

# A stochastic algorithm finding generalized means on compact manifolds

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## Abstract

A stochastic algorithm is proposed, finding the set of generalized means associated to a probability measure  $\nu$  on a compact Riemannian manifold and a continuous cost function  $\kappa$  on  $M \times M$ . Generalized means include  $p$ -means for  $p \in (0, \infty)$ , computed with any continuous distance function, not necessarily the Riemannian distance. They also include means for lengths computed from Finsler metrics, or for divergences.

The algorithm is fed sequentially with independent random variables  $(Y_n)_{n \in \mathbb{N}}$  distributed according to  $\nu$  and this is the only knowledge of  $\nu$  required. It evolves like a Brownian motion between the times it jumps in direction of the  $Y_n$ . Its principle is based on simulated annealing and homogenization, so that temperature and approximations schemes must be tuned up. The proof relies on the investigation of the evolution of a time-inhomogeneous  $\mathbb{L}^2$  functional and on the corresponding spectral gap estimates due to Holley, Kusuoka and Stroock.

**Keywords:** Stochastic algorithms, simulated annealing, homogenization, probability measures on compact Riemannian manifolds, intrinsic means, instantaneous invariant measures, Gibbs measures, spectral gap at small temperature.

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# 1 Introduction

The purpose of this paper is to present a stochastic algorithm finding the generalized means of a probability measure  $\nu$  defined on a compact manifold  $M$ . A Riemannian metric is used, only for the algorithm.

Algorithms for finding means, medians or minimax centers have been the object of many investigations, see e.g. [20], [25], [13], [14], [7], [26], [9], [1], [5], [10], [6]. In these references a gradient descent algorithm is used, or a stochastic version of it avoiding to compute the gradient of the functional to minimize. Either the functional to minimize has only one local minimum which is also global, or ([9]) a local minimum is sought. The case of Karcher means in the circle is investigated in [11] and [16]. In this special situation the global minimum of the functional can be found by more or less explicit formula.

For generalized means on compact manifolds the situation is different since the functional (1) to minimize may have many local minima, and no explicit formula for a global minimum can be expected. In [2] the case of compact symmetric spaces has been investigated and a continuous inhomogeneous diffusion process has been constructed which converges in probability to the set of  $p$ -means. In [3] the case of  $p$ -means on the circle is treated. A Markov process is constructed which has Brownian continuous part and more and more frequent jumps in the direction of independent random variables with law  $\nu$ . It is proven that it converges in probability to the set of  $p$ -means of  $\nu$ . Both [2] and [3] use simulated annealing techniques.

The purpose of this paper is to extend the construction in [3] to all compact manifolds, and to all generalized means.

So let be given  $\nu$  a probability measure on  $M$ , a compact Riemannian manifold. Denote by  $\kappa : M \times M \rightarrow \mathbb{R}$  a continuous function and consider the continuous mapping

$$U : M \ni x \mapsto \int_M \kappa(x, y) \nu(dy). \quad (1)$$

A global minimum of  $U$  is called a  $\kappa$ -mean or generalized mean of  $\nu$  and let  $\mathcal{M}$  be their set, which is non-empty in the above compact setting.

In practice the knowledge of  $\nu$  is often given by a sequence  $Y := (Y_n)_{n \in \mathbb{N}}$  of independent random variables, identically distributed according to  $\nu$ . So let us present a stochastic algorithm using this data and enabling to find some elements of  $\mathcal{M}$ . It is based on simulated annealing and homogenization procedures. Thus we will need respectively an inverse temperature evolution  $\beta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , an inverse speed up evolution  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$  and a regularization function  $\delta : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ . Typically,  $\beta_t$  is non-decreasing,  $\alpha_t$  and  $\delta_t$  are non-increasing and we have  $\lim_{t \rightarrow +\infty} \beta_t = +\infty$ ,  $\lim_{t \rightarrow +\infty} \alpha_t = 0$  and  $\lim_{t \rightarrow +\infty} \delta_t = 0$ , but we are looking for more precise conditions so that the stochastic algorithm we describe below finds  $\mathcal{M}$ .

For finding  $\mathcal{M}$ , we will use a regularization of  $\kappa$  with the heat kernel  $p(\delta, x, z)$ . So define for  $\delta > 0$   $\kappa_\delta(x, y) = \int_M p(\delta, x, z) \kappa(z, y) \lambda(dz)$  where  $\lambda$  denotes the Lebesgue measure (namely the unnormalized Riemannian measure) and

$$U_\delta : M \ni x \mapsto \int_M \kappa_\delta(x, y) \nu(dy). \quad (2)$$

Notice any other regularization which would satisfy the estimates (16) and (23) below and which would be easier to compute could be used instead of the heat kernel.

Let  $N := (N_t)_{t \geq 0}$  be a standard Poisson process: it starts at 0 at time 0 and has jumps of length 1 whose interarrival times are independent and distributed according to exponential random variables of parameter 1. The process  $N$  is assumed to be independent from the chain  $Y$ . We define the speeded-up process  $N^{(\alpha)} := (N_t^{(\alpha)})_{t \geq 0}$  via

$$\forall t \geq 0, \quad N_t^{(\alpha)} := N_{\int_0^t \frac{1}{\alpha_s} ds}. \quad (3)$$

Consider the time-inhomogeneous Markov process  $X := (X_t)_{t \geq 0}$  which evolves in  $M$  in the following way: if  $T > 0$  is a jump time of  $N^{(\alpha)}$ , then  $X$  jumps at the same time, from  $X_{T-}$  to  $X_T := \phi_\delta(\beta_T \alpha_T, X_{T-}, Y_{N_T^{(\alpha)}})$ , where

$$s \mapsto \phi_\delta(s, x, y) \in M \quad (4)$$

is the value at time  $s$  of the flow started at  $x$  of the vector field  $z \mapsto -\frac{1}{2} \nabla_z \kappa_\delta(\cdot, y)$ . In particular

$$\phi'_\delta(s, x, y) = -\frac{1}{2} \nabla \kappa_\delta(\cdot, y)(\phi_\delta(s, x, y)) \quad (5)$$

where  $\phi'_\delta$  denotes the derivative with respect to the first variable.

Typically we will have  $\lim_{t \rightarrow +\infty} \alpha_t \beta_t \|\nabla \kappa_{\delta_t}\|_\infty = 0$ , so that for sufficiently large jump-times  $T$ ,  $X_T$  will be "between"  $X_{T-}$  and  $Y_{N_T^{(\alpha)}}$  and quite close to  $X_{T-}$ . To proceed with the construction, we require that between consecutive jump times (and between time 0 and the first jump time),  $X$  evolves as a Brownian motion, relatively to the Riemannian structure of  $M$  (see for instance the book of Ikeda and Watanabe [19]) and independently of  $Y$  and  $N$ . Informally, the evolution of the algorithm  $X$  can be summarized by the Itô equation (in centers of exponential charts)

$$\forall t \geq 0, \quad dX_t = \sigma(X_t) dB_t + \overrightarrow{X_{t-} \phi_{\delta_t}(\beta_t \alpha_t, X_{t-}, Y_{N_t^{(\alpha)}})} dN_t^{(\alpha)}$$

where  $(B_t)_{t \geq 0}$  is a Brownian motion on some  $\mathbb{R}^m$ , for all  $x \in M$   $\sigma(x) : \mathbb{R}^m \rightarrow T_x M$  is linear satisfying  $\sigma \sigma^* = \text{id}$ ,  $x \mapsto \sigma(x)$  is smooth, and where  $(Y_{N_t^{(\alpha)}})_{t \geq 0}$  should be interpreted as a fast auxiliary process. The law of  $X$  is then entirely determined by the initial distribution  $m_0 = \mathcal{L}(X_0)$ . More generally at any time  $t \geq 0$ , denote by  $m_t$  the law of  $X_t$ .

We will prove that the above algorithm  $X$  finds in probability at large times the set of means  $\mathcal{M}$ . Let us define a constant  $b \geq 0$ , coming from the theory of simulated annealing (cf. for instance Holley, Kusuoka and Stroock [15]) in the following way. For any  $x, y \in M$ , let  $\mathcal{C}_{x,y}$  be the set of continuous paths  $p := (p(t))_{0 \leq t \leq 1}$  going from  $p(0) = x$  to  $p(1) = y$ . The elevation  $U(p)$  of such a path  $p$  relatively to  $U$  is defined by

$$U(p) := \max_{t \in [0,1]} U(p(t))$$

and the minimal elevation  $U(x, y)$  between  $x$  and  $y$  is given by

$$U(x, y) := \min_{p \in \mathcal{C}_{x,y}} U(p).$$

Then we consider

$$b(U) := \max_{x, y \in M} U(x, y) - U(x) - U(y) + \min_M U \quad (6)$$

This constant can also be seen as the largest depth of a well not encountering a fixed global minimum of  $U$ . Namely, if  $x_0 \in \mathcal{M}$ , then we have

$$b(U) = \max_{y \in M} U(x_0, y) - U(y) \quad (7)$$

independently of the choice of  $x_0 \in \mathcal{M}$ .

With these notations, the main result of this paper is:

**Theorem 1** *For any scheme of the form*

$$\forall t \geq 0, \quad \begin{cases} \alpha_t & := (1+t)^{-1} \\ \beta_t & := c^{-1} \ln(1+t) \\ \delta_t & := \ln(2+t)^{-1} \end{cases} \quad (8)$$

where  $c > b(U)$ , we have for any neighborhood  $\mathcal{N}$  of  $\mathcal{M}$  and for any  $m_0$ ,

$$\lim_{t \rightarrow +\infty} m_t[\mathcal{N}] = 1 \quad (9)$$

Thus to find a element of  $\mathcal{M}$  with an important probability, one should pick up the value of  $X_t$  for sufficiently large times  $t$ .

A crucial ingredient of the proof of this convergence are the Gibbs measures associated to the potentials  $U_\delta$ . They are defined as the probability measures  $\mu_{\beta,\delta}$  given for any  $\beta \geq 0$  by

$$\mu_{\beta,\delta}(dx) := \frac{\exp(-\beta U_\delta(x))}{Z_{\beta,\delta}} \lambda(dx) \quad (10)$$

where  $Z_{\beta,\delta} := \int \exp(-\beta U_\delta(x)) \lambda(dx)$  is the normalizing factor.

Indeed we will show that  $\mathcal{L}(X_t)$  and  $\mu_{\beta_t,\delta_t}$  become closer and closer as  $t \geq 0$  goes to infinity in the sense of  $\mathbb{L}^2$  variation.

By uniform continuity of  $\kappa$  we can easily prove that  $U_\delta$  converges uniformly to  $U$  as  $\delta \rightarrow 0$ . As a consequence, for any neighbourhood  $\mathcal{N}$  of  $\mathcal{M}$ ,  $\mu_{\beta,\delta}(\mathcal{N})$  converges to 1 as  $\beta \rightarrow \infty$ , uniformly in  $\delta \leq \delta_0$  for some  $\delta_0 > 0$  depending on  $\mathcal{N}$ . All this will prove the theorem.

The main difference in the method between the present work and [3] concerns the definition of the jumps. Instead of following the geodesic from the current position to a realization of  $\nu$ , the process jumps to  $\phi_{\delta_t}(\beta_t \alpha_t, X_{t-}, Y_{N_t^{(\alpha)}})$ . The calculations are much easier, and this allows to consider more general cost functions  $\kappa$ . The drawback is that the implementation may be more complicated.

The cost functions  $\kappa$  are only assumed to be continuous. So this includes distances at the power  $p, \rho^p$ , which lead to  $p$ -means, for all  $p \in (0, \infty)$ . Notice the case  $p \in (0, 1)$  has never been considered in previous works. This also includes lengths for Finsler metrics, all divergences for parametric statistical models (Kullback-Leibler, Jeffrey, Chernoff, Hellinger...).

The paper is constructed on the following plan. In Section 2 we obtain an estimate of  $L_{\alpha,\beta,\delta}^* \mathbb{1}$  where  $L_{\alpha,\beta,\delta}^*$  is the adjoint of  $L_{\alpha,\beta,\delta}$  in  $L^2(\mu_{\beta,\delta})$ ,  $L_{\alpha,\beta,\delta}$  is the generator of the process  $X_t$  described above but with constant  $\alpha, \beta, \delta$ . The proof of Theorem 1 is given in Section 3. For this proof, the estimate of Section 2 is crucial to see how close is the instantaneous invariant measure associated to the algorithm at large times  $t \geq 0$  to the Gibbs measures associated to the potential  $U_{\delta_t}$  and to the inverse temperature  $\beta_t^{-1}$ .

## 2 Regularity issues

Let us consider a general probability measure  $\nu$  on  $M$ . For any  $\alpha > 0$ ,  $\beta \geq 0$  and  $\delta > 0$ , we are interested into the generator  $L_{\alpha,\beta,\delta}$  defined for  $f$  from  $\mathcal{C}^2(M)$  via

$$\forall x \in M, \quad L_{\alpha,\beta,\delta}[f](x) := \frac{1}{2} \Delta f(x) + \frac{1}{\alpha} \int (f(\phi_\delta(\beta \alpha, x, y)) - f(x)) \nu(dy) \quad (11)$$

We will prove in Section 3 that  $\mathcal{L}(X_t)$  gets closer and closer to the Gibbs distribution  $\mu_{\beta_t,\delta_t}$  as  $t \rightarrow \infty$ . Since for large  $\beta \geq 0$ ,  $\mu_{\beta,\delta}$  concentrates around  $\mathcal{M}$  uniformly in  $\delta$  sufficiently small, this will be sufficient to establish Theorem 1.

The remaining part of this section is devoted to a quantification of what separates  $\mu_{\beta,\delta}$  from being an invariant probability of  $L_{\alpha,\beta,\delta}$ , for  $\alpha > 0$ ,  $\delta > 0$  and  $\beta \geq 0$ . As it will become clearer in the next section, a practical way to measure this discrepancy is through the evaluation of  $\mu_{\beta,\delta}[(L_{\alpha,\beta,\delta}^* \mathbb{1})^2]$ , where  $L_{\alpha,\beta,\delta}^*$  is the dual operator of  $L_{\alpha,\beta,\delta}$  in  $\mathbb{L}^2(\mu_{\beta,\delta})$  and where  $\mathbb{1}$  is the constant function taking the value 1. Indeed, it can be seen that  $L_{\alpha,\beta,\delta}^* \mathbb{1} = 0$  in  $\mathbb{L}^2(\mu_{\beta,\delta})$  if and only if  $\mu_{\beta,\delta}$  is invariant for  $L_{\alpha,\beta,\delta}$ . Before being more precise about the definition of  $L_{\alpha,\beta,\delta}^*$ , we need an elementary result, where we will use the following notations: for any  $s \in \mathbb{R}$ ,  $T_{\delta,y,s}$  is the operator acting on measurable functions  $f$  defined on  $M$  via

$$\forall x \in M, \quad T_{\delta,y,s} f(x) := f(\phi_\delta(s, x, y)) \quad (12)$$

**Lemma 2** For any  $y \in M$ , any  $s \in [0, 1)$  and any measurable and bounded functions  $f, g$ , we have

$$\int_M g(x) T_{\delta, y, s} f(x) \lambda(dx) = \int_M f(z) T_{\delta, y, -s} g(z) |J\phi_\delta(-s, \cdot, y)|(z) \lambda(dz)$$

where  $dx$  and  $dz$  denote Lebesgue measure on  $M$  and  $J\phi_\delta(-s, \cdot, y)(z)$  is the determinant at  $z$  of the Jacobian matrix of  $\phi_\delta(-s, \cdot, y)$ .

**Proof**

Just make the change of variable  $z = \phi_\delta(s, x, y)$  in the first integral, which yields  $x = \phi_\delta(-s, z, y)$ .  $\blacksquare$

This lemma has for consequence the next result, where  $\mathcal{D}$  is the subspace of  $\mathbb{L}^2(\lambda)$  consisting of functions whose second derivatives in the distribution sense belongs to  $\mathbb{L}^2(\lambda)$  (or equivalently to  $\mathbb{L}^2(\mu_{\beta, \delta})$  for any  $\beta \geq 0$  and  $\delta > 0$ ).

**Lemma 3** For  $\alpha > 0$ ,  $\beta \geq 0$  and  $\delta > 0$ , the domain of the maximal extension of  $L_{\alpha, \beta, \delta}$  on  $\mathbb{L}^2(\mu_{\beta, \delta})$  is  $\mathcal{D}$ . Furthermore the domain of its dual operator  $L_{\alpha, \beta, \delta}^*$  in  $\mathbb{L}^2(\mu_{\beta, \delta})$  is also  $\mathcal{D}$  and we have for any  $f \in \mathcal{D}$ ,

$$\begin{aligned} L_{\alpha, \beta, \delta}^* f &= \frac{1}{2} \exp(\beta U_\delta) \Delta[\exp(-\beta U_\delta) f] + \frac{\exp(\beta U_\delta)}{\alpha} \int T_{\delta, y, -\alpha\beta}[\exp(-\beta U_\delta) f] |J\phi_\delta(-\alpha\beta, \cdot, y)| \nu(dy) - \frac{f}{\alpha} \end{aligned}$$

**Proof**

With the previous definitions, we can write for any  $\alpha > 0$ ,  $\beta \geq 0$  and  $\delta > 0$ ,

$$L_{\alpha, \beta, \delta} = \frac{1}{2} \Delta + \frac{1}{\alpha} \int T_{\delta, y, \alpha\beta} \nu(dy) - \frac{I}{\alpha}$$

where  $I$  is the identity operator. Note furthermore that the identity operator is bounded from  $\mathbb{L}^2(\lambda)$  to  $\mathbb{L}^2(\mu_{\beta, \delta})$  and conversely. Thus to get the first assertion, it is sufficient to show that  $\int T_{\delta, y, \alpha\beta} \nu(dy)$  is bounded from  $\mathbb{L}^2(\lambda)$  to itself, or even only that  $\|T_{\delta, y, \alpha\beta}\|_{\mathbb{L}^2(\lambda) \rightarrow \mathbb{L}^2(\lambda)}$  is uniformly bounded in  $y \in M$ . To see that this is true, consider a bounded and measurable function  $f$  and assume that  $\alpha\beta \geq 0$ . Since  $(T_{\alpha\beta} f)^2 = T_{\alpha\beta} f^2$ , we can apply Lemma 2 with  $s = \alpha\beta$ ,  $g = \mathbb{1}$  and  $f$  replaced by  $f^2$  to get that

$$\begin{aligned} \int (T_{\delta, y, \alpha\beta} f)^2(x) \lambda(dx) &= \int f^2(z) T_{\delta, y, -\alpha\beta} \mathbb{1} |J\phi_\delta(-\alpha\beta, \cdot, y)|(z) \lambda(dz) \\ &\leq J_{\delta, \infty} \int f^2 d\lambda \end{aligned}$$

with  $J_{\delta, \infty} = \sup_{z, y \in M} |J\phi_\delta(-\alpha\beta, \cdot, y)|(z)$ . This quantity is finite, since  $\kappa_\delta(\cdot, \cdot)$  belongs to the class  $\mathcal{C}^{\infty, 0}$ , due to its definition by convolution with a smooth kernel. Next to see that for any  $f, g \in \mathcal{C}^2(M)$ ,

$$\int g L_{\alpha, \beta, \delta} f d\mu_{\beta, \delta} = \int f L_{\alpha, \beta, \delta}^* g d\mu_{\beta, \delta} \quad (13)$$

where  $L_{\alpha, \beta, \delta}^*$  is the operator defined in the statement of the lemma, we note that, on one hand,

$$\begin{aligned} \int g \Delta f d\mu_{\beta, \delta} &= Z_{\beta, \delta}^{-1} \int \exp(-\beta U_\delta) g \Delta f d\lambda \\ &= \int f \exp(\beta U_\delta) \Delta[\exp(-\beta U_\delta) g] d\mu_{\beta, \delta} \end{aligned}$$

and on the other hand, for any  $y \in M$ ,

$$\begin{aligned} \int g T_{\delta,y,\alpha\beta} f d\mu_{\beta,\delta} &= Z_{\beta,\delta}^{-1} \int \exp(-\beta U_\delta) g T_{\delta,y,\alpha\beta} f d\lambda \\ &= Z_{\beta,\delta}^{-1} \int_M f T_{\delta,y,-\alpha\beta} (\exp(-\beta U_\delta) g) |J\phi_\delta(-\alpha\beta, \cdot, y)|(x) \lambda(dx) \end{aligned}$$

by Lemma 2. After an additional integration with respect to  $\nu(dy)$ , (13) follows without difficulty. To conclude, it is sufficient to see that for any  $f \in \mathbb{L}^2(\mu_{\beta,\delta})$ ,  $L_{\alpha,\beta,\delta}^* f \in \mathbb{L}^2(\mu_{\beta,\delta})$  (where  $L_{\alpha,\beta,\delta}^* f$  is first interpreted as a distribution) if and only if  $f \in \mathcal{D}$ . This is done by adapting the arguments given in the first part of the proof, in particular we get that

$$\left\| \frac{\exp(\beta U_\delta)}{\alpha} \int T_{\delta,y,-\alpha\beta} [\exp(-\beta U_\delta) \cdot] |J\phi_\delta(-\alpha\beta, \cdot, y)| \nu(dy) \right\|_{\mathbb{L}^2(\lambda) \ominus}^2 \leq \frac{J_{\delta,\infty}^2 \exp(2\beta \text{osc}(U_\delta))}{\alpha^2}.$$

■

For any  $\alpha > 0$  and  $\beta \geq 0$ , denote  $\eta = \alpha\beta$ . As a consequence of the previous lemma, we get that for any  $x \in M$ ,

$$\begin{aligned} L_{\alpha,\beta,\delta}^* \mathbf{1}(x) &= \frac{1}{2} \exp(\beta U_\delta(x)) \Delta \exp(-\beta U_\delta(x)) - \frac{1}{\alpha} \\ &\quad + \frac{\exp(\beta U_\delta(x))}{\alpha} \int T_{\delta,y,-\eta} [\exp(-\beta U_\delta)](x) |J\phi_\delta(-\eta, \cdot, y)|(x) \nu(dy) \\ &= \frac{\beta^2}{2} (|\nabla U_\delta|(x))^2 - \frac{\beta}{2} \Delta U_\delta(x) - \frac{1}{\alpha} \\ &\quad + \frac{1}{\alpha} \int_M \exp(\beta[U_\delta(x) - U_\delta(\phi_\delta(-\eta, x, y))]) |J\phi_\delta(-\eta, \cdot, y)|(x) \nu(dy) \end{aligned} \quad (14)$$

It appears that  $L_{\alpha,\beta,\delta}^* \mathbf{1}$  is continuous. The next result evaluates the uniform norm of this function.

**Proposition 4** *There exists a constant  $C > 0$ , depending on  $M$  and  $\|\kappa\|_\infty$ , such that for any  $\beta \geq 1$ ,  $\delta \in (0, 1]$  and  $\alpha \in (0, \delta^2/(2\beta^2))$  we have*

$$\|L_{\alpha,\beta,\delta}^* \mathbf{1}\|_\infty \leq C \alpha \beta^4 \delta^{-4}$$

### Proof

In view of the expression of  $L_{\alpha,\beta,\delta}^* \mathbf{1}(x)$  given before the statement of the proposition, we want to estimate for any fixed  $x \in \mathbb{T}$ , the quantity

$$\int_M \exp(\beta[U_\delta(x) - U_\delta(\phi_\delta(-\eta, x, y))]) |J\phi_\delta(-\eta, \cdot, y)|(x) \nu(dy).$$

Consider the function

$$\psi(s) = U_\delta(x) - U_\delta(\phi_\delta(s, x, y)).$$

It has derivative

$$\psi'(s) = \frac{1}{2} \langle \nabla U_\delta, \nabla \kappa_\delta(\cdot, y) \rangle (\phi_\delta(s, x, y))$$

and second derivative

$$\psi''(s) = -\frac{1}{4} \text{Hess } U_\delta (\nabla \kappa_\delta(\cdot, y)(\phi_\delta(s, x, y)), \nabla \kappa_\delta(\cdot, y)(\phi_\delta(s, x, y))) - \frac{1}{4} \langle \nabla U_\delta, \nabla_{\nabla \kappa_\delta(\cdot, y)(\phi_\delta(s, x, y))} \nabla \kappa_\delta(\cdot, y) \rangle.$$

For any  $\eta = \alpha\beta$ , there exists  $s \in [0, \eta]$  such that

$$\psi(-\eta) = \psi(0) - \eta \psi'(0) + \frac{\eta^2}{2} \psi''(-s).$$

This yields

$$\begin{aligned} \beta (U_\delta(x) - U_\delta(\phi(-\eta, x, y))) &= \frac{-\beta\eta}{2} \langle \nabla U_\delta, \nabla \kappa_\delta(\cdot, y) \rangle(x) \\ &- \frac{\beta\eta^2}{8} (\text{Hess } U_\delta(\nabla \kappa_\delta(\cdot, y)(\phi_\delta(-s, x, y)), \nabla \kappa_\delta(\cdot, y)(\phi_\delta(-s, x, y))) + \langle \nabla U_\delta, \nabla_{\nabla \kappa_\delta(\cdot, y)(\phi_\delta(-s, x, y))} \nabla \kappa_\delta(\cdot, y) \rangle) \end{aligned}$$

Observe that for any  $a, b \in \mathbb{R}$ , we can find  $u, v \in (0, 1)$  such that

$$\exp(a + b) = (1 + a + a^2 \exp(ua)/2)(1 + b \exp(vb)). \quad (15)$$

Apply this equality with

$$a = \frac{-\beta\eta}{2} \langle \nabla U_\delta, \nabla \kappa_\delta(\cdot, y) \rangle(x)$$

and

$$b = -\frac{\beta\eta^2}{8} (\text{Hess } U_\delta(\nabla \kappa_\delta(\cdot, y)(\phi_\delta(s, x, y)), \nabla \kappa_\delta(\cdot, y)(\phi_\delta(-s, x, y))) + \langle \nabla U_\delta, \nabla_{\nabla \kappa_\delta(\cdot, y)(\phi_\delta(-s, x, y))} \nabla \kappa_\delta(\cdot, y) \rangle).$$

Using the bounds

$$\forall \delta > 0, \quad \|\nabla \ln p(\delta, \cdot, y)(x)\| \leq \frac{C'}{\delta} \quad \text{and} \quad \|\nabla d \ln p(\delta, \cdot, y)(x)\| \leq \frac{C'}{\delta^2} \quad (16)$$

for some  $C' > 0$  (see e.g. [17]), writing

$$\nabla \kappa_\delta(\cdot, y)(x) = \int_M \nabla \ln p(\delta, \cdot, z) p(\delta, x, z) \kappa(z, y) \lambda(dz),$$

we get

$$\forall \delta > 0, \quad \|\nabla \kappa_\delta(\cdot, y)(x)\| \leq \frac{C}{\delta} \quad \text{and} \quad \|\nabla d \kappa_\delta(\cdot, y)(x)\| \leq \frac{C}{\delta^2} \quad (17)$$

with  $C = 2C' \|\kappa\|_\infty$ , together with

$$\|\nabla U_\delta(x)\| \leq \frac{C}{\delta} \quad \text{and} \quad \|\nabla d U_\delta(x)\| \leq \frac{C}{\delta^2}. \quad (18)$$

It follows that  $|a| = \mathcal{O}(\alpha\beta^2\delta^{-2})$  and  $|b| = \mathcal{O}(\alpha^2\beta^3\delta^{-4})$ , so in conjunction with the assumption  $\alpha\beta^2\delta^{-2} \leq 1/2$ , we can write with (15) that

$$\exp(\beta[U_\delta(x) - U_\delta(\phi_\delta(-\eta, x, y))]) = 1 - \frac{\beta\eta}{2} \langle \nabla U_\delta, \nabla \kappa_\delta(\cdot, y) \rangle(x) + \mathcal{O}(\alpha^2\beta^4\delta^{-4}). \quad (19)$$

Integrating this expression, we get that

$$\begin{aligned} &\int_M \exp(\beta[U_\delta(x) - U_\delta(\phi(-\eta, x, y))]) |J\phi_\delta(-\eta, \cdot, y)|(x) \nu(dy) \\ &= \int_M |J\phi_\delta(-\eta, \cdot, y)|(x) \nu(dy) \\ &- \frac{\beta\eta}{2} \int_M \langle \nabla U_\delta, \nabla \kappa_\delta(\cdot, y) \rangle(x) |J\phi_\delta(-\eta, \cdot, y)|(x) \nu(dy) + \mathcal{O}(\alpha^2\beta^4\delta^{-4}) \end{aligned}$$

where we used the fact that  $|J\phi_\delta(-\eta, \cdot, y)|(x)$  is uniformly bounded (see (20) below). We can now return to (14) and we obtain that for any  $x \in M$ ,

$$\begin{aligned} L_{\alpha, \beta, \delta}^* \mathbf{1}(x) &= \frac{\beta^2}{2} |\nabla U_\delta(x)|^2 - \frac{\beta}{2} \Delta U_\delta(x) + \frac{1}{\alpha} \int_M (|J\phi_\delta(-\eta, \cdot, y)|(x) - 1) \nu(dy) \\ &- \frac{\beta^2}{2} \int_M \langle \nabla U_\delta, \nabla \kappa_\delta(\cdot, y) \rangle(x) |J\phi_\delta(-\eta, \cdot, y)|(x) \nu(dy) + \mathcal{O}(\alpha\beta^4\delta^{-4}). \end{aligned}$$

Note that for  $\eta/\delta \geq 0$  small enough (up to a universal factor, less than the injection radius of  $M$ ), we have

$$\phi_\delta(-\eta, x, y) = \exp_x \left( \frac{\eta}{2} \nabla \kappa_\delta(\cdot, y)(x) + \eta^2 \int_0^1 (\log_x \circ \phi_\delta)''(-s\eta, x, y)(1-s) ds \right)$$

where  $\log_x$  is the inverse function of  $\exp_x$  and  $(\log_x \circ \phi_\delta)''$  is the second derivative in the first variable. From this equality, in conjunction with (5) and (17), we get

$$|J\phi_\delta(-\eta, \cdot, y)|(x) = 1 + \frac{\eta}{2} \Delta \kappa_\delta(\cdot, y)(x) + \mathcal{O}(\eta^2 \delta^{-4}) \quad (20)$$

first for  $\alpha\beta/\delta^2$  small enough and next by a compactness argument for all  $\alpha, \beta, \delta$  in the range described in the statement of Proposition 4. It also appears that  $|J\phi_\delta(-\eta, \cdot, y)|(x)$  is uniformly bounded when  $\alpha\beta^2\delta^{-2} \leq \frac{1}{2}$ . This yields

$$\begin{aligned} \frac{1}{\alpha} \int_M (|J\phi_\delta(-\eta, \cdot, y)|(x) - 1) \nu(dy) &= \frac{\beta}{2} \int_M \Delta \kappa_\delta(\cdot, y)(x) \nu(dy) + \mathcal{O}(\alpha\beta^2\delta^{-4}) \\ &= \frac{\beta}{2} \Delta U_\delta(x) + \mathcal{O}(\alpha\beta^2\delta^{-4}). \end{aligned}$$

Notice the first term in the right cancels with the second in the right of (14). We also have

$$\begin{aligned} &-\frac{\beta^2}{2} \int_M \langle \nabla U_\delta, \nabla \kappa_\delta(\cdot, y) \rangle(x) |J\phi_\delta(-\eta, \cdot, y)|(x) \nu(dy) \\ &= -\frac{\beta^2}{2} \left\langle \nabla U_\delta, \int_M \nabla \kappa_\delta(\cdot, y)(x) \nu(dy) \right\rangle + \mathcal{O}(\alpha\beta^3\delta^{-4}) \\ &= \frac{-\beta^2}{2} |\nabla U_\delta(x)|^2 + \mathcal{O}(\alpha\beta^3\delta^{-4}). \end{aligned}$$

Here the first term in the right cancels with the first term in the right of (14). The bound announced in the lemma follows at once. ■

In particular, under the hypotheses of the previous proposition we get

$$\sqrt{\mu_{\beta,\delta}[(L_{\alpha,\beta,\delta}^* \mathbf{1})^2]} \leq C\alpha\beta^4\delta^{-4} \quad (21)$$

The l.h.s. will be used in the next section, when  $\alpha\beta^4\delta^{-4}$  is small, as a discrepancy for the fact that  $\mu_{\beta,\delta}$  is not necessarily an invariant measure for  $L_{\alpha,\beta,\delta}$ .

### 3 Proof of Theorem 1

This is the main part of the paper: we are going to prove Theorem 1 by the investigation of the evolution of a  $L^2$  type functional.

On  $M$  consider the algorithm  $X := (X_t)_{t \geq 0}$  described in the introduction. For the time being, the schemes  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ ,  $\beta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  and  $\delta : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$  are assumed to be continuously differentiable. Only later on, in Proposition 9, will we present the conditions insuring the wanted convergence (9). On the initial distribution  $m_0$ , the last ingredient necessary to specify the law of  $X$ , no hypothesis is made. We also denote  $m_t$  the law of  $X_t$ , for any  $t > 0$ . We have that  $m_t$  admits a  $\mathcal{C}^1$  density with respect to  $\lambda$ , which is equally written  $m_t$  (for a proof we refer to the appendix of [3]). As it was mentioned in the previous section, we want to compare these temporal marginal laws with the corresponding instantaneous Gibbs measures, which were defined in (10)



with respect to the potentials  $U_\delta$  given in (2). A convenient way to quantify this discrepancy is to consider the variance of the density of  $m_t$  with respect to  $\mu_{\beta_t, \delta_t}$  under the probability measure  $\mu_{\beta_t, \delta_t}$ :

$$\forall t > 0, \quad I_t := \int \left( \frac{m_t}{\mu_{\beta_t, \delta_t}} - 1 \right)^2 d\mu_{\beta_t, \delta_t} \quad (22)$$

Our goal here is to derive a differential inequality satisfied by this quantity, which implies its convergence to zero under appropriate conditions on the schemes  $\alpha$  and  $\beta$ . More precisely, our purpose is to obtain:

**Proposition 5** *There exists two constants  $c_1, c_2 > 0$  such that for any  $t > 0$  with  $\beta_t \geq 1$  and  $\alpha_t \beta_t^2 \delta_t^{-2} \leq 1/2$ , we have*

$$\begin{aligned} I_t' &\leq -c_1 [(\beta_t \delta_t^{-1})^{2-5m} \exp(-b(U)\beta_t) - \alpha_t \beta_t^3 \delta_t^{-3} - |\beta_t'| - \beta_t \delta_t^{-2} |\delta_t'|] I_t \\ &\quad + c_2 [\alpha_t \beta_t^4 \delta_t^{-4} + |\beta_t'| + \beta_t \delta_t^{-2} |\delta_t'|] \sqrt{I_t} \end{aligned}$$

where  $b(U)$  was defined in (6).

### Proof

At least formally, there is no difficulty to differentiate the quantity  $I_t$  with respect to the time  $t > 0$ . For a rigorous justification of the following computations, we refer to the appendix of [3], where the regularity of the temporal marginal laws in presence of jumps is discussed in detail (it is written in the situation considered there of the circle but can be extended to compact manifolds). Thus we get at any time  $t > 0$ ,

$$\begin{aligned} I_t' &= 2 \int \left( \frac{m_t}{\mu_{\beta_t, \delta_t}} - 1 \right) \frac{\partial_t m_t}{\mu_{\beta_t, \delta_t}} d\mu_{\beta_t, \delta_t} - 2 \int \left( \frac{m_t}{\mu_{\beta_t, \delta_t}} - 1 \right) \frac{m_t}{\mu_{\beta_t, \delta_t}} \partial_t \ln(\mu_{\beta_t, \delta_t}) d\mu_{\beta_t, \delta_t} \\ &\quad + \int \left( \frac{m_t}{\mu_{\beta_t, \delta_t}} - 1 \right)^2 \partial_t \ln(\mu_{\beta_t, \delta_t}) d\mu_{\beta_t, \delta_t} \\ &= 2 \int \left( \frac{m_t}{\mu_{\beta_t, \delta_t}} - 1 \right) \partial_t m_t d\lambda - \int \left( \frac{m_t}{\mu_{\beta_t, \delta_t}} - 1 \right)^2 \partial_t \ln(\mu_{\beta_t, \delta_t}) d\mu_{\beta_t, \delta_t} \\ &\quad - 2 \int \left( \frac{m_t}{\mu_{\beta_t, \delta_t}} - 1 \right) \partial_t \ln(\mu_{\beta_t, \delta_t}) d\mu_{\beta_t, \delta_t} \\ &\leq 2 \int \left( \frac{m_t}{\mu_{\beta_t, \delta_t}} - 1 \right) \partial_t m_t d\lambda + \|\partial_t \ln(\mu_{\beta_t, \delta_t})\|_\infty \left( \int \left( \frac{m_t}{\mu_{\beta_t, \delta_t}} - 1 \right)^2 d\mu_{\beta_t, \delta_t} + 2 \int \left| \frac{m_t}{\mu_{\beta_t, \delta_t}} - 1 \right| d\mu_{\beta_t, \delta_t} \right) \\ &\leq 2 \int \left( \frac{m_t}{\mu_{\beta_t, \delta_t}} - 1 \right) \partial_t m_t d\lambda + \|\partial_t \ln(\mu_{\beta_t, \delta_t})\|_\infty (I_t + 2\sqrt{I_t}) \end{aligned}$$

where we used the Cauchy-Schwarz inequality. The last term is easy to deal with:

**Lemma 6** *There exists  $C_0 \geq 0$ , depending on  $\kappa$ , such that for any  $t \geq 0$ , we have*

$$\|\partial_t \ln(\mu_{\beta_t, \delta_t})\|_\infty \leq C_0 (|\beta_t'| + \beta_t |\delta_t'| \delta_t^{-2}).$$

### Proof

Since for any  $t \geq 0$  we have

$$\forall x \in M, \quad \ln(\mu_{\beta_t, \delta_t})(x) = -\beta_t U_{\delta_t}(x) - \ln \left( \int \exp(-\beta_t U_{\delta_t}(y)) \lambda(dy) \right)$$

it appears that  $\forall x \in M$ ,

$$\begin{aligned} & \partial_t \ln(\mu_{\beta_t})(x) \\ &= \beta'_t \int U_{\delta_t}(y) - U_{\delta_t}(x) \mu_{\beta_t, \delta_t}(dy) \\ & \quad + \beta_t \delta'_t \iint (p(\delta_t, y, z) \partial_\delta \ln p(\delta_t, y, z) - p(\delta_t, x, z) \partial_\delta \ln p(\delta_t, x, z)) \kappa(z, v) \nu(dv) \mu_{\beta_t, \delta_t}(dy) \lambda(dz) \end{aligned}$$

so that

$$\|\partial_t \ln(\mu_{\beta_t})\|_\infty \leq \text{osc}(U_{\delta_t}) |\beta'_t| + 2\beta_t |\delta'_t| \|\partial_\delta \ln p\|_\infty \cdot \|\kappa\|_\infty.$$

Clearly  $\text{osc}(U_{\delta_t}) \leq 2\|\kappa\|_\infty$ . To finish the proof we are left to use the bound

$$|\partial_\delta \ln p(\delta, x, y)| \leq \frac{C'''}{\delta^2} \quad (23)$$

for some  $C''' > 0$  (see e.g. [17]). ■

Denote for any  $t > 0$ ,  $f_t := m_t / \mu_{\beta_t, \delta_t}$ . If this function was to be  $\mathcal{C}^2$ , we would get, by the martingale problem satisfied by the law of  $X$ , that

$$\begin{aligned} \int \left( \frac{m_t}{\mu_{\beta_t, \delta_t}} - 1 \right) \partial_t m_t d\lambda &= \int L_{\alpha_t, \beta_t, \delta_t} [f_t - 1] dm_t \\ &= \int L_{\alpha_t, \beta_t, \delta_t} [f_t - 1] f_t d\mu_{\beta_t, \delta_t} \end{aligned}$$

where  $L_{\alpha_t, \beta_t, \delta_t}$ , described in the previous section, is the instantaneous generator at time  $t \geq 0$  of  $X$ . The interest of the estimate (21) comes from the decomposition of the previous term into

$$\begin{aligned} & \int L_{\alpha_t, \beta_t, \delta_t} [f_t - 1] (f_t - 1) d\mu_{\beta_t, \delta_t} + \int L_{\alpha_t, \beta_t, \delta_t} [f_t - 1] d\mu_{\beta_t, \delta_t} \\ &= \int L_{\alpha_t, \beta_t, \delta_t} [f_t - 1] (f_t - 1) d\mu_{\beta_t, \delta_t} + \int (f_t - 1) L_{\alpha_t, \beta_t, \delta_t}^* [\mathbf{1}] d\mu_{\beta_t, \delta_t} \\ &\leq \int L_{\alpha_t, \beta_t, \delta_t} [f_t - 1] (f_t - 1) d\mu_{\beta_t, \delta_t} + \sqrt{I_t} \sqrt{\mu_{\beta_t, \delta_t} [L_{\alpha_t, \beta_t, \delta_t}^* [\mathbf{1}]]^2} \end{aligned}$$

It follows from these bounds that to prove Proposition 5, it remains to treat the first term in the above r.h.s. A first step is:

**Lemma 7** *There exists a constant  $c_3 > 0$ , such that for any  $\alpha > 0$  and  $\beta \geq 1$  such that  $\alpha\beta^2\delta^{-2} \leq 1/2$ , we have, for any  $f \in \mathcal{C}^2(M)$ ,*

$$\int L_{\alpha, \beta, \delta} [f - 1] (f - 1) d\mu_{\beta, \delta} \leq - \left( \frac{1}{2} - c_3 \alpha \beta^3 \delta^{-3} \right) \int (|\nabla f|)^2 d\mu_{\beta, \delta} + c_3 \alpha \beta^3 \delta^{-3} \int (f - 1)^2 d\mu_{\beta, \delta}$$

**Proof**

For any  $\alpha > 0$ ,  $\beta \geq 0$  and  $\delta > 0$ , we begin by decomposing the generator  $L_{\alpha, \beta, \delta}$  into

$$L_{\alpha, \beta, \delta} = L_{\beta, \delta} + R_{\alpha, \beta, \delta} \quad (24)$$

where

$$L_{\beta, \delta} \cdot := \frac{1}{2} (\Delta \cdot - \beta \langle \nabla U_\delta, \nabla \cdot \rangle) \quad (25)$$

and where  $R_{\alpha,\beta,\delta}$  is the remaining operator. An immediate integration by parts leads to

$$\begin{aligned} \int L_{\beta,\delta} [f - 1] (f - 1) d\mu_{\beta,\delta} &= -\frac{1}{2} \int |\nabla(f - 1)|^2 d\mu_{\beta,\delta} \\ &= -\frac{1}{2} \int |\nabla f|^2 d\mu_{\beta,\delta} \end{aligned} \quad (26)$$

Thus our main task is to find a constant  $c_3 > 0$ , such that for any  $\alpha > 0$ ,  $\beta \geq 1$  and  $\delta > 0$  with  $\alpha\beta^2\delta^{-2} \leq 1/2$ , we have, for any  $f \in \mathcal{C}^2(M)$ ,

$$\left| \int R_{\alpha,\beta,\delta} [f - 1] (f - 1) d\mu_{\beta,\delta} \right| \leq c_3 \alpha \beta^3 \delta^{-3} \left( \int |\nabla f|^2 d\mu_{\beta,\delta} + \int (f - 1)^2 d\mu_{\beta,\delta} \right) \quad (27)$$

By definition, we have for any  $f \in \mathcal{C}^2(M)$  (but what follows is valid for  $f \in \mathcal{C}^1(M)$ ),

$$\forall x \in M, \quad R_{\alpha,\beta,\delta}[f](x) = \frac{1}{\alpha} \int f(\phi_\delta(\alpha\beta, x, y)) - f(x) \nu(dy) + \frac{\beta}{2} \langle \nabla U_\delta(x), \nabla f(x) \rangle$$

To evaluate this quantity, on one hand, recall that we have for any  $x \in M$ ,

$$\nabla U_\delta(x) = \int_M \nabla \kappa_\delta(\cdot, y)(x) \nu(dy)$$

and on the other hand, write that for any  $x, y \in M$ ,

$$f(\phi_\delta(\alpha\beta, x, y)) - f(x) = \alpha\beta \int_0^1 \left\langle \nabla f(\phi_\delta(\alpha\beta u, x, y)), -\frac{1}{2} \nabla \kappa_\delta(\cdot, y)(\phi_\delta(\alpha\beta u, x, y)) \right\rangle du$$

It follows that

$$\begin{aligned} &\int R_{\alpha,\beta,\delta} [f - 1] (f - 1) d\mu_{\beta,\delta} \\ &= \frac{\beta}{2} \int_0^1 du \int \nu(dy) \int \mu_{\beta,\delta}(dx) (\langle \nabla f, \nabla \kappa_\delta(\cdot, y)(x) \rangle - \langle \nabla f, \nabla \kappa_\delta(\cdot, y)(\phi_\delta(\alpha\beta u, x, y)) \rangle) (f(x) - 1) \\ &= \frac{\beta}{2} \int_0^1 du \int \nu(dy) \int \lambda(dx) \langle \nabla f, \nabla \kappa_\delta(\cdot, y)(x) \rangle \left[ (f(x) - 1) \mu_{\beta,\delta}(x) \right. \\ &\quad \left. - (f(\phi_\delta(-\alpha\beta u, x, y)) - 1) \mu_{\beta,\delta}(\phi_\delta(-\alpha\beta u, x, y)) |J\phi_\delta(-\alpha\beta u, \cdot, y)|(x) \right] \end{aligned}$$

where we used the change of variable  $z \mapsto \phi_\delta(-\alpha\beta u, z, y)$  for the second term in the right. So

$$\begin{aligned} &\int R_{\alpha,\beta,\delta} [f - 1] (f - 1) d\mu_{\beta,\delta} \\ &= \frac{\beta}{2} \int_0^1 du \int \nu(dy) \int \lambda(dx) \langle \nabla f, \nabla \kappa_\delta(\cdot, y)(x) \rangle I_\delta(\alpha\beta u, x, y) \end{aligned}$$

where

$$\begin{aligned} I_\delta(s, x, y) &= |J\phi_\delta(-s, \cdot, y)|(x) \{ (f(x) - 1) \mu_{\beta,\delta}(x) - (f(\phi_\delta(-s, x, y)) - 1) \mu_{\beta,\delta}(\phi_\delta(-s, x, y)) \} \\ &\quad + (f(x) - 1) \mu_{\beta,\delta}(x) (1 - |J\phi_\delta(-s, \cdot, y)|(x)). \end{aligned}$$

Write

$$\int R_{\alpha,\beta,\delta} [f - 1] (f - 1) d\mu_{\beta,\delta} = J_1 + J_2$$

where  $J_1$  is the integral containing the term  $(1 - |J\phi_\delta(-s, \cdot, y)|(x))$ . From the validity of

$$\phi_\delta(-s, x, y) = \exp_x \left( s \int_0^1 (\log_x \circ \phi_\delta)'(-vs, x, y) dv \right)$$

for  $s$  small enough, we get that for any  $s$ ,

$$|1 - |J\phi_\delta(-s, \cdot, y)|(x)| \leq c_4 s \delta^{-2}$$

for some  $c_4 > 0$ , which yields

$$\begin{aligned} J_1 &\leq \frac{1}{2} c_4 \alpha \beta^2 \delta^{-2} \int_0^1 du \int \nu(dy) \int \mu_{\beta, \delta}(dx) \|\nabla_1 \kappa_\delta\|_\infty |\nabla f|(x) \cdot |f(x) - 1| \\ &\leq \frac{1}{2} c_4 \alpha \beta^2 \delta^{-2} \|\nabla_1 \kappa_\delta\|_\infty \|\nabla f\|_{\mathbb{L}^2(\mu_\beta)} \sqrt{\int (f-1)^2 d\mu_\beta} \\ &\leq \frac{1}{4} c_4 \alpha \beta^2 \delta^{-3} \left( \|\nabla f\|_{\mathbb{L}^2(\mu_{\beta, \delta})}^2 + \int (f-1)^2 d\mu_{\beta, \delta} \right) \\ &= c_5 \alpha \beta^2 \delta^{-3} \left( \|\nabla f\|_{\mathbb{L}^2(\mu_{\beta, \delta})}^2 + \int (f-1)^2 d\mu_{\beta, \delta} \right) \end{aligned}$$

with  $c_5 = \frac{1}{4} c_4$ , using again (17). Moreover, we have

$$\begin{aligned} J_2 &= \frac{\beta}{2} \int_0^1 du \int \nu(dy) \int \sqrt{\mu_{\beta, \delta}(x)} dx \langle \nabla f, \nabla \kappa_\delta(\cdot, y) \rangle(x) |J\phi_\delta(-\alpha\beta u, \cdot, y)|(x) (-\alpha\beta u) \int_0^1 dv \\ &\quad \left( \langle \nabla f, -\frac{1}{2} \nabla \kappa_\delta(\cdot, y) \rangle(\phi_\delta(-\alpha\beta uv, x, y)) \sqrt{\mu_{\beta, \delta}(\phi_\delta(-\alpha\beta uv, x, y))} \sqrt{\frac{\mu_{\beta, \delta}(\phi_\delta(-\alpha\beta uv, x, y))}{\mu_{\beta, \delta}(x)}} \right. \\ &\quad \left. + f(\phi_\delta(-\alpha\beta uv, x, y)) - 1 \right) \langle \nabla \ln \mu_{\beta, \delta}, -\frac{1}{2} \nabla \kappa_\delta(\cdot, y) \rangle(\phi_\delta(-\alpha\beta uv, x, y)) \\ &\quad \sqrt{\mu_{\beta, \delta}(\phi_\delta(-\alpha\beta uv, x, y))} \sqrt{\frac{\mu_{\beta, \delta}(\phi_\delta(-\alpha\beta uv, x, y))}{\mu_{\beta, \delta}(x)}} \Big). \end{aligned}$$

Recalling (20), notice that  $|J\phi_\delta(-\alpha\beta u, \cdot, y)|(x)$  and  $\sqrt{\frac{\mu_{\beta, \delta}(\phi_\delta(-\alpha\beta uv, x, y))}{\mu_{\beta, \delta}(x)}}$  are uniformly bounded, since  $\alpha\beta^2\delta^{-2} \leq 1/2$ . Putting the integral with respect to  $v$  on the left, using Cauchy-Schwartz inequality for the integral in the right and making the change of variable  $z = \phi_\delta(-\alpha\beta uv, x, y)$  we get

$$\begin{aligned} J_2 &\leq c_6 \alpha \beta^2 \|\nabla_1 \kappa_\delta\|_\infty^2 \|\nabla f\|_{\mathbb{L}^2(\mu_{\beta, \delta})} \left( \|\nabla f\|_{\mathbb{L}^2(\mu_{\beta, \delta})} + \beta \|\nabla_1 \kappa_\delta\|_\infty \sqrt{\int (f-1)^2 d\mu_{\beta, \delta}} \right) \\ &\leq c_6 \alpha \beta^2 \|\nabla_1 \kappa_\delta\|_\infty^2 \left[ \left( 1 + \frac{1}{2} \beta \|\nabla_1 \kappa_\delta\|_\infty \right) \|\nabla f\|_{\mathbb{L}^2(\mu_{\beta, \delta})}^2 + \frac{1}{2} \beta \|\nabla_1 \kappa_\delta\|_\infty^2 \int (f-1)^2 d\mu_{\beta, \delta} \right] \end{aligned} \tag{28}$$

where  $c_6 \geq 0$  is a constant independent from  $\alpha, \beta, \delta$ . Up to a change of this constant, we obtain

$$J_2 \leq c_6 \alpha \beta^3 \delta^{-3} \left( \|\nabla f\|_{\mathbb{L}^2(\mu_{\beta, \delta})}^2 + \int (f-1)^2 d\mu_{\beta, \delta} \right) \tag{29}$$

where we used (17).

So putting together (26) with the bounds for  $J_1$  and  $J_2$  we get the wanted result. ■

To conclude the proof of Proposition 5, we must be able to compare, for any  $\beta \geq 0$  and any  $f \in \mathcal{C}^1(M)$ , the energy  $\mu_{\beta, \delta}[\|\nabla f\|^2]$  and the variance  $\text{Var}(f, \mu_{\beta, \delta})$ . This task was already done by Holley, Kusuoka and Stroock [15], let us recall their result:

**Proposition 8** *Let  $\tilde{U}$  be a  $\mathcal{C}^1$  function on a compact Riemannian manifold  $M$  of dimension  $m \geq 1$ . Let  $b(\tilde{U}) \geq 0$  be the associated constant as in (6). For any  $\beta \geq 0$ , consider the Gibbs measure  $\tilde{\mu}_\beta$  given similarly to (10). Then there exists a constant  $C_M > 0$ , depending only on  $M$ , such that the following Poincaré inequalities are satisfied:*

$$\forall \beta \geq 0, \forall f \in \mathcal{C}^1(M), \quad \text{Var}(f, \tilde{\mu}_\beta) \leq C_M [1 \vee (\beta \|\nabla \tilde{U}\|_\infty)]^{5m-2} \exp(b(\tilde{U})\beta) \tilde{\mu}_\beta[|\nabla f|^2]$$

We can now come back to the study of the evolution of the quantity  $I_t = \text{Var}(f_t, \mu_{\beta_t, \delta_t})$ , for  $t > 0$ . Indeed applying Lemma 7 and Proposition 8 with  $\alpha = \alpha_t$ ,  $\beta = \beta_t$ ,  $\delta = \delta_t$  and  $f = f_t$ , we get at any time  $t > 0$  such that  $\beta_t \geq 1$ ,  $\delta_t \in (0, 1]$  and  $\alpha_t \beta_t^2 \delta_t^{-2} \leq 1/2$ ,

$$\begin{aligned} & \int L_{\alpha_t, \beta_t, \delta_t} [f_t - 1] (f_t - 1) d\mu_{\beta_t, \delta_t} \\ & \leq -c_7 (\beta_t \delta_t^{-1})^{2-5m} \exp(-b(U_{\delta_t})\beta_t) (1 - 2c_3 \alpha_t \beta_t^3 \delta_t^{-3}) I_t + c_3 \alpha_t \beta_t^3 \delta_t^{-3} I_t \\ & \leq -[c_7 (\beta_t \delta_t^{-1})^{2-5m} \exp(-b(U_{\delta_t})\beta_t) - c_8 \alpha_t \beta_t^3 \delta_t^{-3}] I_t \end{aligned}$$

for some constants  $c_7, c_8 > 0$ .

Taking into account Lemma 6, the computations preceding Lemma 7 and (21), one can find constants  $c_1, c_2 > 0$  such that Proposition 5 is satisfied. This achieves the proof of Proposition 5.  $\blacksquare$

This result leads immediately to conditions insuring the convergence toward 0 of the quantity  $I_t$  for large times  $t > 0$ :

**Proposition 9** *Let  $\alpha, \delta : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$  and  $\beta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be schemes as at the beginning of this section and assume:*

$$\begin{aligned} \lim_{t \rightarrow +\infty} \alpha_t &= 0 \\ \lim_{t \rightarrow +\infty} \beta_t &= +\infty \\ \lim_{t \rightarrow +\infty} \delta_t &= 0 \end{aligned}$$

$$\int_0^{+\infty} (1 \vee (\beta_t \delta_t^{-1}))^{2-5m} \exp(-b(U_{\delta_t})\beta_t) dt = +\infty$$

and that for large times  $t > 0$ ,

$$\max\{\alpha_t \beta_t^4 \delta_t^{-4}, |\beta_t'|, \beta_t \delta_t^{-2} |\delta_t'|\} \ll (\beta_t \delta_t^{-1})^{2-5m} \exp(-b(U_{\delta_t})\beta_t)$$

Then we are assured of

$$\lim_{t \rightarrow +\infty} I_t = 0$$

**Proof**

The differential equation of Proposition 5 can be rewritten under the form

$$F_t' \leq -\eta_t F_t + \epsilon_t \tag{30}$$

where for any  $t > 0$ ,

$$\begin{aligned} F_t &:= \sqrt{I_t} \\ \eta_t &:= c_1 ((\beta_t \delta_t^{-1})^{2-5m} \exp(-b(U_{\delta_t})\beta_t) - \alpha_t \beta_t^3 \delta_t^{-3} - |\beta_t'| - \beta_t |\delta_t'| \delta_t^{-2})/2 \\ \epsilon_t &:= c_2 (\alpha_t \beta_t^4 \delta_t^{-4} + |\beta_t'| + \beta_t \delta_t^{-2} |\delta_t'|)/2 \end{aligned}$$

The assumptions of the above proposition imply that for  $t \geq 0$  large enough,  $\beta_t \geq 1$  and  $\alpha_t \beta_t^2 \delta_t^{-2} \leq 1/2$ . This insures that there exists  $T > 0$  such that (30) is satisfied for any  $t \geq T$  (and also  $F_T < +\infty$ ). We deduce that for any  $t \geq T$ ,

$$F_t \leq F_T \exp\left(-\int_T^t \eta_s ds\right) + \int_T^t \epsilon_s \exp\left(-\int_s^t \eta_u du\right) ds \quad (31)$$

It appears that  $\lim_{t \rightarrow +\infty} F_t = 0$  as soon as

$$\begin{aligned} \int_T^{+\infty} \eta_s ds &= +\infty \\ \lim_{t \rightarrow +\infty} \epsilon_t / \eta_t &= 0 \end{aligned}$$

The above assumptions were chosen to insure these properties. ■

In particular, the schemes given in (8) satisfy the hypotheses of the previous proposition (notice that  $b(U_\delta) \rightarrow b(U)$  as  $\delta \rightarrow 0$ , due to the uniform convergence of  $U_\delta$  to  $U$ ), so that under the conditions of Theorem 1, we get

$$\lim_{t \rightarrow +\infty} I_t = 0$$

Let us deduce (9) for any neighborhood  $\mathcal{N}$  of the set  $\mathcal{M}$  of the global minima of  $U$ . From Cauchy-Schwartz inequality we have for any  $t > 0$ ,

$$\begin{aligned} \|m_t - \mu_{\beta_t, \delta_t}\|_{\text{tv}} &= \int |f_t - 1| \mu_{\beta_t, \delta_t} \\ &\leq \sqrt{I_t} \end{aligned}$$

An equivalent definition of the total variation norm states that

$$\|m_t - \mu_{\beta_t}\|_{\text{tv}} = 2 \max_{A \in \mathcal{T}} |m_t(A) - \mu_{\beta_t, \delta_t}(A)|$$

where  $\mathcal{T}$  is the Borelian  $\sigma$ -algebra of  $M$ . It follows that (9) reduces to

$$\lim_{\beta, \delta^{-1} \rightarrow +\infty} \mu_{\beta, \delta}(\mathcal{N}) = 1$$

for any neighborhood  $\mathcal{N}$  of  $\mathcal{M}$  and  $\delta$  sufficiently small, property which is immediate from the definition (10) of the Gibbs measures  $\mu_{\beta, \delta}$  for  $\beta \geq 0$  and  $\delta > 0$ .

**Remark 10** Similarly to the approach presented for instance in [21, 23], we could have studied the evolution of  $(E_t)_{t>0}$ , which are the relative entropies of the time marginal laws with respect to the corresponding instantaneous Gibbs measures, namely

$$\forall t > 0, \quad E_t := \int \ln\left(\frac{m_t}{\mu_{\beta_t, \delta_t}}\right) dm_t$$

To get a differential inequality satisfied by these functionals, the spectral gap estimate of Holley, Kusuoka and Stroock [15] recalled in Proposition 8 must be replaced by the corresponding logarithmic Sobolev constant estimate. □

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