{l_n}{1-\rho_1}\mu_n(x)
\end{aligned}
$$

so we end up with

$$
B_{n,+} \le \frac{2(p+1)^2}{p}\left(\frac{l_n}{1-\rho_1}\right)^2
$$

Then we look at

$$
B_{n,-} = \max_{x<m_n}\left(\sum_{y=x}^{m_n-1}\frac{1}{\mu_n(y)a_n(y)}\right)\mu_n([1,x])
$$

and we first consider the case of $x\in\{k_n,\cdots,m_n-1\}$, because it is then possible to find explicitly a $n_1\ge n_0$ large enough such that for all $n\ge n_1$, we are assured of $\inf_{k_n\le y<m_n} a_n(y)\ge p/(2(2+p)^2)$, and thus we have as before, for $n\ge n_1$,

$$
\begin{aligned}
\frac{p}{2(p+2)^2}\sum_{y=x}^{m_n-1}\frac{1}{\mu_n(y)a_n(y)} &\le \frac{l_n}{1-\rho_2}\frac{1}{\mu_n(x)} \\
\mu_n([1,x]) &\le \frac{l_n}{1-\rho_2}\mu_n(x)
\end{aligned}
$$

so

$$
\max_{k_n\le x<m_n}\left(\sum_{y=x}^{m_n-1}\frac{1}{\mu_n(y)a_n(y)}\right)\mu_n([1,x]) \le \frac{2(p+2)^2}{p}\left(\frac{l_n}{1-\rho_2}\right)^2
$$

For $x\in\{1,\cdots,k_n\}$, we use the following bounds, which are proved in same way

$$
\begin{aligned}
\sum_{y=x}^{k_n-1}\frac{1}{\mu_n(y)} &\le \frac{1}{1-1/2}\frac{1}{\mu_n(x)} \\
\mu_n([1,x]) &\le \frac{1}{1-1/2}\mu_n(x)
\end{aligned}
$$

to obtain, via the inequality $a_n(y)\ge 1/n$, satisfied for all $1\le y<n$,

$$
\max_{1\le x<k_n}\sum_{y=x}^{k_n-1}\frac{1}{\mu_n(y)a_n(y)}\mu_n([1,x]) \le 4n
$$

but we are also assured of

$$\max_{1\leq x<k_n} \sum_{y=k_n}^{m_n-1} \frac{1}{\mu_n(y)a_n(y)}\mu_n([1,x]) \;\leq\; \sum_{y=k_n}^{m_n-1} \frac{1}{\mu_n(y)a_n(y)}\mu_n([1,k_n])$$
$$\leq\; \frac{2(p+2)^2}{p}\left(\frac{l_n}{1-\rho_2}\right)^2$$

so in the end

$$B_{n,-} \;\leq\; 4n + \frac{2(p+2)^2}{p}\left(\frac{l_n}{1-\rho_2}\right)^2$$

from where we easily get the existence of a constant $C_1 > 0$ such that

$$B_{n,-} \vee B_{n,+} \;\leq\; C_1 n$$

i.e. the above mentionned lower bound for λ_n is proved with $c_1 = 1/(4C_1)$, at least for $n \geq n_1$.

To prove a related upper bound, just use the definition of λ_n with the particular function $\mathbb{1}_{\{1\}}$, it gives $\lambda_n \leq (n[1-pn/((1+p)^n-1)])^{-1} \sim 1/n$, for n large.

3 Logarithmic-Sobolev inequalities

The weighted Hardy's inequalities are also an efficient method to evaluate logarithmic Sobolev constants on trees. By using some ideas and results of Bobkov and Götze [1], who have seen logarithmic Sobolev inequalities as Poincaré's inequalities in the Orlicz space associated to the Young function $\Psi : \mathbb{R}_+ \ni x \mapsto x\ln(1+x)$, we have presented this approach for the discrete setting in [9]. Let us give these estimations in the special case of $\mathbb{Z}$ endowed with its natural tree structure. So we come back to the general birth and death process of positive rates $(a(i))_{i\in\mathbb{Z}}$ and $(b(i))_{i\in\mathbb{Z}}$, reversible with respect to probability μ. The associated logarithmic Sobolev constant is defined by

$$\alpha \;=\; \inf_{f\in\mathbb{L}^2(\mu)\setminus\mathrm{Vect}(\mathbb{1})} \frac{\mathcal{E}(f,f)}{\mathrm{Ent}(f^2,\mu)}$$

where the entropy of the function f^2 with respect to μ is the quantity $\mathrm{Ent}(f^2,\mu) = \int f^2 \ln(f^2/\|f\|^2_{\mathbb{L}^2(\mu)})\,d\mu$.

To give an approximation of this ergodic constant α, we introduce for all $i \in \mathbb{Z}$, the numbers

$$C_+(i) \;=\; \sup_{x>i}\left(\sum_{y=i+1}^{x} \frac{1}{\mu(y)b(y)}\right)\mu([x,+\infty[)\,|\ln(\mu([x,+\infty[))|$$
$$C_-(i) \;=\; \sup_{x<i}\left(\sum_{y=x}^{i-1} \frac{1}{\mu(y)a(y)}\right)\mu(]-\infty,x])\,|\ln(\mu(]-\infty,x]))|$$

and we consider

$$C \;=\; \inf_{i\in\mathbb{Z}}(C_-(i)\vee C_+(i))$$

Then it can be shown that

Proposition 4

> The constant C^{-1} is of the same order as α, since the following bounds are always satisfied
>
> $$\frac{1}{20}\frac{1}{C} \le \alpha \le \frac{4}{3}\left(1-\frac{\sqrt{5}}{2\sqrt{2}}\right)^{-2}\frac{1}{C}$$

If we restrict ourselves to the case of a finite interval, one deduces from the lower bound that

$$\alpha \quad \geq \quad \frac{\lambda}{40\,|\ln(\mu_*)|}$$

where μ_* is the minimum value taken by μ on the finite interval.

But there is a better result in this direction, due to Diaconis and Saloff-Coste [3], saying that for all finite and irreducible Markov kernels (without assumptions on the underlying graph),

$$\alpha \quad \geq \quad \frac{1-2\mu_*}{\ln(\mu_*^{-1}-1)}\lambda$$

where λ, α and μ_* are respectively the spectral gap, the logarithmic Sobolev constant and the minimum value of the invariant probability.

Of course, there is not a related upper bound (up to an universal constant, but we are assured of $\alpha \le \lambda/2$), nevertheless, in the example of Meise, this gives the right order of the logarithmic Sobolev constants α_n: there are two constants $0 < c_3 < c_4 < +\infty$, such that for all $n \in I\!\!N^* \setminus \{1\}$,

$$\frac{c_3}{n^2} \le \alpha_n \le \frac{c_4}{n^2}$$

To get the lower bound, just note that $\mu_{n,*} = \mu_n(1) \wedge \mu(n) = (pn \wedge p^n)/((1+p)^n - 1)$, and the upper bound can be obtained by considering the function $\mathbf{1}_{\{1\}}$ in the definition of α_n.

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