About projections of logarithmic Sobolev inequalities

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Abstract

We derive bounds for the logarithmic Sobolev constant associated to some finite type Fleming-Viot operators different from those recently obtained by Stannat. We will verify that on large subdomains of the space of underlying parameters, our estimates are of the right order, nevertheless, the result of Stannat remains better for some small values. Our approach is based on an uplifting of the problem to gamma distribution product spaces, leading us to evaluate the logarithmic Sobolev constant corresponding to Laguerre operators via Hardy's inequalities techniques.

Keywords: Logarithmic Sobolev inequalities, finite type Fleming-Viot generators, Laguerre operators, gamma distributions, Hardy's inequalities.

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1 Introduction

Recently, Stannat [13] gave a lower bound on the logarithmic Sobolev constant of Fleming-Viot operators with parent-independent jump mutation and without selection nor recombination, when the type space is finite. Our objective is to present an alternative approach to this problem which will result in different estimates, which will be of the right order for large values of the underlying parameters. Nevertheless, we cannot expect our lower bound to be appropriate everywhere, since in some situations it is worse than the estimate of Stannat and this will lead us to present a conjecture for a general behavior and what is still missing to obtain it.

Our basic idea is to take advantage of some projection properties inherent to the model and leading to the consideration of products of one-dimensional objects appropriate for the use of Hardy's inequalities. This feature enables to avoid the iterative method of Stannat and we believe it could be applied to other related contexts.

We begin by recalling the setting and the result of Stannat we are interested in here, for their links with the theory of population genetics, we refer to the original article [13].

For $d \in \mathbb{N}^*$, we consider the simplex

$$\triangle_d = \{x = (x_1, ..., x_d) \in \mathbb{R}_+^d \ : \ |x| \leq 1\}$$

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where $|\cdot|$ will always stand for the l^1 norm (independentely of the dimension, for instance above $|x| = \sum_{1\leq i\leq d} x_i$), and we designate by $C^\infty(\triangle_d)$ the set of functions which are restrictions to $\triangle_d$ of C^∞ mappings on $\mathbb{R}^d$.

For $q = (q_1, ..., q_{d+1}) \in (\mathbb{R}_+^*)^{d+1}$, we denote L_q the operator on $C^\infty(\triangle_d)$ acting by

$$\forall\, f \in C^\infty(\triangle_d),\ \forall\, x \in \triangle_d, \qquad L_q[f](x) = \frac{1}{2}\sum_{1\leq i\leq d} x_i\partial_i^2 f(x) - \frac{1}{2}\sum_{1\leq i,j\leq d} x_i x_j \partial_i\partial_j f(x) + \frac{1}{2}\sum_{1\leq i\leq d} (q_i - |q|\, x_i)\partial_i f(x)$$

where for $1 \leq i \leq d$, ∂_i symbolizes the partial differentiation with respect to x_i.

We also introduce the Borelian probability ν_q defined on $\triangle_d$ by

$$\nu_q(dx) = \frac{\Gamma(|q|)}{\prod_{1\leq i\leq d+1}\Gamma(q_i)}(1-|x|)^{q_{d+1}-1}\prod_{1\leq i\leq d} x_i^{q_i-1}\, dx_1\cdots dx_d \tag{1}$$

where Γ is the usual gamma function;

$$\forall\, p > 0, \qquad \Gamma(p) = \int_{\mathbb{R}_+} t^{p-1}\exp(-t)\, dt$$

It is well-known that this measure is symmetrizing for the operator L_q and thus we are led to look at the bilinear form $\mathcal{E}_q$ defined on $C^\infty(\triangle_d)$ by

$$\begin{aligned}\forall\, f, g \in C^\infty(\triangle_d), \qquad \mathcal{E}_q(f,g) &= -\nu_q(f L_q[g]) \\ &= \frac{1}{2}\int \sum_{1\leq i,j\leq d} x_i(\delta_{i,j} - x_j)\partial_i f(x)\partial_j g(x)\, \nu_q(dx)\end{aligned}$$

Then the associated logarithmic Sobolev constant is given by

$$\alpha(q) := \inf_{f\in C^\infty(\triangle_d)\setminus \mathrm{Vect}(\mathbf{I})} \frac{\mathcal{E}_q(f,f)}{\mathrm{Ent}(f^2,\nu_q)}$$

where as usual the previous entropy is $\mathrm{Ent}(f^2,\nu_q) = \nu_q[f^2\ln(f^2/\nu_q[f^2])]$. We have chosen the inverse definition of that of Stannat, for whom the logarithmic Sobolev constant is rather $\alpha^{-1}(q)$, because it makes it closer to the notion of spectral gap, which will also be considered latter on.

Stated as his theorem 2.8 p. 676 of [13], Stannat has proved the following interesting estimate:

$$\forall\, q = (q_1, ..., q_{d+1}) \in (\mathbb{R}_+^*)^{d+1}, \qquad \alpha(q) \geq \frac{q_*}{320} \tag{2}$$

with $q_* := \min\{q_1, ..., q_{d+1}\}$.

Our main result in this note can now be formulated as:

Proposition 1.1 *There exists an universal constant* $0 < c_1 < +\infty$ *such that for all* $d \in \mathbb{N}^*$ *and all* $q = (q_1, ..., q_{d+1}) \in (\mathbb{R}_+^*)^{d+1}$, *we are assured of*

$$\alpha(q) \geq c_1 \frac{|q|}{\ln[1/(q_* \wedge e^{-1})]}$$

This lower bound can be worse than that of Stannat, for instance if we consider the case of a fixed dimension $d \in \mathbb{N}^*$ and $q_i = q_1$, for all $1 < i \leq d+1$, with q_1 arbitrary small. Nevertheless, the next result shows that in particular for all fixed $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that for all $d \in \mathbb{N}^*$ and all $q = (q_1, ..., q_{d+1}) \in (\mathbb{R}_+^*)^{d+1}$ verifying $|q| \geq \epsilon$, we have

$$\alpha(q) \leq C_\epsilon \frac{|q|}{\ln[1/(q_* \wedge e^{-1})]}$$

Indeed, if $|q| \geq \epsilon$, it appears that

$$\begin{aligned} \ln\left(\frac{|q| \wedge 1}{q_* \wedge e^{-1}}\right) &\geq \ln\left(\frac{\epsilon \wedge 1}{q_* \wedge e^{-1}}\right) \\ &= \ln(\epsilon \wedge 1) - \ln(q_* \wedge e^{-1}) \\ &\geq -(1 + |\ln(\epsilon \wedge 1)|) \ln(q_* \wedge e^{-1}) \end{aligned}$$

thus we can take $C_\epsilon = c_2(1 + |\ln(\epsilon \wedge 1)|)$, using the following statement:

Proposition 1.2 *There exists a constant $c_2 > 0$ such that for all $d \in \mathbb{N}^*$ and all $q = (q_1, ..., q_{d+1}) \in (\mathbb{R}_+^*)^{d+1}$, we are insured of the upper bound*

$$\alpha(q) \leq c_2 \frac{|q|}{\ln[(|q| \wedge 1)/(q_* \wedge e^{-1})]}$$

The proof of these results will be based on a study of the spectral gap and the logarithmic Sobolev constant associated to Laguerre operators and gamma distributions on $\mathbb{R}_+$. This investigation, made in the next two sections, will itself rely on the Hardy's inequalities approach to ergodic constants developed by Bobkov and Götze [2] and can be seen as having its own interest.

At the end of section 5, we will explain why we believe that the logarithmic Sobolev constant should always be of the order of the upper bound given in proposition 1.2. Namely, we think that the following is true:

Conjecture 1.3 *There exists a constant $c_3 > 0$ such that for all $d \in \mathbb{N}^*$ and all $q = (q_1, ..., q_{d+1}) \in (\mathbb{R}_+^*)^{d+1}$, we are insured of the lower bound*

$$\alpha(q) \geq c_3 \frac{|q|}{\ln[(|q| \wedge 1)/(q_* \wedge e^{-1})]}$$

At least there is no contradiction with the result of Stannat, since we note that for all parameters q such that $0 < |q| \leq 1$, we have

$$\begin{aligned} \frac{|q|}{\ln[(|q| \wedge 1)/(q_* \wedge e^{-1})]} &= \frac{|q|/(q_* \wedge e^{-1})}{\ln[(|q|)/(q_* \wedge e^{-1})]}(q_* \wedge e^{-1}) \\ &\geq c_4 q_* \end{aligned}$$

with $c_4 = e^{-1} \min_{r \geq 2} r/\ln(r) = 1$.

Notice: The lacking estimate to conclude rigorously to the above conjecture has recently been obtained in [10].

2 About the spectral gap of Laguerre operators

Our first step consists in giving a complicated proof of a weak version of a classical result! The advantage of this alternative approach is that it can be extended to treat the logarithmic Sobolev constant. Furthermore, it introduces natural quantities that will be useful latter on. Nevertheless in this section, we will only be concerned with the spectral gap, which is another ergodic constant, sharing some similarities with the latter one but easier to manipulate. We will also restrict ourself to the simple one-dimensional situation of Laguerre operators.

More precisely, our setting is the following: for fixed $p > 0$, we introduce the Laguerre operator $\widetilde{L}_p$ acting on $C^\infty_{\mathrm{p}}(\mathbb{R}_+)$ (which will denote the set of restrictions to $\mathbb{R}_+$ of C^∞ functions on $\mathbb{R}$ whose derivatives are polynomially bounded) by

$$\forall\, f \in C^\infty_{\mathrm{p}}(\mathbb{R}_+),\ \forall\, x \geq 0, \qquad \widetilde{L}_p[f](x) \ := \ xf''(x) + (p-x)f'(x)$$

Its name comes from the famous Laguerre-Sonin polynomials $(P_{p,n})_{n\geq 0}$, defined by

$$\forall\, n \in \mathbb{N},\ \forall\, x \in \mathbb{R}_+, \qquad P_{p,n}(x) \ := \ \frac{1}{n!}\exp(x)x^{-p+1}(\exp(-x)x^{n+p-1})^{(n)}$$

which are eigenvectors for $\widetilde{L}_p$ (cf for instance [14]):

$$\forall\, n \in \mathbb{N}, \qquad \widetilde{L}_p[P_{p,n}] \ = \ -nP_{p,n}$$

In fact, the gamma distribution μ_p of parameter p (on $\mathbb{R}_+$, remember that $\mu_p(ds) = \Gamma(p)^{-1}s^{p-1}\exp(-s)\,ds$) is symmetrizing for $\widetilde{L}_p$ and the sequence $(-n, \mathrm{Vect}(P_{p,n}))_{n\geq 0}$ forms the spectral decomposition of the minimal auto-adjoint extension of this operator on $\mathbb{L}^2(\mu_p)$. So we are induced to have a look at the bilinear form $\widetilde{\mathcal{E}}_p$ defined on $C^\infty_{\mathrm{p}}(\mathbb{R}_+)$ by

$$\begin{aligned}\forall\, f,g \in C^\infty_{\mathrm{p}}(\mathbb{R}_+), \qquad \widetilde{\mathcal{E}}_p(f,g) \ &:= \ -\mu_p(f\widetilde{L}_p[g]) \\ &= \ \int xf'(x)g'(x)\,\mu_p(dx)\end{aligned}$$

and to consider the associated spectral gap

$$\widetilde{\lambda}(p) \ := \ \inf_{f\in C^\infty_{\mathrm{p}}(\mathbb{R}_+)\setminus\mathrm{Vect}(\mathbf{1})} \frac{\widetilde{\mathcal{E}}_p(f,f)}{\mathrm{Var}(f,\mu_p)}$$

where $\mathrm{Var}(f,\mu_p) := \mu_p[(f-\mu_p(f))^2]$ designates the variance of f with respect to μ_p (one would have noted that $C^\infty_{\mathrm{p}}(\mathbb{R}_+) \subset \mathbb{L}^2(\mu_p)$ and in fact this inclusion is dense).

Then the above considerations show (up to an easy closure argument) that there is no mystery for this quantity,

$$\forall\, p > 0, \qquad \widetilde{\lambda}(p) \ = \ 1$$

What may seem strange indeed, is that our objective here is to recover that

Proposition 2.1 *There exists a finite constant $c > 0$ such that*

$$\forall\, p > 0, \qquad \widetilde{\lambda}(p) \;\geq\; c$$

Even worse, we shall need several steps to derive this conclusion. We begin by having interest in the cases where $0 < p \leq 1/2$.

Lemma 2.2 *For all $0 < p < 1$, we have*

$$4\widetilde{\lambda}(p) \;\geq\; \left[\sup_{t>0} \int_0^t s^{-p} \exp(s)\, ds \int_t^{+\infty} s^{p-1} \exp(-s)\, ds\right]^{-1}$$

This estimation is well-known and is a consequence of the Hardy's inequalities saying that (cf [11])

$$\forall\, p > 0, \qquad B_p \;\leq\; A_p \;\leq\; 4B_p$$

where

$$A_p \;:=\; \sup_{f\in C_{\mathrm{p}}^\infty(\mathbb{R}_+)} \frac{\mu_p[(f-f(0))^2]}{\widetilde{\mathcal{E}}_p(f,f)}$$

and where B_p is the inverse of the right hand side of the lemma's bound.

To conclude it is sufficient then to take into account the trivial relation $\mathrm{Var}(f,\mu_p) \leq \mu_p[(f-f(0))^2]$ for any $f \in \mathbb{L}^2(\mu_p)$.

Rigorously, the above lemma is verified for every $p > 0$, except that for $p \geq 1$ it gives no information, since for all $t > 0$, $\int_0^t s^{-p} \exp(s)\, ds = +\infty$ and thus $B_p = +\infty$. Nevertheless, let us recall that it is possible to insure similar matching lower and upper bounds for $\widetilde{\lambda}(p)$, if the median m_p of μ_p is asked to play a role: we have (as an easy consequence of arguments from [2] and [9])

$$[B_{p,-} \vee B_{p,+}]^{-1}/4 \;\leq\; \widetilde{\lambda}(p) \;\leq\; 2[B_{p,-} \vee B_{p,+}]^{-1}$$

with

$$B_{p,-} \;:=\; \sup_{0\leq t<m_p} \int_t^{m_p} s^{-p} \exp(s)\, ds \int_0^{m_p} s^{p-1} \exp(-s)\, ds$$

$$B_{p,+} \;:=\; \sup_{t>m_p} \int_{m_p}^{t} s^{-p} \exp(s)\, ds \int_t^{+\infty} s^{p-1} \exp(-s)\, ds$$

Indeed, for any $p > 0$, the quantity B_p^{-1} is a good estimation (up to a factor between $1/4$ and 2) of the spectral gap $\widehat{\lambda}(p)$ associated to the symmetrization $\widehat{\mu}_p$ of μ_p (on $\mathbb{R}$, $\widehat{\mu}_p(ds) = 1/(2\Gamma(p))|s|^{p-1}\exp(-|s|)\, ds$);

$$\widehat{\lambda}(p) \;:=\; \inf_{f\in C_{\mathrm{p}}^\infty(\mathbb{R})} \frac{\int |s|(f'(s))^2\, \widehat{\mu}_p(ds)}{\mathrm{Var}(f,\widehat{\mu}_p)}$$

which in particular is null for $p \geq 1$.

Coming back to our objective, we notice that for all $0 < p \leq 1/2$, we have

$$B_p \;\leq\; B_0^{1-2p} B_{1/2}^{2p}$$

due to the Hölder inequalities, valid for all $t > 0$,

$$\begin{aligned}\int_0^t s^{-p}\exp(s)\,ds &\leq \left(\int_0^t s^{-1/2}\exp(s)\,ds\right)^{2p}\left(\int_0^t \exp(s)\,ds\right)^{1-2p}\\ \int_t^{+\infty} s^{p-1}\exp(-s)\,ds &\leq \left(\int_t^{+\infty} s^{-1/2}\exp(-s)\,ds\right)^{2p}\left(\int_t^{+\infty} s^{-1}\exp(-s)\,ds\right)^{1-2p}\end{aligned}$$

Thus it is sufficient to study B_0 and $B_{1/2}$ to get a lower bound on $\inf_{0<p\leq 1/2}\widetilde{\lambda}(p)$ (namely $(B_0 \vee B_{1/2})^{-1}/4$).

Lemma 2.3 *We are assured of $B_0 < +\infty$ and $B_{1/2} < +\infty$.*

Proof:

Using the Cauchy-Schwarz inequality, we obtain that for any $t > 0$,

$$\begin{aligned}\int_t^{+\infty} s^{-1}\exp(-s)\,ds &\leq \sqrt{\int_t^{+\infty}\exp(-2s)\,ds}\sqrt{\int_t^{+\infty} s^{-2}\,ds}\\ &= \frac{\exp(-t)}{\sqrt{2t}}\end{aligned}$$

and so that

$$B_0 \leq \sup_{t>0}\frac{1-\exp(-t)}{\sqrt{2t}}$$

As the expression appearing in the rhs is continuous in $t > 0$ and converges to zero for small or large t, we can conclude to the first affirmation.

For the finiteness of $B_{1/2}$, we could also study the behaviour of the relevant quantities in the neighbourhood of 0 and $+\infty$, but it is sufficient to rewrite $B_{1/2}$ as

$$4\sup_{t>0}\int_0^t \exp(s^2)\,ds\int_t^{+\infty}\exp(-s^2)\,ds$$

and to recognize the Hardy's bound associated to the spectral gap of the centralized Gaussian law of variance $1/2$, to end up immediately with $B_{1/2} \leq 4$.

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In order to finish the proof of the proposition, we take into account the convolution semigroup property satisfied by the gamma distributions:

$$\forall\, p_1, p_2 > 0, \qquad \mu_{p_1} * \mu_{p_2} = \mu_{p_1+p_2}$$

which implies that

$$\forall\, p_1, p_2 > 0, \qquad \widetilde{\lambda}(p_1+p_2) \geq \widetilde{\lambda}(p_1)\wedge\widetilde{\lambda}(p_2) \tag{3}$$

Indeed, by the product property of the spectral gap, we have that for all $f \in C_{\mathrm{p}}^{\infty}(\mathbb{R}_+^2)$,

$$(\widetilde{\lambda}(p_1)\wedge\widetilde{\lambda}(p_2))\mathrm{Var}(f, \mu_{p_1}\otimes\mu_{p_2}) \leq \int s_1(\partial_1 f)^2 + s_2(\partial_2 f)^2\,\mu_{p_1}\otimes\mu_{p_2}(ds_1, ds_2)$$

and thus if we consider

$$f\,:\,\mathbb{R}_+^2 \ni (s_1,s_2) \;\mapsto\; g(s_1+s_2)$$

with $g \in C_{\mathrm{p}}^\infty(\mathbb{R}_+)$, we obtain

$$(\widetilde{\lambda}(p_1)\wedge\widetilde{\lambda}(p_2))\mathrm{Var}(g,\mu_{p_1+p_2}) \;\leq\; \widetilde{\mathcal{E}}_{p_1+p_2}(g,g)$$

and (3) is insured by considering an infimum over all such functions g.

This relation is extended at once into

$$\forall\, n\in\mathbb{N}^*,\ \forall\, p_1,\ldots,p_n\in\mathbb{R}_+^*, \qquad \widetilde{\lambda}(p_1+\cdots+p_n) \;\geq\; \min\{\widetilde{\lambda}(p_i)\,:\,1\leq i\leq n\}$$

which enables to attain the annouced result, since any positive real number can be written as a finite sum of elements belonging to $]0,1/2]$.

Remark 2.4: In quite the same "algebraic" spirit, the conclusion that $\widetilde{\lambda}(p)\geq 1/2$ for all $p\in\mathbb{N}^*/2$ can be reached without much effort by using a nice interpretation of μ_p in that case; let γ be the standard Gaussian law on $\mathbb{R}$, then $\mu_p = R(\gamma^{\otimes 2p})$, where R is the mapping defined by

$$\begin{array}{rcl} R\,:\,\mathbb{R}^{2p} & \longrightarrow & \mathbb{R}_+ \\ (z_1,\ldots,z_{2p}) & \mapsto & \dfrac{z_1^2+\cdots+z_{2p}^2}{2} \end{array}$$

But recall that the Gaussian law satisfies a Poincaré's inequality:

$$\forall\, f\in C_{\mathrm{p}}^\infty(\mathbb{R}), \qquad \mathrm{Var}(f,\gamma) \;\leq\; \int (f')^2\,d\gamma$$

which implies immediately that

$$\forall\, f\in C_{\mathrm{b}}^\infty(\mathbb{R}^{2p}), \qquad \mathrm{Var}(f,\gamma^{\otimes 2p}) \;\leq\; \int \sum_{1\leq i\leq 2p}(\partial_i f)^2\,d\gamma^{\otimes 2p}$$

Now let $f\in C_{\mathrm{p}}^\infty(\mathbb{R}_+)$ be given and introduce $F=f\circ R$ on $\mathbb{R}^{2p}$. This function belongs to $C_{\mathrm{p}}^\infty(\mathbb{R}^{2p})$ and we compute that for all $1\leq i\leq 2p$,

$$\forall\, z\in\mathbb{R}^{2p}, \qquad \partial_i F(z) \;=\; z_i f'(R(z))$$

in particular

$$\sum_{1\leq i\leq 2p}(\partial_i F)^2 \;=\; 2R(f'(R))^2$$

It remains to integrate this relation with respect to $\gamma^{\otimes 2p}$ and to consider the infimum over all such functions f to see that $\widetilde{\lambda}(p)\geq 1/2$ (remark that we are "far" from the true constant $\widetilde{\lambda}(p)=1$, a fact showing that if we consider only symmetrical functions in the infimum defining the spectral gap for the standard Gaussian law, we will not recover the right value, this is quite normal, as the eigenspace associated to the spectral gap is generated by the antisymmetrical identity function).

3 Logarithmic Sobolev constant for gamma distributions

We adapt here the study of the latter section to the logarithmic Sobolev constant, which is defined by

$$\widetilde{\alpha}(p) \quad = \quad \inf_{f\in C_{\mathrm{p}}^{\infty}(\mathbb{R}_+)\setminus \mathrm{Vect}(\mathbf{1})} \frac{\widetilde{\mathcal{E}}_p(f,f)}{\mathrm{Ent}(f^2,\mu_p)}$$

for fixed $p>0$.

We first notice that the approach presented in the above remark remains valid for that constant for the particular values $p\in\mathbb{N}^*/2$, by using the corresponding well-known inequality for γ:

$$\forall\, f\in C_{\mathrm{p}}^{\infty}(\mathbb{R}), \qquad \mathrm{Ent}(f^2,\gamma) \;\leq\; 2\int (f')^2\, d\gamma$$

(the proof of this result given by Gross [5] is itself based on a product/projection procedure, starting from the symmetric Bernoulli law). Unfortunately, for $p\in\mathbb{R}_+^*\setminus(\mathbb{N}^*/2)$, we did not found such a convenient representation of μ_p. In some sense it is not so strange, as a new phenomenon will appear for the small values of p: the logarithmic Sobolev constant will no longer be equal to half the spectral gap, as in the Gaussian situation, nor will these constants be comparable (one would have noted that for positive half integers p, we already have $1/4\leq\widetilde{\alpha}(p)\leq\widetilde{\lambda}(p)/2=1/2$).

In fact, using his famous Γ_2 criterion, Bakry [1] has shown that for all $p\geq 1/2$, the identity $\widetilde{\alpha}(p)=1/4$ holds. This equality in the case $p=1$ was discovered by Korzeniowski and Stroock [7] and was the first example for which the logarithmic Sobolev constant is not equal to half the spectral gap. Stannat [13] also partially used the Γ_2 criterion but noted that it does not permit to treat small values of the parameter q. The same observation can be made for gamma distributions with a small parameter. This situation is indeed the interesting part of the following estimate, which is the goal of this section:

Proposition 3.1 *There exists a finite constant $c>0$ such that*

$$\forall\, p>0, \qquad \widetilde{\alpha}(p) \;\geq\; c/\ln(e\vee(1/p))$$

This result will be seen latter on to be of the right order.

Proof:

Noting that

$$[1/2,+\infty[\;\;=\;\; \sqcup_{n\in\mathbb{N}}[2^{n-1},2^n[$$

we see that any real number $p\geq 1$ can be written as $2^n x$ with $n\in\mathbb{N}^*$ and $x\in[1/2,1[$. Thus taking into account the convolution semigroup property satisfied by the gamma distributions and the fact that the logarithmic Sobolev constant shares the same nice behavior as the spectral gap under products, we obtain as in the previous section that

$$\inf_{p\geq 1/2}\widetilde{\alpha}(p) \;\;=\;\; \inf_{1/2\leq p\leq 1}\widetilde{\alpha}(p)$$

This observation shows that it is sufficient to prove the above proposition for $0 < p \le 1/2$. Alternatively, this is also a consequence of the subtler method and sharper results of Bakry [1], that one would have to take into account if not working only up to "universal constants" (nevertheless, as already alluded to, the Γ_2 criterion is not verified for $0 < p < 1/2$, and thus won't help us to prove the conjecture 1.3, see the end of section 5).

Let $\widehat{\alpha}(p)$ be the logarithmic Sobolev constant associated to the symmetrized probability $\widehat{\mu}_p$ introduced in the latter section, namely

$$\widehat{\alpha}(p) \;:=\; \inf_{f\in C_{\mathrm{p}}^{\infty}(\mathbb{R})} \frac{\int |s|(f'(s))^2\,\widehat{\mu}_p(ds)}{\operatorname{Ent}(f^2,\widehat{\mu}_p)}$$

If in the above definition we consider only symmetric functions, we recover $\widetilde{\alpha}(p)$, so we are assured of the bound $\widetilde{\alpha}(p) \ge \widehat{\alpha}(p)$. But using the fact that 0 is the median of $\widehat{\mu}_p$, theorem 5.3 of [2] tells us that

$$C_p^{-1}/720 \;\le\; \widehat{\alpha}(p) \;\le\; 75C_p^{-1}$$

with

$$C_p \;=\; \sup_{t>0} \int_0^t s^{-p}\exp(s)\,ds \int_t^{+\infty} s^{p-1}\exp(-s)\,ds\,|\ln[\widehat{\mu}_p([t,+\infty[)]|$$

As the referee mentioned it in his report, theorem 5.3 of [2] is only stated for constant diffusion coefficient, so we should first consider the change of variable $s \mapsto s^2$ to come back to this situation. But we can notice that the proof of Bobkov and Götze is also valid with a Dirichlet form of the general type $\int (f')^2\,dm$ where m is any non-negative Borel measure on $\mathbb{R}$, as in the case of evaluation of Poincaré's constants, on which is based the approach for the logarithmic Sobolev constant.

In order to evaluate this quantity, we note that an elementary function analysis shows that for $0 < p \le 1$ fixed, we have $(p-1)\ln(s) \ge -s$ for any $s \ge 0$, thus we obtain for $t > 0$,

$$\begin{aligned}
|\ln[\widehat{\mu}_p([t,+\infty[)]| &= \ln(2\Gamma(p)) - \ln\left(\int_t^{+\infty} s^{p-1}\exp(-s)\,ds\right)\\
&\le \ln(2\Gamma(p)) - \ln\left(\int_t^{+\infty} \exp(-2s)\,ds\right)\\
&= \ln(2\Gamma(p)) - \ln\left(\exp(-2t)/2\right)\\
&= \ln(4\Gamma(p)) + 2t
\end{aligned}$$

and then it appears that

$$C_p \;\le\; \ln(4\Gamma(p))B_p + 2D_p$$

with

$$D_p \;=\; \sup_{t>0} t\int_0^t s^{-p}\exp(s)\,ds \int_t^{+\infty} s^{p-1}\exp(-s)\,ds$$

But in the previous section we have shown that $\sup_{0<p\leq 1/2} B_p < +\infty$ and we have that for $0 < p \leq 1/2$,

$$\Gamma(p) \;=\; p^{-1}\Gamma(1+p) \;\leq\; p^{-1}\sup_{1\leq q\leq 3/2}\Gamma(q) \;=\; \Gamma(1)p^{-1} \;=\; p^{-1}$$

(the last but one inequality comes from the fact that the function Γ is convex and that $\Gamma(1) = \Gamma(2) = 1$).

These facts admit as a consequence that in order to obtain the estimate given in the above proposition it is sufficient for instance to prove that

$$\sup_{0<p\leq 1/2} D_p \;<\; +\infty$$

For that purpose, we take into account the Hölder inequality already considered in the latter section in conjunction with the observation that the additional factor t does not depend on p, which reduces the problem to the finiteness of D_0 and $D_{1/2}$.

For the bound $D_{1/2} < +\infty$, we note that via an obvious change of variable, we can write,

$$\begin{aligned} D_{1/2} &= 4\sup_{t>0} t^2\int_0^t \exp(s^2)\,ds\int_t^{+\infty}\exp(-s^2)\,ds \\ &\leq 4\sup_{t>0}\left|\ln\left(\frac{2}{\sqrt{\pi}}\int_t^{+\infty}\exp(-s^2)\,ds\right)\right|\int_0^t\exp(s^2)\,ds\int_t^{+\infty}\exp(-s^2)\,ds \\ &\leq 4\sup_{t>0}\int_0^t\exp(s^2)\,ds\int_t^{+\infty}\exp(-s^2)\,ds\left|\ln\left(\int_t^{+\infty}\exp(-s^2)\frac{ds}{\sqrt{\pi}}\right)\right| \end{aligned}$$

and we recognize the last rhs as the expression appearing in the Hardy's caracterization of the logarithmic Sobolev inequality for the centralized Gaussian law of variance $1/2$, which enables us to conclude to the finiteness of $D_{1/2}$.

To be convinced of that of D_0, instead of the Cauchy-Schwarz inequality used at the beginning of the proof of lemma 2.3, we rather have to resort to the trivial bound

$$\int_t^{+\infty} s^{-1}\exp(-s)\,ds \;\leq\; t^{-1}\int_t^{+\infty}\exp(-s)\,ds \;=\; t^{-1}\exp(-t)$$

which permits to see that

$$D_0 \;\leq\; \sup_{t>0}(1-\exp(-t)) \;=\; 1$$

■

4 On product spaces

Our next step in the proof of proposition 1.1 uplifts the problem to a larger but nicer state space, a mechanism which has already be seen to be useful in the previous sections and which is quite classical (even in discrete settings, cf for instance [3]).

The main ingredient is a traditional interpretation of ν_q, for $q = (q_1, ..., q_{d+1}) \in (\mathbb{R}_+^*)^{d+1}$ which is assumed to be fixed from now on.

Let μ_q be the Borelian product probability $\mu_{q_1} \otimes \mu_{q_2} \otimes \cdots \otimes \mu_{q_{d+1}}$ on $\mathbb{R}_+^{d+1}$ (recall that for $p > 0$, μ_p designate the gamma distribution of parameter p on $\mathbb{R}_+$).

We introduce the mapping

$$\begin{array}{rcl} (S,T)\,:\,\mathbb{R}_+^{d+1} & \rightarrow & \triangle_d \times \mathbb{R}_+ \\ y & \mapsto & (S(y), T(y)) \end{array}$$

with for all $y = (y_1, \ldots, y_{d+1}) \in \mathbb{R}_+^{d+1}$,

$$\begin{array}{rcl} S(y) & = & y_1 + \cdots + y_{d+1} \\ T(y) & = & (y_1/S(y), \ldots, y_d/S(y)) \end{array}$$

(the classical convention $0 \cdot (+\infty) = 0$ is enforced for $y = 0$).

Then it is well-known (cf for instance [4]) that S and T are independent under μ_q, that $S(\mu_q) = \mu_{|q|}$ and that $T(\mu_q) = \nu_q$.

Next we consider the bilinear form $\widetilde{\mathcal{E}}_q$ defined on $C_{\mathrm{p}}^\infty(\mathbb{R}_+^{d+1})$ by

$$\forall\, f, g \in C_{\mathrm{p}}^\infty(\mathbb{R}_+^{d+1}), \qquad \widetilde{\mathcal{E}}_q(f,g) \;=\; \int \sum_{1 \le i \le d+1} y_i \partial_i f(y) \partial_i g(y)\, \mu_q(dy)$$

and the associated logarithmic Sobolev constant

$$\widetilde{\alpha}(q) \;=\; \inf_{f \in C_{\mathrm{p}}^\infty(\mathbb{R}_+^{d+1}) \setminus \mathrm{Vect}(\mathbb{1})} \frac{\widetilde{\mathcal{E}}_q(f,f)}{\mathrm{Ent}(f^2, \mu_q)} \tag{4}$$

(these notions are just multidimensional versions of those defined in the two previous sections).

The advantage of μ_q and $\widetilde{\mathcal{E}}_q$ is that the latter quantity is easily computed due to their product structures; we obtain

$$\widetilde{\alpha}(q) \;=\; \min_{1 \le i \le d+1} \widetilde{\alpha}(q_i)$$

and that observation reduces the problem to one-dimensional estimations.

In particular, the previous section shows that there exists a constant $c > 0$ such that for any $q = (q_1, ..., q_{d+1}) \in (\mathbb{R}_+^*)^{d+1}$, we have

$$\widetilde{\alpha}(p) \;\ge\; \frac{c}{\max(1, \ln(1/q_1), ..., \ln(1/q_{d+1}))} \tag{5}$$

In order to make a link with our quantity of interest $\alpha(q)$, we begin by working with the underlying diffusive *carrés du champ*: first to any C^1 function f on the simplex $\triangle_d$, we associate the continuous mapping $\gamma_q(f,f)$ defined by

$$\forall\, x = (x_1, ..., x_d) \in \triangle_d, \qquad \gamma_q(f,f)(x) \;:=\; \frac{1}{2} \sum_{1 \le i,j \le d} x_i(\delta_{i,j} - x_j) \partial_i f(x) \partial_j f(x)$$

Next, for any couple of C^1 functions F, G : $\mathbb{R}_+^{d+1} \setminus \{0\} \rightarrow \mathbb{R}$, we introduce the continuous mapping

$$\Gamma_q(F,G)\,:\,\mathbb{R}_+^{d+1} \setminus \{0\} \ni y = (y_1, \ldots, y_{d+1}) \;\mapsto\; \sum_{1 \le i \le d+1} y_i \partial_i F(y) \partial_i G(y)$$

In all what follows, we will mainly consider functions of the type $F := F_1 F_2$, where $F_1 := g \circ S$ and $F_2 := f \circ T$, with $f \in C^1(\triangle_d)$ and $g \in C^1(\mathbb{R}_+^*)$.

For instance for our first result of decomposition:

Lemma 4.1 *With the previous notations, we have*

$$\Gamma_q(F_1F_2, F_1F_2) = F_2^2\Gamma_{|q|}(g)\circ S + 2\frac{F_1^2}{S}\gamma_q(f,f)\circ T$$

Proof:

As a general result for bilinear mappings which are acting by derivation on each of its arguments, we have a priori that

$$\Gamma_q(F_1F_2, F_1F_2) = F_1^2\Gamma_q(F_2,F_2) + F_2^2\Gamma_q(F_1,F_1) + 2F_1F_2\Gamma_q(F_1,F_2)$$

but in our particular situation, we note that the last term is null: using that for all $y = (y_1, \ldots, y_{d+1}) \in (\mathbb{R}_+^*)^{d+1} \setminus \{0\}$,

$$\begin{aligned}
\forall\, 1 \le i \le d, \qquad \partial_i F_2(y) &= \frac{1}{S(y)}\partial_i f(T(y)) - \sum_{1\le j\le d}\frac{y_j}{S^2(y)}\partial_j f(T(y)) \\
\partial_{d+1}F_2(y) &= -\sum_{1\le j\le d}\frac{y_j}{S^2(y)}\partial_j f(T(y))
\end{aligned}$$

and that

$$\forall\, 1 \le i \le d+1, \qquad \partial_i F_1 = g'\circ S$$

we compute that

$$\begin{aligned}
\Gamma_q(F_1,F_2) &= g'(S)\left[\sum_{1\le i\le d}(T_i\partial_i f(T) + y_i\partial_{d+1}F_2) + y_{d+1}\partial_{d+1}F_2\right] \\
&= g'(S)\left[\sum_{1\le i\le d} T_i\partial_i f(T) + S\partial_{d+1}F_2\right] \\
&= 0
\end{aligned}$$

(with the obvious notation $T_i(y) = y_i/S(y)$, for $1 \le i \le d$ and $y \in (\mathbb{R}_+)^{d+1}\setminus\{0\}$).

Furthermore, it appears at once that

$$\begin{aligned}
\Gamma_q(F_1,F_1) &= S(g'(S))^2 \\
&= \Gamma_{|q|}(g)\circ S
\end{aligned}$$

and we compute that for all $y \in (\mathbb{R}_+)^{d+1}\setminus\{0\}$,

$$\begin{aligned}
&\Gamma_q(F_2,F_2)(y) \\
&= \sum_{1\le i\le d} y_i\left(\frac{\partial_i f(T(y))}{S(y)} + \partial_{d+1}F(y)\right)^2 + y_{d+1}(\partial_{d+1}F(y))^2 \\
&= \sum_{1\le i\le d}\frac{y_i}{S^2(y)}(\partial_i f(T(y)))^2 + 2\sum_{1\le i\le d}\frac{y_i}{S(y)}\partial_i f(T(y))\partial_{d+1}F(y) + S(y)(\partial_{d+1}F(y))^2
\end{aligned}$$

but the intermediate term is just

$$2\sum_{1\le i\le d}\frac{y_i}{S(y)}\partial_i f(T(y))\partial_{d+1}F(y) = -2S(y)(\partial_{d+1}F(y))^2$$

and so we end up with the following relation between our *carrés du champs* of interest

$$\begin{aligned}\Gamma_q(F_2,F_2) &= \frac{1}{S}\left[\sum_{1\le i\le d} T_i(\partial_i f(T))^2 - \sum_{1\le i,j\le d} T_iT_j\partial_i f(T)\partial_j f(T)\right]\\ &= 2\frac{\gamma_q(f,f)\circ T}{S}\end{aligned}$$

which enables us to conclude.

■

By definition, for any functions $f \in C^1(\triangle_d)$ and $F \in C^1(\mathbb{R}_+^{d+1}\setminus\{0\})$, we have respectively

$$\begin{aligned}\mathcal{E}_q(f,f) &= \int \gamma_q(f,f)\,d\nu_q\\ \widetilde{\mathcal{E}}_q(F,F) &= \int \Gamma_q(F,F)\,d\mu_q\end{aligned}$$

(there the domains of definition have been extended, note in particular that the last integral can now be infinite) so remembering the properties of S and T, we obtain as a direct consequence of the above lemma and with its notations,

$$\widetilde{\mathcal{E}}_q(F_1F_2,F_1F_2) = \widetilde{\mathcal{E}}_{|q|}(g,g)\nu_q(f^2) + 2\mu_{|q|}(g^2/s)\mathcal{E}_q(f,f)$$

where s denotes the identity on $\mathbb{R}_+$.

We also mention the following well-known identities of decomposition for the variance and entropy of independent variables:

$$\begin{aligned}\mathrm{Var}(F_1F_2,\mu_q) &= \mu_q[(F_1F_2)^2] - \mu_q[F_1F_2]^2\\ &= \mu_q(F_1^2)\mathrm{Var}(F_2,\mu_q) + \mu_q(F_2)^2\mathrm{Var}(F_1,\mu_q)\\ &= \mu_{|q|}(g^2)\mathrm{Var}(f,\nu_q) + \mathrm{Var}(g,\mu_{|q|})\nu_q(f)^2\end{aligned}$$

and

$$\begin{aligned}\mathrm{Ent}((F_1F_2)^2,\mu_q) &= \mu_q(F_1^2)\mathrm{Ent}(F_2^2,\mu_q) + \mu_q(F_2^2)\mathrm{Ent}(F_1^2,\mu_q)\\ &= \mu_{|q|}(g^2)\mathrm{Ent}(f^2,\nu_q) + \nu_q(f^2)\mathrm{Ent}(g^2,\mu_{|q|})\end{aligned}$$

To finish this section, let us consider, always for fixed $q=(q_1,\ldots,q_{d+1}) \in (\mathbb{R}_+^*)^{d+1}$, the spectral gap $\lambda(q)$ corresponding to ν_q and $\mathcal{E}_q$, which is defined as the quantity

$$\lambda(q) := \inf_{f\in C^1(\triangle_d)\setminus \mathrm{Vect}(\mathbf{1})} \frac{\mathcal{E}_q(f,f)}{\mathrm{Var}(f,\nu_q)}$$

It follows from the complete spectral decomposition of the minimal auto-adjoint extension of the generator L_q, which was obtained by Shimakura [12] (maybe there is a hidden relation with the Laguerre operators), that $\lambda(q) = |q|/2$. Nevertheless, we will show how the above manipulations enable to recover this equality. To do so, we will take into account the information that $\widetilde{\lambda}(p) = 1$ for all $p > 0$ (ie we need more than just the rough proposition 2.1). Once again, due to the product structures, we deduce from these identities that $\widetilde{\lambda}(q) = \min_{1\leq i\leq d+1} \widetilde{\lambda}(q_i) = 1$, where

$$\widetilde{\lambda}(q) \quad := \quad \inf_{F\in C^1(\mathbb{R}_+^{d+1}\setminus\{0\})\setminus\mathrm{Vect}(\mathbf{1})} \frac{\widetilde{\mathcal{E}}_q(F,F)}{\mathrm{Var}(F,\mu_q)}$$

Our first task is to see that

Proposition 4.2 *For any* $q = (q_1, ..., q_{d+1}) \in (\mathbb{R}_+^*)^{d+1}$*, we have*

$$\lambda(q) \quad \geq \quad |q|/2$$

Proof:

Let a generical $f \in C^1(\triangle_d)$ be given such that $\nu_q(f) = 0$. With the previous notations, we consider $F_1 = f\circ T$ and $F_2 = S$, so that the relation $\widetilde{\lambda}(q) = 1$ implies in particular that

$$\mathrm{Var}(F_1F_2, \nu_q) \quad \leq \quad \widetilde{\mathcal{E}}_q(F_1F_2, F_1F_2)$$

an inequality which can be rewritten as

$$\mu_{|q|}(s^2)\mathrm{Var}(f,\nu_q) \quad \leq \quad \mu_{|q|}(s)\nu_q(f^2) + 2\mu_{|q|}(s)\mathcal{E}_q(f,f)$$

Using now that

$$\frac{\mu_{|q|}(s)}{\mu_{|q|}(s^2)} \quad = \quad \frac{\Gamma(|q|+1)}{\Gamma(|q|+2)} \quad = \quad \frac{1}{|q|+1}$$

and that $\mathrm{Var}(f,\nu_q) = \nu_q(f^2)$, it follows that

$$|q|\mathrm{Var}(f,\nu_q) \quad \leq \quad 2\mathcal{E}_q(f,f)$$

As this inequality is indeed satisfied for all $f \in C^1(\triangle_d)$, the announced bound is proved.

■

To prove an upper bound for an ergodic constant is often easier than a lower bound, since it is sufficient to choose a convenient test function. Here if we consider the first coordinate,

$$f_0 \,:\, \triangle_d \ni x \quad \mapsto \quad x_1 \in \mathbb{R}$$

we compute that

$$\begin{aligned}
\mathcal{E}_q(f_0,f_0) \quad &= \quad \frac{1}{2}\int \sum_{1\leq i,j\leq d} x_i(\delta_{i,j} - x_j)\partial_i f_0(x)\partial_j f_0(x)\,\nu_q(dx) \\
&= \quad \frac{1}{2}\int x_1(1-x_1)\,\nu_q(dx) \\
&= \quad \frac{q_1(|q|-q_1)}{2\,|q|\,(|q|+1)}
\end{aligned}$$

as can be seen by using the form of the renormalization factor of ν_q in (1) (which itself is directly deduced from that of μ_q).

On the other hand, it appears similarly that

$$\begin{aligned}\mathrm{Var}(f_0,\nu_q) &= \frac{q_1(q_1+1)}{|q|(|q|+1)} - \left(\frac{q_1}{|q|}\right)^2 \\ &= \frac{q_1(|q|-q_1)}{|q|^2(|q|+1)}\end{aligned}$$

so we end up with $\lambda(q) \leq |q|/2$ and finally $\lambda(q) = |q|/2$.

5 Estimates for the logarithmic Sobolev constant

Here we will finally take advantage of the previous sections to prove the bounds announced in the introduction.

In order to show the proposition 1.1, let us introduce a new quantity for $q \in (\mathbb{R}_+^*)^{d+1}$:

$$\bar{\alpha}(q) \quad := \quad \inf_{f\in C^1(\Delta_d)\setminus\{0\}\,:\,\nu_q(f)=0} \frac{\mathcal{E}_q(f,f)}{\mathrm{Ent}(f^2,\nu_q)}$$

Clearly we have $\bar{\alpha}(q) \geq \alpha(q)$ and taking into account an inequality due to Deuschel (cf [6]), saying that

$$\forall\, f \in \mathbb{L}^2(\nu_q), \qquad \mathrm{Ent}(f^2,\nu_q) \;\leq\; 2\mathrm{Var}(f,\nu_q) + \mathrm{Ent}((f-\nu_q(f))^2,\nu_q)$$

it appears that up to universal factors, $\alpha(q)$ is of the same order as $\lambda(q) \wedge \bar{\alpha}(q) = (|q|/2) \wedge \bar{\alpha}(q)$. Thus we are led to evaluate the latter constant, and the next result enables us to conclude the validity of the estimate presented in the proposition 1.1.

Proposition 5.1 *There exists a constant $c_0 > 0$ such that for all $q = (q_1, ..., q_{d+1}) \in (\mathbb{R}_+^*)^{d+1}$, we are assured of*

$$\bar{\alpha}(q) \;\geq\; c_0 \frac{|q|}{\max\{1, \ln(1/q_1), ..., \ln(1/q_{d+1})\}}$$

Proof:

Let $f \in C^1(\Delta_d)$ be such that $\nu_q(f) = 0$. Coming back to the notations of section 4, we consider $F = F_1F_2$ with $F_1 = S$ and $F_2 = f \circ T$. Then we obtain

$$\begin{aligned}\mu_{|q|}(s^2)\mathrm{Ent}(f^2,\nu_q) &\leq \mathrm{Ent}(F^2,\mu_q) \\ &\leq \frac{1}{\widetilde{\alpha}(q)}\widetilde{\mathcal{E}}_q(F,F) \\ &= \frac{1}{\widetilde{\alpha}(q)}[\widetilde{\mathcal{E}}_{|q|}(s,s)\nu_q(f^2) + 2\mu_{|q|}(s)\mathcal{E}_q(f,f)] \\ &= \frac{\mu_{|q|}(s)}{\widetilde{\alpha}(q)}[\nu_q(f^2) + 2\mathcal{E}_q(f,f)] \\ &\leq \frac{\mu_{|q|}(s)}{\widetilde{\alpha}(q)}\left[\frac{1}{\lambda(q)} + 2\right]\mathcal{E}_q(f,f) \\ &= 2\frac{\mu_{|q|}(s)}{\widetilde{\alpha}(q)}\frac{|q|+1}{|q|}\mathcal{E}_q(f,f)\end{aligned}$$

and by consequence

$$\mathrm{Ent}(f,\nu_q) \leq \frac{2}{|q|\widetilde{\alpha}(q)}\mathcal{E}_q(f,f)$$

So the expected result follows at once from (5).

■

Remark 5.2: the a priori natural embedding $C^1(\triangle_d) \ni f \mapsto F := f\circ T \in C^1((\mathbb{R}_+^*)^{d+1})$ (ie with $F_1 \equiv 1$) does not give any information for $|q| \leq 1$, because then we have $\mu_{|q|}(1/s) = +\infty$.

We now come to the second objective of this section: as usual, an estimation as that of proposition 1.2 will mainly be the consequence of an appropriate choice of test function. We will consider two cases, corresponding respectively to $q_* = \min(q_i : 1 \leq i \leq d)$ or $q_* = q_{d+1}$. We begin by treating the first situation, where by symmetry, we can and will assume that indeed $q_* = q_1$.

Lemma 5.3 *Under the above setting, there exists a constant $a \geq 0$, such that we have*

$$\frac{\mathcal{E}_q(f_0,f_0)}{\mathrm{Ent}(f_0^2,\nu_q)} \leq \frac{|q|}{[\ln((|q|\wedge 1)/(q_1\wedge e^{-1})) - a]_+}$$

where we recall that f_0 is the first coordinate mapping

$$f_0 : \triangle_d \ni x = (x_1,\ldots,x_d) \mapsto x_1$$

Proof:

We have already computed at the end of the previous section that

$$\mathcal{E}_q(f_0,f_0) = \frac{q_1(|q|-q_1)}{2|q|(|q|+1)}$$

Thus we only need to evaluate the entropy $\mathrm{Ent}(f_0^2,\nu_q)$. For that, we introduce $F_0 := Sf_0\circ T : \mathbb{R}_+^{d+1} \ni y = (y_1,\ldots,y_{d+1}) \mapsto y_1$, since we have

$$\begin{aligned}
\mathrm{Ent}(f_0^2,\nu_q) &= \mathrm{Ent}(F_0^2/S^2,\mu_q)\\
&= \frac{\mathrm{Ent}(F_0^2,\mu_q) - \mu_q(F_0^2/S^2)\mathrm{Ent}(S^2,\mu_q)}{\mu_q(S^2)}\\
&= \frac{\mathrm{Ent}(F_0^2,\mu_{|q|}) - \mu_q(F_0^2)\mathrm{Ent}(S^2,\mu_q)/\mu_q(S^2)}{\mu_q(S^2)}\\
&= \frac{\mathrm{Ent}(s^2,\mu_{q_1}) - \mu_{q_1}(s^2)\mathrm{Ent}(s^2,\mu_{|q|})/\mu_{|q|}(s^2)}{\mu_{|q|}(s^2)}
\end{aligned}$$

It is quite clear that for the denominator, we have

$$\mu_{|q|}(s^2) = \frac{\Gamma(|q|+2)}{\Gamma(|q|)} = |q|\,(|q|+1)$$

and in the same manner we obtain $\mu_{q_1}(s^2) = q_1(q_1+1)$.

For the first appearing entropy, we have

$$\begin{aligned}
\mathrm{Ent}(s^2,\mu_{q_1}) &= \int s^2 \ln(s^2)\,\mu_{q_1}(ds) - \int s^2\,\mu_{q_1}(ds)\ln\left[\int s^2\,\mu_{q_1}(ds)\right] \\
&= 2\frac{\Gamma'(2+q_1)}{\Gamma(q_1)} - q_1(q_1+1)\ln[q_1(q_1+1)] \\
&= q_1(q_1+1)\left(2\frac{\Gamma'(2+q_1)}{q_1(q_1+1)\Gamma(q_1)} - \ln[q_1(q_1+1)]\right) \\
&= q_1(q_1+1)\left(2(\ln[\Gamma(2+q_1)])' - \ln[q_1(q_1+1)]\right)
\end{aligned}$$

This computation is also valid for $\mathrm{Ent}(s^2,\mu_{|q|})$, replacing q_1 by $|q|$, and so we get

$$\mathrm{Ent}(f_0^2,\nu_q) = \frac{q_1(q_1+1)}{|q|(|q|+1)}\left(2\left[(\ln[\Gamma(2+q_1)])' - (\ln[\Gamma(2+|q|)])'\right] + \ln\left[\frac{|q|(|q|+1)}{q_1(q_1+1)}\right]\right)$$

and

$$\begin{aligned}
\frac{\mathcal{E}_q(f_0,f_0)}{\mathrm{Ent}(f_0^2,\nu_q)} &= \frac{|q|-q_1}{q_1+1}\left(2\left[(\ln[\Gamma(2+q_1)])' - (\ln[\Gamma(2+|q|)])'\right] + \ln\left[\frac{|q|(|q|+1)}{q_1(q_1+1)}\right]\right)^{-1} \\
&\leq |q|\left(2\left[(\ln[\Gamma(2+q_1)])' - (\ln[\Gamma(2+|q|)])'\right] + \ln\left[\frac{|q|(|q|+1)}{q_1(q_1+1)}\right]\right)^{-1}
\end{aligned}$$

This leads us to look for a lower bound for the expression between the big parentheses, which we will designate by $A(|q|,q_1)$. In view of the previous manipulations, this quantity is clearly nonnegative, so to obtain the announced result, we only need to find a constant $a \geq 0$ such that for all parameters q as specified above, we are assured of

$$A(|q|,q_1) \geq \ln\left(\frac{|q|\wedge 1}{q_1\wedge e^{-1}}\right) - a \qquad (6)$$

To do so, we distinguish two situations.

- The simplest one is when $0 < |q| \leq 1$: using Hölder's inequalities, it is quite easy to be convinced that the mapping

$$\mathbb{R}_+^* \ni p \mapsto \ln(\Gamma(p))$$

is convex, and taking into account that $\Gamma(1) = \Gamma(2) = 1$, it follows that $\ln(\Gamma(p))'$ is increasing in $p \geq 2$. We deduce from that observation that we can take in (6)

$$a := 2[\ln(\Gamma(3))' - \ln(\Gamma(2))']$$

since due to the a priori bound $0 < q_1 \leq |q|/2$, we are insured on the one hand that $q_1 \leq 1$ and on the other hand that $(|q|+1)/(q_1+1) \geq 1$.

- The other cases correspond to $|q| > 1$: coming back to previous computations, we have

$$-2(\ln[\Gamma(2+|q|)])' + \ln[|q|(|q|+1)] = -\frac{\mathrm{Ent}(s^2,\mu_{|q|})}{\mu_{|q|}(s^2)}$$

Now, with $\widetilde{q} = (1, |q|-1) \in (\mathbb{R}_+^*)^2$ and obvious notations, we also have

$$\begin{aligned}
\mathrm{Ent}(s^2, \mu_1) &= \mathrm{Ent}(y_1^2, \mu_{\widetilde{q}}) \\
&= \mathrm{Ent}\left(S^2 \frac{y_1^2}{S^2}, \mu_{\widetilde{q}}\right) \\
&= \mu_{|q|}(s^2)\mathrm{Ent}\left(\frac{y_1^2}{S^2}, \mu_{\widetilde{q}}\right) + \mu_{\widetilde{q}}\left(\frac{y_1^2}{S^2}\right)\mathrm{Ent}(s^2, \mu_{|q|}) \\
&\geq \frac{\mu_1(s^2)}{\mu_{|q|}(s^2)}\mathrm{Ent}(s^2, \mu_{|q|})
\end{aligned}$$

Thus we obtain that

$$A(|q|, q_1) \geq -\frac{\mathrm{Ent}(s^2, \mu_1)}{\mu_1(s^2)} + 2(\ln[\Gamma(2+q_1)])' - \ln[q_1(q_1+1)]$$

We furthermore note that the sum of the last two terms, reinterpreted once more as $\mathrm{Ent}(s^2, \mu_{q_1})/\mu_{q_1}(s^2)$, is positive and as the same is true for $\ln[\Gamma(2+q_1)])'$, we can bound it below by

$$\begin{aligned}
\ln_+[1/(q_1(q_1+1))] &= -\ln[(q_1(q_1+1)) \wedge 1] \\
&\geq -\mathbf{1}_{q_1 \leq 1} \ln(q_1(q_1+1)) \\
&\geq -\mathbf{1}_{q_1 \leq 1}[\ln(2) + \ln(q_1)] \\
&\geq -\ln(2) - \ln(q_1 \wedge 1) \\
&\geq -\ln(2) - 1 - \ln(q_1 \wedge e^{-1})
\end{aligned}$$

Finally, it appears that (6) is verified in this situation with

$$a := \frac{\mathrm{Ent}(s^2, \mu_1)}{\mu_1(s^2)} + \ln(2) + 1 = 2\ln(\Gamma(3))' + 1$$

and more generally note that this choice of a is good for all cases.

■

Let us now consider the remaining situation, where the minimum q_* is attained in q_{d+1}. A convenient choice for the test function is then

$$\widetilde{f}_0 : \triangle_d \ni x = (x_1, \dots, x_d) \mapsto 1 - x_1 - \cdots - x_d$$

because the associated natural uplift $\widetilde{F}_0 := S\widetilde{f}_0 \circ T$ is immediately seen to be the coordinate mapping $\mathbb{R}_+^{d+1} \ni y = (y_1, \dots, y_{d+1}) \mapsto y_{d+1}$. Thus the proof of the above lemma shows (indeed, we have just to replace q_1 by q_{d+1}) that

$$\frac{\mathcal{E}_q(\widetilde{f}_0, \widetilde{f}_0)}{\mathrm{Ent}(\widetilde{f}_0^2, \nu_q)} \leq \frac{|q|}{[\ln((|q| \wedge 1)/(q_{d+1} \wedge e^{-1})) - a]_+}$$

As a conclusion of these considerations, we have that for any $q \in (\mathbb{R}_+^*)^{d+1}$,

$$\alpha(q) \leq \frac{|q|}{[\ln((|q| \wedge 1)/(q_* \wedge e^{-1})) - a]_+}$$

and by consequence,

$$\alpha(q) \leq \frac{2|q|}{[\ln((|q| \wedge 1)/(q_* \wedge e^{-1}))}$$

as soon as $(|q| \wedge 1)/(q_* \wedge e^{-1}) \geq \exp(2a)$.

But as a general bound (cf for instance [3]), the logarithmic Sobolev constant is always smaller than half the spectral gap, so if $(|q| \wedge 1)/(q_* \wedge e^{-1}) < \exp(2a)$, we have

$$\begin{aligned} \alpha(q) &\leq \frac{|q|}{4} \\ &\leq \frac{\exp(2a)}{4} \frac{|q|}{[\ln((|q| \wedge 1)/(q_* \wedge e^{-1}))} \end{aligned}$$

and the proposition 1.2 follows.

Remark 5.4: at least for $|q| \geq 1$, the estimate of proposition 1.1 is of the right order and it is quite straightforward to deduce from this, by a contradiction argument, that the same has to be true for the bound of proposition 3.1.

To end up this section, let us present a clue leading us to think that $\alpha(q)$ should always be of the order of its upper bound of proposition 1.2. To back up this affirmation, we make, for instance, the conjecture that

$$\limsup_{p \to 0_+} \widetilde{\alpha}^{-1}(p) - \frac{\text{Ent}(s^2, \mu_p)}{\widetilde{\mathcal{E}}_p(s,s)} < + \infty \tag{7}$$

This relation is just saying that the identity mapping is almost a minimizer in the definition of the logarithmic Sobolev constant for the one-dimensional gamma distribution, asymptotically for small values of the underlying parameter (remember that this mapping is a true minimizer for the corresponding spectral gap). In particular, via above computations showing that

$$\limsup_{p \to 0_+} \left| \frac{\text{Ent}(s^2, \mu_p)}{\widetilde{\mathcal{E}}_p(s,s)} - \ln(1/p) \right| < +\infty$$

the condition (7) implies that

$$\limsup_{p \to 0_+} \widetilde{\alpha}^{-1}(p) - \ln(1/p) < + \infty$$

(we also have $\liminf_{p \to 0_+} \widetilde{\alpha}^{-1}(p) - \ln(1/p) > -\infty$, in particular, $\widetilde{\alpha}^{-1}(p) \sim \ln(1/p)$ for $p > 0$ small). Note that this result is out of reach by Hardy's inequalities techniques, which only give estimates up to universal factors.

We take into account this supplementary information to deduce from (7) that there exists a constant $c_3 > 0$ independent of the dimension $d \in \mathbb{N}^*$ such that

$$\forall\, q \in (\mathbb{R}_+^*)^{d+1}, \qquad \alpha(q) \geq c_3 \frac{|q|}{\ln[(|q| \wedge 1)/(q_* \wedge e^{-1})]}$$

Indeed, in view of what we have already done, it is sufficient to show this bound for $|q| \leq 1$.

But then let $K \geq 0$ be such that for all $0 < p \leq e^{-1}$, we are assured of

$$\begin{aligned} \widetilde{\alpha}^{-1}(p) &\leq \ln(1/p) + K \\ \ln(1/p) &\leq \frac{\mathrm{Ent}(s^2, \mu_p)}{\widetilde{\mathcal{E}}_p(s,s)} + K \end{aligned}$$

Coming back to the beginning of the proof of proposition 5.1, we rather write (with its notations),

$$\begin{aligned} &\mu_{|q|}(s^2)\mathrm{Ent}(f^2, \nu_q) \\ &= \mathrm{Ent}(F^2, \mu_q) - \nu_q(f^2)\mathrm{Ent}(s^2, \mu_{|q|}) \\ &\leq \frac{1}{\widetilde{\alpha}(q)}\widetilde{\mathcal{E}}_q(F, F) - \nu_q(f^2)\mathrm{Ent}(s^2, \mu_{|q|}) \\ &= \left[\frac{\widetilde{\mathcal{E}}_{|q|}(s,s)}{\widetilde{\alpha}(q)} - \mathrm{Ent}(s^2, \mu_{|q|})\right]\nu_q(f^2) + 2\frac{\mu_{|q|}(s)}{\widetilde{\alpha}(q)}\mathcal{E}_q(f,f) \\ &\leq [\ln(|q|/q_*) + 2K]\,\widetilde{\mathcal{E}}_{|q|}(s,s)\nu_q(f^2) + 2(\ln(1/|q|) + K)\mu_{|q|}(s)\mathcal{E}_q(f,f) \\ &\leq [\ln(|q|/q_*) + 2K]\,\mu_{|q|}(s)\lambda^{-1}(q)\mathcal{E}_q(f,f) + 2(\ln(1/|q|) + K)\mu_{|q|}(s)\mathcal{E}_q(f,f) \\ &\leq 2\mu_{|q|}(s)(\ln(|q|/q_*) + 2K)(|q|^{-1} + 1)\mathcal{E}_q(f,f) \end{aligned}$$

and the announced result follows quite easily (remember that $\mu_{|q|}(s^2) = (|q|+1)\mu_{|q|}(s)$).

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