On ergodic diffusions on continuous graphs whose centered resolvent admits a trace

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Abstract

We consider ergodic and reversible diffusions on continuous and connected graphs $\mathbb{G}$ with a finite number of bifurcation vertices and some rays going to infinity. A necessary and sufficient condition is presented for the spectrum of the associated generator L to be without continuous part and for the sum of the inverses of its eigenvalues (except 0) to be finite. This criterion is easily computable in terms of the coefficients of L and does not depend on the transition kernels at the vertices. Its motivation is that it is conjectured to be also a necessary and sufficient condition for the diffusion to admit strong stationary times whatever its initial distribution (this is known to be true if $\mathbb{G}$ is the real line).

The above criterion for the centered resolvent to be of trace class is next extended to Markov processes on denumerable connected graphs with only a finite number of vertices of degree larger than or equal to 3.

Keywords: Diffusions on quantum graphs, trace class centered resolvents, reversibility, Dirichlet forms.

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1 Introduction

The purpose of this paper is to give an explicit criterion for the centered resolvent associated to an ergodic and reversible diffusion on a continuous and connected graph (often called a quantum graph, see the recent book of Berkolaiko and Kuchment [2]) to admit a trace. The graph is assumed to have a finite number of bifurcating points and in particular a finite number of infinite rays. This result is already interesting when the state space is $\mathbb{R}$, since it enables to deduce from [9] that a one-dimensional ergodic diffusion has strong stationary times, whatever its initial distribution, if and only if the corresponding centered resolvent has a trace. We strongly believe that the latter equivalence result is also true when $\mathbb{R}$ is replaced by the state spaces we are to consider here, but the techniques needed to approach this conjecture are different from those we use in this paper, so the investigation of this question is postponed to a future work.

We adopt a convenient framework for diffusion processes on graphs adapted from the article of Freidlin and Sheu [4]. A finite connected graph $G := (V, E)$ is given, where V and E are respectively the sets of vertices and unoriented edges. To each edge $e \in E$, is associated a segment $R_e := [e^-, e^+] \subset \mathbb{R} \sqcup \{+\infty\}$, which is either bounded if $e^+ < +\infty$, or semi-infinite if $e^+ = +\infty$. The boundary points e^- and e^+ correspond respectively to the vertices from V adjacent to e, denoted $v^-(e)$ and $v^+(e)$, and by writing $e = \{v^-(e), v^+(e)\}$, an orientation is provided to E (but except for the "semi-infinite edges", this orientation is arbitrary). If the segment is semi-infinite, we assume that $v^+(e)$ is a leaf, namely a vertex of degree 1. This is not really a restriction, because in the setting described below, the vertex $v^+(e)$ is (a.s.) not reachable by the underlying process from e, so this vertex can be detached from the "other vertices coinciding at $v^+(e)$", up to considering only the connected component of G to which e belongs, if the graph ends up being disconnected after this operation. We denote by ∞_1, ..., ∞_N, with $N \in \mathbb{Z}_+$ their number, the leaves which correspond to an infinite end and we call ∞ their set. The structure (G, E, ∞), where a subset ∞ of the leaves has been distinguished, is sometimes called a graph with boundary. By assumption, for each $k \in [\![N]\!]$, there exists a unique edge, say $e_k \in E$, to which ∞_k is adjacent. Let $\bar{G}$ be the union of the segments R_e for $e \in E$, seen as disjoint sets except that the end points corresponding to a same vertex are identified. Thus formally $\bar{G} := (\sqcup_{e\in E} R_e \times \{e\})/\simeq$, where $\simeq$ is the equivalence relation enabling to identify the end points corresponding to a same vertex, but we will often abuse notations by simply writing $\bar{G}$ as $\cup_{e\in E} R_e$. Define $\mathbb{G} = \bar{G} \backslash \infty$, which will be the state space of the diffusion processes we are interested in.

To describe the evolution of these processes on the interiors $\mathring{R}_e := (e^-, e^+)$ of the segments R_e, for $e \in E$, we assume that we are given a second order elliptic Markov generator defined on $\mathring{R}_e$ by

$$L_e \quad := \quad a_e(x)\frac{d^2}{d^2x} + b_e(x)\frac{d}{dx} \tag{1}$$

where a priori a_e is a continuous and positive function on $R_e \cap \mathbb{R}$ and b_e is a measurable and locally integrable function on $R_e \cap \mathbb{R}$. These regularity hypotheses enable to consider

$$\forall\ x \in R_e, \qquad \left\{ \begin{array}{lcl} c_e(x) & := & \displaystyle\int_{e^-}^{x} \frac{b_e(y)}{a_e(y)}\,dy \\[2ex] \mu_e(x) & := & \dfrac{\exp(c_e(x))}{a_e(x)} \end{array} \right. \tag{2}$$

We denote by the same symbol μ_e the measure on R_e admitting the function μ_e as density with respect to the restriction of the Lebesgue measure on R_e. This measure will also be seen as a measure on $\mathbb{G}$ (endowed with the σ-field $\mathcal{G}$ inherited from those of the segments) with the convention that it vanishes on the other segments.
If e is not one of the e_k, $k \in [\![N]\!]$, necessarily the measure μ_e has a finite mass: $\mu_e(R_e) < +\infty$. Otherwise we assume that

$$\forall\ k \in [\![N]\!], \qquad \mu_{e_k}(R_{e_k}) \quad < \quad +\infty \tag{3}$$

This finiteness is not sufficient to prevent the processes associated to the generators L_{e_k}, with $k \in [\![N]\!]$, to explode at ∞_k, we need furthermore that

$$\forall\ k \in [\![N]\!], \qquad \int_{e_k^-}^{\infty} \mu_{e_k}([e_k^-, y]) \exp(-c_{e_k}(y))\, dy \quad = \quad +\infty \tag{4}$$

To complete the description of a diffusion process on $\mathbb{G}$, we must specify its behavior at the vertices of V. To do so, let be given for all $v \in V \backslash \infty$ a probability $\alpha(v, \cdot)$ on the set of edges adjacent to v, whose set is denoted E_v. It is assumed that $\alpha(v, e) > 0$ for all $v \in V$ and $e \in E_v$. It is convenient to extend this probability to the whole set E by making it vanish on the other edges, so that α can be seen as a Markov kernel from $V \backslash \infty$ to E. Heuristically, when the process reaches $v \in V \backslash \infty$, it chooses its next excursion in the direction of edge e with probability $\alpha(v, e)$. More rigorously, we consider $\mathcal{C}^{\infty}_{c,\alpha}(\mathbb{G})$ the space of continuous functions f with compact support in $\mathbb{G}$, which are smooth on each of the segments R_e, for $e \in E$, and which satisfy

$$\forall\ v \in V \backslash \infty, \qquad \sum_{e \in E_v} \alpha(v, e) f'(v, e) \quad = \quad 0 \tag{5}$$

where for any $v \in V \backslash \infty$ and $e \in E_v$,

$$f'(v, e) \quad \coloneqq \quad \lim_{x \to v,\, x \in \mathring{R}_e} \frac{f(x) - f(v)}{x - v} \tag{6}$$

(where the slight abuse of notation "$\mathbb{G} = \cup_{e \in E} R_e$" is effective). Equivalently, we have

$$f'(v, e) \quad = \quad \begin{cases} \partial_e f_e(e^-) & \text{, if } v = v^-(e) \\ -\partial_e f_e(e^+) & \text{, if } v = v^+(e) \end{cases}$$

where f_e is the restriction of f to R_e and where ∂_e is a shorthand for d/dx on R_e. When v is a leaf of G which does not belong to ∞, one recognizes in (5) the Neumann reflecting boundary condition at v.

An operator L is then defined on $\mathcal{C}^{\infty}_{c,\alpha}(\mathbb{G})$ by imposing that

$$\forall\ f \in \mathcal{C}^{\infty}_{c,\alpha}(\mathbb{G}),\ \forall\ x \in \mathbb{G}, \qquad L[f](x) \quad \coloneqq \quad \begin{cases} L_e[f_e](x) & \text{, if } x \in \mathring{R}_e \\ \sum_{e \in E_x} \alpha(x, e) L_e[f_e](x) & \text{, if } x \in V \end{cases} \tag{7}$$

Freidlin and Sheu have proven in [4] that the martingale problem associated to L is well-posed: for any initial distribution m_0 on $\mathbb{G}$, there exists a unique (in law) diffusion process $X \coloneqq (X_t)_{t \geqslant 0}$ on $\mathbb{G}$ with $\mathcal{L}(X_0) = m_0$ such that for all $f \in \mathcal{C}^{\infty}_{c,\alpha}(\mathbb{G})$, the process $\mathcal{M}^f \coloneqq (\mathcal{M}^f_t)_{t \geqslant 0}$ defined by

$$\forall\ t \geqslant 0, \qquad \mathcal{M}^f_t \quad \coloneqq \quad f(X_t) - f(X_0) - \int_0^t L[f](X_s)\, ds$$

is a martingale. Due to the assumption (4), the points of ∞ are never reached (a.s.), that is why it is not necessary to specify the behavior of X once it would reach them. But the hypotheses (3) and (4) furthermore imply that the process X is ergodic, let us call μ its invariant probability. There is only one such invariant probability, because L is irreducible, as a consequence of the positivity of the diffusion coefficients a_e, $e \in E$, and of the transmission coefficients $(\alpha(v, e))_{v \in V \backslash \infty,\, e \in E_v}$ (see also Freidlin and Sheu [4]).

In this paper we are only interested in reversible processes, so we begin by giving a necessary and sufficient condition for this property. Introduce the generator matrix κ on $V \backslash \infty$ via

$$\forall\ u \neq v \in V \backslash \infty, \qquad \kappa(u, v) \quad \coloneqq \quad \begin{cases} \alpha(u, \{u, v\}) \exp(-c_e(u)) & \text{, if } \{u, v\} \in E \\ 0 & \text{, otherwise} \end{cases} \tag{8}$$

(the values on the diagonal are prescribed by the fact that the sums of the lines must be zero).

Proposition 1 *The probability μ is reversible for L if and only if the generator κ admits a reversible probability measure, say ν on $V\backslash\infty$ (it is then unique by irreducibility of κ on $V\backslash\infty$). Define β on E via*

$$\forall\ e = \{x, y\} \in E, \quad \beta(e) \ := \ \begin{cases} \nu(x)\kappa(x, y) \ = \ \nu(y)\kappa(y, x) & , \text{ if } x, y \in V\backslash\infty \\ \nu(x)\alpha(x, e_k) & , \text{ if } y = \infty_k \text{ for some } k \in [\![1, N]\!] \end{cases}$$

The probability μ associated to L is given by

$$\forall\ A \in \mathcal{G}, \qquad \mu(A) \ := \ \frac{\sum_{e\in E} \beta(e)\mu_e(A)}{\sum_{e'\in E} \beta(e')\mu_{e'}(R_{e'})} \tag{9}$$

Remarks 2

(a) Note that the definition of κ a priori depends on the chosen orientation of E and it is instructive to figure out the modifications brought by reversing the orientation of an edge $e = \{u, v\} \in E\backslash\{e_1, ..., e_N\}$. This flip changes the function c_e by adding a constant to it, because the origin of the integral in (2) has been modified. So $\kappa(u, v)$ and $\kappa(v, u)$ are both multiplied by a common factor. This does not change its property to admit a reversible probability or not, nor its probability ν if it does. Finally $\beta(e)$ is modified by a certain factor, but since μ_e is changed by the inverse factor, $\beta(e)\mu_e$ remains the same, as well as μ, as it should be.

(b) It will be shown in Remark 9 of next section that the invariant measure μ is proportional to μ_e on R_e, for all $e \in E$, if and only if μ is reversible.

□

The reversibility of μ with respect to L enables to extend this generator into a self-adjoint operator in $\mathbb{L}^2(\mu)$, named its Freidrich extension (that we still denote by L), say defined on the domain $\mathcal{D}(L) \supset \mathcal{C}^\infty_{c,\alpha}(\mathbb{G})$. Consider

$$\mathcal{F} \ := \ \{f \in \mathbb{L}^2(\mu) \ : \ \mu[f] = 0\} \tag{10}$$

and $L_\mathcal{F}$ the restriction of L to $\mathcal{D}(L) \cap \mathcal{F}$. By invariance of μ, the image of $\mathcal{D}(L) \cap \mathcal{F}$ by L is included into $\mathcal{F}$, so that $L_\mathcal{F}$ can be seen as a self-adjoint operator densely defined in $\mathcal{F}$. Furthermore, by ergodicity, the kernel of $L_\mathcal{F}$ is reduced to $\{0\}$. Functional calculus enables then to consider the operator $L_\mathcal{F}^{-1}$ (a priori defined on a dense subspace of $\mathcal{F}$), which is called the reduced resolvent (or sometimes the reduced Green operator) associated to $L_\mathcal{F}$. Our goal is to characterize the reversible operators L which are such that $L_\mathcal{F}^{-1}$ admits a trace. This is equivalent to the fact that L has no continuous spectrum and that the sum of the inverses of its eigenvalues (except 0) is finite.

To present our criterion, we introduce the following quantities

$$\begin{aligned} \forall\ k \in [\![N]\!], \qquad I_k \ &:= \ \int_{e_k^-}^{\infty} \left(\int_{e_k^-}^{x} \exp(-c_{e_k}(y))\, dy \right) \mu_{e_k}(dx) \\ I \ &:= \ \max(I_k \ : \ k \in [\![N]\!]) \end{aligned} \tag{11}$$

We can now state the main result of this paper:

Theorem 3 *The operator $L_\mathcal{F}^{-1}$ is of trace class if and only if $I < +\infty$.*

Remark 4 The previous setting of reversible diffusions on continuous and connected graphs can be extended by allowing the vertices to be gluey, see Freidlin and Wentzel [5]. So let be given for all $v \in V\backslash\infty$ a number $\alpha(v) \geqslant 0$, in addition to the transmission kernel $(\alpha(v, e))_{v\in V\backslash\infty,\, e\in E_v}$. When the underlying process X reaches $v \in V\backslash\infty$, it stays there for an exponential time of parameter $1/\alpha(v)$, before choosing its next excursion as before. The previous setting correspond to $\alpha(v) = 0$: reaching $v \in V$, the process X instantaneously leaves it (but with overwhelming probability, it returns very rapidly to v, due to the important weight of the small excursions, see for instance Chapter 12 of the book of Revuz and Yor [10]). The definition of $\mathcal{C}^\infty_{c,\alpha}(\mathbb{G})$ must be modified in the following way: it still consists of continuous functions f with compact support in $\mathbb{G}$, which are smooth on each of the segments R_e, for $e \in E$, which satisfy the equality given in (5) for all $v \in V\backslash\infty$ with $\alpha(v) = 0$, but for any $v \in V\backslash\infty$ with $\alpha(v) > 0$ and $e \in E_v$, as $x \in \mathring{R}_e$ goes to v, $L_e[f](x)$ must converge toward

$$\frac{1}{\alpha}\sum_{e'\in E_v}\alpha(v,e')f'(v,e') \quad =: \quad L[f](v)$$

The condition for L to admit a reversible probability measure μ is the same as that of Proposition 1, but μ now gives positive weights to the vertices $v \in V\backslash\infty$ which are such that $\alpha(v) > 0$. Nevertheless the above theorem can be shown to be also true in this extended setting.

□

Theorem 3 is an extension of a result of Cheng and Mao [3] which corresponds to the case where $\mathbb{G} = \mathbb{R}_+$. Then L is given as the second order elliptic Markov generator defined on $\mathbb{R}_+$ by $L := a(x)\frac{d^2}{d^2x} + b(x)\frac{d}{dx}$, where a is a continuous and positive function on $\mathbb{R}_+$ and b is a measurable and locally integrable function on $\mathbb{R}_+$ such that

$$\begin{aligned}\mu(\mathbb{R}_+) &< +\infty\\ \int_0^\infty \mu([0,y])\exp(-c(y))\,dy &= +\infty\end{aligned}$$

where c and μ are defined in terms of a and b as in (2), with e^- is taken to be 0. Cheng and Mao have proven in [3] that the operator $L_{\mathcal{F}}^{-1}$ (where $\mathcal{F}$ is defined as in (10)) is of trace class if and only if

$$\int_0^{+\infty}\left(\int_0^x \exp(-c(y))\,dy\right)\mu(dx) \quad < \quad +\infty \tag{12}$$

To come back to the above setting, we take $V := \{0,+\infty\}$, $E = \{e_1\}$, with $e_1 := \{0,+\infty\}$, $R_{e_1} = [0,+\infty)$, $\infty := \{+\infty\}$ and $L_{e_1} = L$. The transmission kernel is reduced to the Dirac mass δ_{e_1} at 0. The quantity I is just the l.h.s. of (12).

In a similar spirit, our motivation for Theorem 3 comes from the case $\mathbb{G} = \mathbb{R}$, endowed with the generator

$$L \quad := \quad a(x)\frac{d^2}{d^2x} + b(x)\frac{d}{dx} \tag{13}$$

where a is a continuous and positive function on $\mathbb{R}$ and b is a measurable and locally integrable function on $\mathbb{R}$ such that

$$\begin{aligned}\mu(\mathbb{R}) &< +\infty\\ \int_{-\infty}^0 \mu([y,0])\exp(-c(y))\,dy &= +\infty\\ \int_0^{+\infty} \mu([0,y])\exp(-c(y))\,dy &= +\infty\end{aligned} \tag{14}$$

(again c and μ are defined in terms of a and b as in (2), with e^- is taken to be 0). In [9], we were wondering if $L_{\mathcal{F}}^{-1}$ is of trace class if and only if

$$\int_{-\infty}^0\left(\int_x^0 \exp(-c(y))\,dy\right)\mu(dx) \;<\; +\infty \quad\text{and}\quad \int_0^{+\infty}\left(\int_0^x \exp(-c(y))\,dy\right)\mu(dx) \;<\; +\infty$$

Theorem 3 shows that it is indeed true, by considering as above, $V := \{-\infty, 0, +\infty\}$, $\infty := \{-\infty, +\infty\}$, $E := \{\{-\infty, 0\}, \{0, +\infty\}\}$ and the transmission kernel α given by $\alpha(0, \{-\infty, 0\}) = 1/2 = \alpha(0, \{0, +\infty\})$. The interest of this result is that we can conclude from [9] that the diffusion process generated by L admits a strong stationary time, whatever its initial distribution, if and only if $L_{\mathcal{F}}^{-1}$ is of trace class. In [9] it was also observed that this equivalence cannot be true for all reversible Markov processes.

Remark 5 Up to some regular transformations of the intervals R_e (and of the functions a_e and b_e, which should be locally integrable on $[0,1)$), for $e \in E$, we could have assumed that they are all equal to $[0,1]$ (or all equal to $[0,+\infty]$ or that all the functions a_e are identically equal to 1). But we find it more telling to associate semi-infinite segments to edges which are adjacent to vertices which cannot be reached by the underlying process. It furthermore facilitate the analogy with the discrete setting described below.

□

The previous result also holds when the diffusions on segments are replaced by birth and death processes. We equally start with a finite graph $G := (V, E)$ but to each edge $e \in E$, we associate a discrete segment $S_e := [\![e^-, e^+]\!] \subset \mathbb{Z} \sqcup \{+\infty\}$, which is finite or semi-infinite. This leads again to a subset $\infty = \{\infty_1, ..., \infty_N\}$ of the leaves of G, to the corresponding edges e_1, ..., e_N and to the state space denoted $\mathfrak{G}$ instead of $\mathbb{G}$. This state space has naturally a graph structure, with a finite or denumerable set of vertices $\mathfrak{V}$ and all the notions relative to this structure will be written using gothic letters, to make the difference with the initial structure of G. Note that any at most denumerable graph which has only a finite number of vertices of degree larger or equal to 3 is of the preceding form. In fact only the graph structure $\mathfrak{G} := (\mathfrak{V}, \mathfrak{E})$ is important for the following development and G could be forgotten, except for the analogy with the continuous framework. Or more accurately, the minimal graph G can be deduced from $\mathfrak{G}$ by defining its vertices as being the vertices of $\mathfrak{V}$ of degree larger or equal to 3 (note that this is not necessarily true in the continuous case, where some points "of degree 2" from $\mathbb{G}$ have to be vertices of G because at them the diffusion has more chance, instantaneously, to go in one direction than in the other one).

Next, on each $S_e \cap \mathbb{Z}$, for $e \in E$, we are given a birth and death generator, namely positive birth rates $(b_e(k))_{k \in [\![e^-, e^+-1]\!] \cap \mathbb{Z}}$, positive death rates $(d_e(k))_{k \in [\![e^-+1, e^+]\!] \cap \mathbb{Z}}$ and the corresponding generator L_e acting on the functions f which vanish outside a finite set of S_e by

$$\forall\, x \in S_e, \qquad L_e[f](x) \;:=\; b_e(x)(f(x+1) - f(x)) + d_e(x)(f(x-1) - f(x))$$

(with the convention $b_e(e^+) = 0$ if $e^+ \neq +\infty$, and $d_e(e^-) = 0$). In this discrete framework, no transmission kernel is needed: any oriented edge $\mathfrak{e}$ of $\mathfrak{E}$ is an oriented edge $(x, x+1)$ or $(x, x-1)$ of exactly one of the discrete segments S_e and receives its corresponding weight $L(\mathfrak{e})$, namely $b_e(x)$ or $d_e(x)$. A global generator L is then defined on functions f which vanish outside a finite number of vertices of $\mathfrak{V}$ via

$$\forall\, \mathfrak{u} \in \mathfrak{V}, \qquad L[f](\mathfrak{u}) \;:=\; \sum_{\mathfrak{v} \sim \mathfrak{u}} L(\mathfrak{u}, \mathfrak{v})(f(\mathfrak{v}) - f(\mathfrak{u})) \tag{15}$$

where $\mathfrak{u} \sim \mathfrak{v}$ indicates that $\mathfrak{u}$ and $\mathfrak{v}$ are adjacent vertices of $\mathfrak{V}$. The jump process associated to L is ergodic if and only if the following conditions (which are the analogues of (3) and (4)) are satisfied

$$\mu(\mathfrak{G}) \;<\; +\infty \tag{16}$$

$$\forall\, k \in [\![N]\!], \qquad \sum_{n \in S_{e_k}} \frac{1}{\mu_{e_k}(n) b_{e_k}(n)} \sum_{j \in [\![e_k^-, n]\!]} \mu_{e_k}(j) \;=\; +\infty \tag{17}$$

where for $k \in [\![1, N]\!]$, the measures μ_{e_k} are defined on S_{e_k} by

$$\forall\, n \in S_{e_k}, \qquad \mu_{e_k}(n) \;=\; \frac{\prod_{j \in [\![e_k^-, n-1]\!]} b_{e_k}(j)}{\prod_{j \in [\![e_k^-, n-1]\!]} d_{e_k}(j+1)} \tag{18}$$

From now on, (16) and (17) are assumed to be enforced in this discrete setting. As a consequence, there exists a unique invariant probability μ for L. We make the assumption that μ is furthermore reversible, it can then be described as in Proposition 1. By this reversibility assumption, L can be extended into a self-adjoint operator on $\mathbb{L}^2(\mu)$, still denoted L. We consider again its restriction $L_{\mathcal{F}}$ to $\mathcal{F}$, which is defined as in (10). The operator $L_{\mathcal{F}}$ is self-adjoint in $\mathcal{F}$, so functional calculus enables to define the operator $L_{\mathcal{F}}^{-1}$ (on an appropriate dense domain of $\mathcal{F}$). To state the result similar to Theorem 3, we need to define the quantity analogous to (11).

$$\begin{aligned} \forall\, k \in [\![N]\!], \qquad J_k &:= \sum_{n \in S_{e_k}} \frac{1}{\mu(n) b_{e_k}(n)} \sum_{j \in [\![n+1,+\infty[\![} \mu(j) \\ J &:= \max(J_k \,:\, k \in [\![N]\!]) \end{aligned} \tag{19}$$

Note that in the above definition of J_k, μ can be replaced by μ_{e_k} which is more explicitly given by (18). Then we have:

Theorem 6 *The operator $L_{\mathcal{F}}^{-1}$ is of trace class if and only if $J < +\infty$.*

Again, this result was first obtained by Mao [8] when $\mathfrak{G} = \mathbb{N}$.

The interest of the conditions $I < +\infty$ and $J < +\infty$ is that they are conjectured to be equivalent, in their respective frameworks, to the fact that the associated Markov processes admit strong stationary times, whatever their initial distribution. This would be a natural extension of the one-dimensional diffusion case.

The plan of the paper is as follows. In next section, the reversibility of the generators mentioned above is investigated and Proposition 1 is proven. The final section is devoted to the proof of Theorems 3 and 6, by transcribing the setting in terms of Dirichlet forms and by comparing several spectra, in particular those associated to certain Dirichlet and Neumann boundary conditions.

2 On reversibility

Our goal here is to prove the criterion for reversibility of diffusions on quantum graphs given in Proposition 1.

The framework and the notations are those described in the first part of the introduction. We begin by checking that if L admits a reversible probability measure μ, it is necessary of the form given in (9).

Lemma 7 *If the probability μ is reversible for L, then for any $e \in E$, the restriction of μ on the Borelian subsets of R_e is proportional to μ_e.*

Proof

Fixing $e \in E$, we begin by considering ξ the measure which admits $1/\mu_e$ for density with respect to the restriction of μ on the interior of R_e. By the reversibility assumption, for any f, g smooth functions with compact support in $\mathring{R}_e$, we have

$$\xi[f\mu_e L_e[g]] \;=\; \xi[g\mu_e L_e[f]]$$

Note that $L_e[g]$ can be written under the factorized form $a_e \exp(-c_e)\partial_e \exp(c_e)\partial_e[g]$, thus the above equality means that in the distribution sense,

$$(\partial_e \exp(c_e)\partial_e(f\xi))[g] \;=\; ((\partial_e \exp(c_e)\partial_e f)\xi)[g]$$

Since this is true for any smooth function g with compact support in $\mathring{R}_e$, we get

$$\partial_e \exp(c_e)\partial_e(f\xi) \;=\; (\partial_e \exp(c_e)\partial_e f)\xi$$

or equivalently

$$f(\partial_e \exp(c_e)\partial_e\xi) + 2(\partial_e f)\exp(c_e)(\partial_e\xi) \quad = \quad 0$$

Multiplying by f and denoting $\widetilde{\xi} \coloneqq \exp(c_e)(\partial_e\xi)$, it follows that

$$\partial_e f^2\widetilde{\xi} \quad = \quad 0$$

namely $f^2\widetilde{\xi}$ is a constant function, as a distribution. Since this is true for any smooth function f with compact support in $\mathring{R}_e$, necessarily $\widetilde{\xi}$ vanishes identically. We deduce that ξ is a constant function, which just means that the restriction of μ on $\mathring{R}_e$ is equal to the restriction on $\mathring{R}_e$ of $s_e\mu_e$, for some $s_e \geqslant 0$.

To conclude to the statement of the above lemma, it remains to check that μ does not contain Dirac masses at the vertices of $V\backslash\infty$. So let fix $v \in V\backslash\infty$. Let φ be a smooth function on $\mathbb{R}_+$, which vanishes on $[1, +\infty)$ and which satisfies

$$\begin{aligned} \varphi(0) &= 1 \\ \varphi'(0) &= 0 \\ \varphi''(0) &= -1 \end{aligned}$$

For any $e = \{v, u\} \in E_v$, assume that e^+ corresponds to u, to simplify the notations. Next consider the smooth function f_e on $R_e \cap \mathbb{R}$, defined by

$$\forall\ x \in R_e, \qquad f_e(x) \quad \coloneqq \quad \varphi((x - e^-)/(r_e))$$

where $r_e \coloneqq \min(1, (e^+ - e^-)/2)$. Finally, let f be the function from $\mathcal{C}^\infty_{\mathrm{c},\alpha}(\mathbb{G})$ which coincides with f_e on R_e for $e \in E_v$ and which vanishes on the other edges (note that condition (5) is in particular satisfied at v, because $f'(v, e) = 0$ for $e \in E_v$). We also need a function $g \in \mathcal{C}^\infty_{\mathrm{c},\alpha}(\mathbb{G})$ which is equal to 1 in a neighborhood of the support of f. Then the supports of f and $L[g]$ are disjoint, so that, by assumption on μ,

$$\mu[gL[f]] \;=\; \mu[fL[g]] \;=\; 0$$

Furthermore, the l.h.s. is equal to

$$\mu(\{v\})L[f](v) + \sum_{e\in E_v} s_e\mu_e[L_e[f_e]]$$

For any $e \in E_v$, we have

$$\begin{aligned} \mu_e[L_e[f_e]] &= \int_{e^-}^{e^+} L_e[f_e]\, d\mu_e \\ &= \int_{e^-}^{e^+} \partial_e \exp(c_e)\partial_e[f_e](x)\, dx \\ &= -\exp(c_e(e-))\partial_e[f_e](e^-) \\ &= -\frac{1}{r_e}\varphi'(0) \\ &= 0 \end{aligned}$$

On the other hand, it appears that

$$\begin{aligned} L[f](v) &= \sum_{e\in E_v} \alpha(v, e)L_e[f_e](v) \\ &= \sum_{e\in E_v} \alpha(v, e)a_e(v)f_e''(v, e) \\ &= \varphi''(0)\sum_{e\in E_v} \alpha(v, e)\frac{a_e(v)}{r_e^2} \\ &= -\sum_{e\in E_v} \alpha(v, e)\frac{a_e(v)}{r_e^2} \end{aligned}$$

This expression being non zero, it follows that $\mu(\{v\}) = 0$, which ends the proof that the restriction of μ on the Borelian subsets of R_e is proportional to μ_e, for any $e \in E$.

■

The end of the above proof explains the convention taken for the definition of the generator on V in (7). Other conventions are possible (even if it would be less natural, the transmission probability kernel $(\alpha(v,e))_{v\in V\setminus\infty,\, e\in E_v}$ could be replaced there by any transmission probability kernel) and the validity of the well-posedness of the martingale problem proven by Freidlin and Sheu [4] is independent from the choice of the value of the generator on V. This observation also reflects the fact that the associated invariant measure cannot charge V.

Now let be given a probability measure μ of the form described in (9), for some non-negative weights $(\beta(e))_{e\in E}$ (not all of them vanishing and not necessarily those defined in Proposition 1). Next result provides a necessary and sufficient condition for its reversibility with respect to L.

Lemma 8 *The probability μ is reversible for L if and only if for any $v \in V\setminus\infty$, the two mappings $E_v \ni e \mapsto \beta(e)$ and $E_v \ni e \mapsto \alpha(v,e)\exp(-c_e(v))$ are proportional.*

Proof

The reversibility of μ with respect to L amounts to the symmetry of $\mu[fL[g]]$ with respect to $f, g \in \mathcal{C}^\infty_{c,\alpha}(\mathbb{G})$.

Let us begin by computing $\mu_e[f_e L_e[g_e]]$ for $e \in E$ given. Recall that $L_e[g_e] = \mu_e^{-1}\partial_e \exp(c_e)\partial_e[g_e]$, so an integration by parts gives us

$$\begin{aligned}\mu_e[f_e L_e[g_e]] &= \int_{e^-}^{e^+} f_e(x) L_e[g_e](x)\,\mu_e(dx)\\ &= \int_{e^-}^{e^+} f_e(x)\partial_e \exp(c_e)\partial_e[g_e](x)\,dx\\ &= [f_e(x)\partial_e[g_e](x)\exp(c_e(x))]_{e^-}^{e^+} - \int_{e^-}^{e^+} \partial_e[f_e](x)\partial_e[g_e](x)\ \exp(c_e(x))dx\end{aligned}$$

In particular the expression $\mu[fL[g]]$ is symmetric in f, g if

$$\sum_{e\in E} \beta(e)[f_e(x)\partial_e[g_e](x)\exp(c_e(x))]_{e^-}^{e^+} = 0$$

Since f has compact support, the above sum can be rewritten under the form

$$\sum_{v\in V\setminus\infty} f(v)\iota(g,v)$$

with, for any $v \in V\setminus\infty$,

$$\begin{aligned}\iota(g,v) &\coloneqq \sum_{e\in E\,:\,v^+(e)=v} \beta(e)\exp(c_e(v))\partial_e g_e(v) - \sum_{e\in E\,:\,v^-(e)=v} \beta(e)\exp(c_e(v))\partial_e g_e(v)\\ &= \sum_{e\in E_v} \beta(e)\exp(c_e(v))g'(v,e)\end{aligned}$$

with the notation introduced in (6). Recalling Condition (5) satisfied by elements of $\mathcal{C}^\infty_{c,\alpha}(\mathbb{G})$, the latter expression vanishes for any $v \in V\setminus\infty$, if for any such v, the two mappings $E_v \ni e \mapsto \beta(e)\exp(c_e(v))$ and $E_v \ni e \mapsto \alpha(v,e)$ are proportional, which is equivalent to the statement of the above lemma.

This shows the sufficient part. Conversely, for the necessity part, we deduce from the above computations that if μ is reversible for L, then for any $f, g \in \mathcal{C}^\infty_{c,\alpha}(\mathbb{G})$,

$$\sum_{v\in V\setminus\infty} f(v)\iota(g,v) = \sum_{v\in V\setminus\infty} g(v)\iota(f,v)$$

Let be given two families of real numbers $(h(v))_{v\in V\setminus\infty}$ and $h(v,e))_{v\in V\setminus\infty,\, e\in E_v}$, the latter one satisfying the condition analogous to (5):

$$\forall\ v\in V\setminus\infty,\qquad \sum_{e\in E_v}\alpha(v,e)h(v,e) \quad=\quad 0 \tag{20}$$

One easily check that there exist functions $f,g\in\mathcal{C}^{\infty}_{\mathrm{c},\alpha}(\mathbb{G})$ such that for all $v\in V\setminus\infty$ and $e\in E_v$,

$$g(v) \;=\; f'(v,e) \;=\; 0,\quad f(v) \;=\; h(v),\quad g'(v,e) \;=\; h(v,e)$$

Since the choice of $(h(v))_{v\in V\setminus\infty}$ is arbitrary, the above computations imply that for all $v\in V\setminus\infty$,

$$\sum_{e\in E_v}\beta(e)\exp(c_e(v))h(v,e) \quad=\quad 0$$

The fact that this identity must be true for all families $h(v,e))_{v\in V\setminus\infty,\, e\in E_v}$ satisfying (20) enables to conclude to the validity of the condition given in the lemma.

■

We now have at our disposal all the prerequisites necessary to the

Proof of Proposition 1

According to the previous lemmas, the problem of the existence of a reversible measure μ for L reduces to the existence of weights $(\beta(e))_{e\in E}$ such that the condition of Lemma 8 is satisfied. Note that there is no difficulty in finding the weights of the edges e_1, ..., e_N, linked to the vertices of ∞, once the other weights have been constructed accordingly to this condition.

Recall the definition of the $(V\setminus\infty)\times(V\setminus\infty)$ generator matrix κ given in (8). The question of the existence of a reversible measure μ for L is equivalent to the existence of a family of non-negative weights $(\nu(v))_{v\in V\setminus\infty}$, not all of the them zero (they are the proportionality factors in Lemma 8), such that if one defines

$$\forall\ (u,v)\in\widetilde{E},\qquad \beta(u,v) \quad\coloneqq\quad \nu(u)\kappa(u,v)$$

with

$$\widetilde{E} \quad\coloneqq\quad \{(u,v)\in(V\setminus\infty)\times(V\setminus\infty)\ :\ \{u,v\}\in E\}$$

then $\beta(u,v)$ is symmetric with respect to u and v such that $(u,v)\in\widetilde{E}$, because we are thus led to a mapping β defined on the unoriented edges.

One then recognizes the problem of the existence of a reversible measure for κ, which induces the results stated in Proposition 1.

■

In practice it remains to check the existence of a reversible probability measure for κ to get that for L. This can be done via Kolmogorov's criterion (see for instance of book of Kelly [7]), which requires that for all distinct $v_0,v_1,...,v_n\in V\setminus\infty$, $n\in\mathbb{N}\setminus\{1\}$, such that $\{v_k,v_{k+1}\}\in E$ for $k\in[\![0,n]\!]$, with the convention that $v_{n+1}=v_0$, we have

$$\prod_{k\in[\![0,n]\!]}\kappa(v_k,v_{k+1}) \quad=\quad \prod_{k\in[\![0,n]\!]}\kappa(v_{k+1},v_k)$$

In particular if the graph G is a tree, the invariant measure μ is automatically reversible.

It is time now to justify Remark 2 (b):

Remark 9 Let μ be the unique invariant probability associated to L and assume that it is not reversible, then there exists $e\in E\setminus\{e_1,...,e_k\}$ such that the restriction of μ to R_e is not proportional to μ_e. Indeed, suppose that for all $e \in E$, the restriction of μ to R_e is proportional to μ_e. Let $g \in \mathcal{C}^\infty_{c,\alpha}(\mathbb{G})$ be given and consider $f \in \mathcal{C}^\infty_{c,\alpha}(\mathbb{G})$ such that $\{f = 1\}$ contains the support of g. The computations of the proof of Lemma 8, enable us to see that

$$\begin{aligned} 0 &= \mu[L[g]] \\ &= \mu[fL[g]] \\ &= \sum_{v \in V\setminus\infty} \iota(g, v) \end{aligned}$$

Fix $v \in V\setminus\infty$, by considering $g \in \mathcal{C}^\infty_{c,\alpha}(\mathbb{G})$ whose support is contained in a neighborhood of v not intersecting the other vertices, it appears that we must have $\iota(g, v) = 0$. And since this holds for all such g and v, we end up with the condition of Lemma 8 being satisfied. It follows that μ is reversible for L.

Let us give the simplest example of a non-reversible L in the context of quantum graphs. Consider the graph $G \coloneqq (\{0\}, \{e\})$, where e is a loop attached to the vertex 0 and let $R_e \coloneqq [0, 1]$ be endowed with a second order elliptic operator L. Formally, we did not allowed in the introduction the graph G to have loops or multiple edges. But we are brought back to this situation by choosing two new vertices on $\mathring{R}_e$, say 1/3 and 2/3, by cutting the edge $[0, 1]$ into three edges $e_1 \coloneqq [0, 1/3]$, $e_2 \coloneqq [1/3, 2/3]$ and $e_3 \coloneqq [2/3, 1]$ and by considering transmission kernels α satisfying $\alpha(1/3, e_1) = \alpha(1/3, e_2) = 1/2$ and $\alpha(2/3, e_2) = \alpha(2/3, e_3) = 1/2$. At 0 (identified with 1), take $\alpha(0, e_1) \neq \alpha(0, e_3)$, then by applying Kolmogorov's criterion, L does not admit a reversible probability. We could even consider the critical case where $\alpha(0, e_1) = 1$ and $\alpha(0, e_3) = 0$. It appears that the corresponding invariant measure μ is the quasi-stationary probability measure associated to L with a Neumann condition at 0 and a Dirichlet condition at 1 (this does really not enter in our framework, where the transmission coefficients were assumed to be positive, the definition of the core $\mathcal{C}^\infty_{c,\alpha}(\mathbb{G})$ has then to be modified by imposing a Dirichlet condition at 1).

□

We end this section with a bibliographic remark:

Remark 10 In their book [2], Berkolaiko and Kuchment investigated a related problem: instead of diffusion operators, they are given Schrödinger operators on each of the edges and they provide a necessary and sufficient condition on the vertex conditions for the self-adjointness of the global operator with respect to the measure which coincides with the Lebesgue measure on each of the edges. The condition at a vertex of degree d consists in d general linear relations involving the values of the functions and of their first derivatives. This is also our case, with $d - 1$ relations coming from the continuity of the function at the vertex and the last relation corresponding to (5). So we could have tried to come back to their framework by first locally stretching the edges to get the corresponding diffusion coefficients to be equal to 1 (see Remark 5) and next by replacing the diffusion operators by Schrödinger operators via ground state transforms. Nevertheless, we preferred to keep working in a Markovian framework, with natural vertex conditions for this setting.

□

3 On Dirichlet forms

The key to Theorems 3 and 6 is the rewriting of their respective frameworks in terms of Dirichlet forms, which enables us to come back to the results of Cheng and Mao [8, 3].

3.1 The continuous case

Indeed, our interest in reversibility in the previous section comes from that it allows for the definition of a (symmetric) Dirichlet form. Another consequence of the proof of Lemma 8 is the computation of the pre-Dirichlet form associated to L. Denote by ν the measure on $\mathbb{G}$ which admits on each R_e, for $e \in E$, with respect to the Lebesgue measure, the density

$$\mathring{R}_e \ni x \quad \mapsto \quad \frac{\beta(e)}{\sum_{e' \in E} \beta(e') \mu_{e'}(R_{e'})} \exp(c_e(x))$$

where β is as in Proposition 1. We have for all $f, g \in \mathcal{C}^\infty_{c,\alpha}(\mathbb{G})$,

$$-\mu[f L[g]] \quad = \quad \nu[\partial[f]\partial[g]]$$

(where ∂ is the first order differentiation operator, say on $\mathbb{G}\backslash V$, coinciding with ∂_e on $\mathring{R}_e$ for all $e \in E$). One can next consider the closure of this form in $\mathbb{L}^2(\mu)$, which can be characterized in the following way. Define for any function f which admits a weak derivative (still denoted $\partial_e f$ or ∂f) on each of the segments $\mathring{R}_e$, with $e \in E$,

$$\mathcal{E}(f) \quad := \quad \int (\partial f)^2 \, d\nu \ \in \ \mathbb{R}_+ \sqcup \{+\infty\}$$

and let $\mathcal{D}(\mathcal{E})$ be the space of functions $f \in \mathbb{L}^2(\mu)$ which are such that $\mathcal{E}(f) < +\infty$. The space $\mathcal{D}(\mathcal{E})$ is Hilbertian, once it is endowed with the norm $\mathcal{D}(\mathcal{E}) \ni f \mapsto \sqrt{\mu[f^2] + \mathcal{E}(f)}$.

The restriction of $\mathcal{E}$ to $\mathcal{D}(\mathcal{E})$ is the Dirichlet form associated to L and it is in fact equivalent to the knowledge of the Freidrich extension of L (for the general theory, see for instance Fukushima, Ōshima and Takeda [6] and Athreya, Eckhoff and Winter [1] for continuous trees extending those obtained through the definition of quantum graphs considered here). In some sense, the vertex conditions given by (5) are encapsulated in μ, especially in its weights $(\beta(e))_{e \in E}$.

For any $k \in \mathbb{N}$, define

$$\lambda_k \quad := \quad \inf_{H \subset \mathcal{D}(\mathcal{E}),\, \dim(H) = k} \ \max_{f \in H \backslash \{0\}} \frac{\mathcal{E}(f)}{\mu(f^2)} \tag{21}$$

where the infimum is over all subspaces of $\mathcal{D}(\mathcal{E})$ of dimension k. These quantities are non-decreasing with respect to $k \in \mathbb{N}$ and they are equal to the eigenvalues of $-L$ up to the "time" they reach the bottom l of the continuous spectrum of $-L$, after what they are all equal to l. This is an easy consequence of the spectral decomposition of the self-adjoint Freidrich extension of L in $\mathbb{L}^2(\mu)$. Alternatively the definition (21) can be replaced by

$$\lambda_k \quad := \quad \sup_{H \subset \mathcal{D}(\mathcal{E}),\, \mathrm{codim}(H) = k-1} \ \inf_{f \in H \backslash \{0\}} \frac{\mathcal{E}(f)}{\mu(f^2)} \tag{22}$$

where the above supremum is taken over all subspaces of $\mathcal{D}(\mathcal{E})$ of co-dimension $k-1$.

The interest of the quantities $(\lambda_k)_{k \in \mathbb{N}}$ is that the operator $L_{\mathcal{F}}^{-1}$ is of trace class if and only if

$$\sum_{k \geqslant 2} \frac{1}{\lambda_k} \quad < \quad +\infty$$

Unfortunately, the domain $\mathcal{D}(\mathcal{E})$ is not so easy to manipulate, because if we are given some functions f_e defined on the edges R_e, for $e \in E$, and satisfying $\int (\partial_e f_e(x))^2 \exp(c_e(x)) dx < +\infty$, in general we cannot put them together to get a function from $\mathcal{D}(E)$. This observation leads us to introduce another Dirichlet form $\mathcal{E}_0$, obtained by imposing Dirichlet conditions on $V\backslash\infty$: since the functions from $\mathcal{D}$ are absolutely continuous, their values are well defined at any point of $\mathbb{G}$ and in particular on $V\backslash\infty$. We can thus consider

$$\mathcal{D}(\mathcal{E}_0) \;:=\; \{f\in\mathcal{D}(\mathcal{E})\,:\,\forall\, v\in V\backslash\infty,\ f(v)=0\} \tag{23}$$

$$\forall\, f\in\mathbb{L}^2(\mu),\qquad \mathcal{E}_0(f) \;=\; \begin{cases}\mathcal{E}(f) & \text{, if } f\in\mathcal{D}(\mathcal{E}_0)\\ +\infty & \text{, otherwise}\end{cases} \tag{24}$$

Similarly to what we have done in (21) and (22), we consider

$$\lambda_{0,k} \;:=\; \inf_{H\subset\mathcal{D}(\mathcal{E}_0),\,\dim(H)=k}\ \max_{f\in H\backslash\{0\}} \frac{\mathcal{E}_0(f)}{\mu(f^2)} \tag{25}$$

$$= \sup_{H\subset\mathcal{D}(\mathcal{E}_0),\,\mathrm{codim}(H)=k-1}\ \inf_{f\in H\backslash\{0\}} \frac{\mathcal{E}_0(f)}{\mu(f^2)} \tag{26}$$

This definition through variational principles leads to immediate comparisons:

Lemma 11 *For all $k\in\mathbb{N}$, we have*

$$\lambda_k \;\leqslant\; \lambda_{0,k} \;\leqslant\; \lambda_{k+n}$$

where n is the cardinal of $V\backslash\infty$.

Proof

The first inequality comes from (21), (25) and the fact that $\mathcal{D}(\mathcal{E}_0)\subset\mathcal{D}(\mathcal{E})$ and that $\mathcal{E}_0$ and $\mathcal{E}$ coincide on $\mathcal{D}(\mathcal{E}_0)$. For the second inequality, consider a subspace $H\subset\mathcal{D}(\mathcal{E}_0)$ of codimension $k-1$. Since $\mathcal{D}(\mathcal{E}_0)$ is of codimension v in $\mathcal{D}(\mathcal{E})$, H is of codimension $n+k-1$ in $\mathcal{D}(\mathcal{E})$. It follows that

$$\begin{aligned}\inf_{f\in H\backslash\{0\}} \frac{\mathcal{E}_0(f)}{\mu(f^2)} &= \inf_{f\in H\backslash\{0\}} \frac{\mathcal{E}(f)}{\mu(f^2)}\\ &\leqslant \sup_{H'\subset\mathcal{D}(\mathcal{E}),\,\mathrm{codim}(H')=n+k-1}\ \inf_{f\in H'\backslash\{0\}} \frac{\mathcal{E}(f)}{\mu(f^2)}\\ &= \lambda_{k+n}\end{aligned}$$

■

To fully exploit the previous bounds, we need a simple observation:

Lemma 12 *We have for any $r\in\mathbb{N}$,*

$$\sum_{k\geqslant 2}\frac{1}{\lambda_k}<+\infty \;\Leftrightarrow\; \sum_{k\geqslant 2+r}\frac{1}{\lambda_k}<+\infty$$

and

$$\sum_{k\geqslant 1}\frac{1}{\lambda_{0,k}}<+\infty \;\Leftrightarrow\; \sum_{k\geqslant 1+r}\frac{1}{\lambda_{0,k}}<+\infty$$

Proof

Of course, only the reverse implications have to be checked. Assume that for some $r\in\mathbb{N}$, $\sum_{k\geqslant 2+r}\frac{1}{\lambda_k}<+\infty$. A first consequence of this finiteness is that the spectrum of L is without continuous part. On the other hand, if $\sum_{k\geqslant 2+r}\frac{1}{\lambda_k}$ is infinite, it means that $\lambda_2=0$. It would imply that 0 is an eigenvalue of multiplicity 2 of L, which is in contradiction with its irreducibility. More precisely, if $f\in\mathbb{L}^2(\mu)$ satisfies $L[f]=0$, then $\mu[fL[f]]=0$, so that $f\in\mathcal{D}(\mathcal{E})$ and $\mathcal{E}(f)=0$. It follows that $\partial f = 0$ ν-a.s. and f is a constant function. The multiplicity of the eigenvalue 0 must thus be 1.

The same argument shows that if $\lambda_{0,1}$ corresponds to an eigenvalue (of the operator L with Dirichlet conditions on $V\backslash\infty$), then it cannot vanish. The previous reasoning then leads to the second equivalence stated in the above lemma.

■

As an immediate consequence of the two previous lemmas, we get:

Corollary 13 *We have*

$$\sum_{k\geqslant 2}\frac{1}{\lambda_k} < +\infty \quad\Leftrightarrow\quad \sum_{k\geqslant 1}\frac{1}{\lambda_{0,k}} < +\infty$$

The advantage of $\mathcal{E}_0$ is it easily related to the Dirichlet forms on the edges with Dirichlet conditions on their boundaries. As above, we begin by considering Neumann boundary conditions. Fix $e\in E$ and denote ν_e the measure on R_e which admits $\exp(c_e)$ as density with respect to the Lebesgue measure. For absolutely continuous functions $f\in\mathbb{L}^2(\mu_e)$ defined on $\mathring{R}_e$, we consider

$$\mathcal{E}^e(f) \quad\coloneqq\quad \int_{R_e}(\partial_e f)^2\,d\nu_e \;\in\; \mathbb{R}_+\sqcup\{+\infty\}$$

and let $\mathcal{D}(\mathcal{E}^e)$ be the space of such functions f with $\mathcal{E}^e(f) < +\infty$. For $k\in\mathbb{N}$, define again

$$\lambda_k^e \quad\coloneqq\quad \inf_{H\subset\mathcal{D}(\mathcal{E}^e),\,\dim(H)=k}\ \max_{f\in H\backslash\{0\}}\frac{\mathcal{E}^e(f)}{\mu_e(f^2)}$$

The corresponding Dirichlet "eigenvalues" are given by

$$\lambda_{0,k}^e \quad\coloneqq\quad \inf_{H\subset\mathcal{D}(\mathcal{E}_0^e),\,\dim(H)=k}\ \max_{f\in H\backslash\{0\}}\frac{\mathcal{E}^e(f)}{\mu_e(f^2)}$$

where if $e\in\{e_1,...,e_N\}$,

$$\mathcal{D}(\mathcal{E}_0^e) \quad\coloneqq\quad \{f\in\mathcal{D}(\mathcal{E}^e)\,:\,f(e^-)=0\}$$

while if $e\in E\backslash\{e_1,...,e_N\}$,

$$\mathcal{D}(\mathcal{E}_0^e) \quad\coloneqq\quad \{f\in\mathcal{D}(\mathcal{E}^e)\,:\,f(e^-)=0=f(e^+)\}$$

As already mentioned, the interest of the previous Dirichlet eigenvalues is that they are in direct relation with the family $(\lambda_{0,k})_{k\in\mathbb{N}}$: the latter is the beginning of the ordering of the multi-set (set with multiplicities) obtained by putting together the sequences $(\lambda_{0,k}^e)_{k\in\mathbb{N}}$, with $e\in E$. This observation is a consequence, on one hand, of the fact that for any $f\in\mathcal{D}(\mathcal{E}_0)$, its restrictions f_e to $\mathring{R}_e$, for $e\in E$, belong respectively to $\mathcal{D}(\mathcal{E}_0^e)$ and that

$$\begin{aligned}\mu[f^2] &= \sum_{e\in E}\frac{\beta(e)}{\sum_{e'\in E}\beta(e')\mu_{e'}(R_{e'})}\mu_e[f_e^2]\\ \mathcal{E}_0(f) &= \sum_{e\in E}\frac{\beta(e)}{\sum_{e'\in E}\beta(e')\mu_{e'}(R_{e'})}\mathcal{E}_0^e(f_e)\end{aligned}$$

And on the other hand, that conversely, starting from a family $(f_e)_{e\in E}\in\prod_{e\in E}\mathcal{D}(\mathcal{E}_0^e)$, by putting them together, we obtain a function $f\in\mathcal{D}(\mathcal{E}_0)$ which is as above.

Note that it may happen that $(\lambda_{0,k})_{k\in\mathbb{N}}$ does not exhaust the whole family $(\lambda_{0,k}^e)_{k\in\mathbb{N},\,e\in E}$ (with multiplicities), this is for instance the case if the operator L_{e_1} (with a Dirichlet condition at e_1^-)

has a continuous spectrum and if there exists $e \in E$ with $\lim_{k\to\infty} \lambda^e_{0,k} = +\infty$ (the latter condition is satisfied as soon as $E \neq \{e_1, ..., e_N\}$).

Nevertheless, it appears that we always have

$$\sum_{k\geqslant 1} \frac{1}{\lambda_{0,k}} \quad = \quad \sum_{e\in E}\sum_{k\geqslant 1} \frac{1}{\lambda^e_{0,k}}$$

because if $(\lambda_{0,k})_{k\in\mathbb{N}}$ does not exhaust the whole family $(\lambda^e_{0,k})_{k\in\mathbb{N},\, e\in E}$, then both terms are infinite. It follows that

$$\sum_{k\geqslant 1} \frac{1}{\lambda_{0,k}} < +\infty \quad\Leftrightarrow\quad \max_{e\in E} \sum_{k\geqslant 1} \frac{1}{\lambda^e_{0,k}} < +\infty$$

We are thus led to consider the sums appearing in the r.h.s. We begin with the compact case:

Lemma 14 *If $e \in E\backslash\{e_1, ..., e_k\}$, we have*

$$\sum_{k\geqslant 1} \frac{1}{\lambda^e_{0,k}} \quad < \quad +\infty$$

Proof

This can be seen as a consequence of Weyl's law on the counting of eigenvalues of the Beltrami Laplacian. For the sake of completeness, let us give an elementary rough argument. Note that there exists a constant $\epsilon(e) \in (0, 1]$ such that $\epsilon(e) \leqslant \nu_e \leqslant 1/\epsilon(e)$ and $\epsilon(e) \leqslant \mu_e \leqslant 1/\epsilon(e)$. It follows that for any $k \in \mathbb{N}$,

$$(\epsilon(e))^2 \widetilde{\lambda}^e_{0,k} \;\leqslant\; \lambda^e_{0,k} \;\leqslant\; (\epsilon(e))^{-2} \widetilde{\lambda}^e_{0,k}$$

where $\widetilde{\lambda}^e_{0,k}$ is the k^{th} eigenvalue of the usual (positive) Laplacian on R_e. It is well-known that

$$\forall\; k \in \mathbb{N}, \qquad \widetilde{\lambda}^e_{0,k} \quad = \quad \left(\frac{\pi k}{e^+ - e^-}\right)^2$$

so that $\sum_{k\geqslant 1} \frac{1}{\lambda^e_{0,k}} < +\infty$.

■

It remains to treat the semi-infinite edges e_1, ..., e_N. For that, recall that with the notation introduced above (11), Cheng and Mao [3] have proven that for all $l \in [\![N]\!]$,

$$\sum_{k\geqslant 1} \frac{1}{\lambda^{e_l}_k} \;<\; +\infty \quad\Leftrightarrow\quad I_l < +\infty$$

As a consequence, we just need to see that

$$\sum_{k\geqslant 1} \frac{1}{\lambda^e_{0,k}} \;<\; +\infty \quad\Leftrightarrow\quad \sum_{k\geqslant 1} \frac{1}{\lambda^e_k} \;<\; +\infty$$

and this can be shown as in Corollary 13.

3.2 The discrete case

This situation is technically simpler than the continuous case and overall the approach is the same, so we won't develop it in details. We simply describe the corresponding Dirichlet forms. Recall that the invariant probability measure μ of the jump generator L defined in (15) is assumed to be reversible (property which can be checked via Kolmogorov's criterion, note that under our hypotheses, there is only a finite number of "simple looping paths"). The associated Dirichlet form is given by

$$\mathcal{E}(f) \quad \coloneqq \quad \sum_{\{\mathfrak{v},\mathfrak{u}\}\in\mathfrak{E}} \mu(\mathfrak{u})L(\mathfrak{u},\mathfrak{v})(f(\mathfrak{v})-f(\mathfrak{u}))^2$$

for any $f \in \mathcal{D}(\mathcal{E})$, which is the space of functions f from $\mathbb{L}^2(\mu)$ such that the above sum is finite.

The definitions (21) and (22) are still equivalent, $\mathcal{D}(\mathcal{E}_0)$, $\mathcal{E}_0$ can be constructed exactly as in (23) and (24) and the $(\lambda_{0,k})_{k\in\mathbb{N}}$ as in (25) and (26).

Lemmas 11 and 12 as well as Corollary 13 are valid verbatim, with the same proofs.

For any $e \in E$, let μ_e be the restriction of μ on S_e, so we differ by a factor from the definition given in (18), if e is one of the e_k, for $k \in [\![N]\!]$. Next we consider the Dirichlet form $\mathcal{E}^e$ given by

$$\mathcal{E}^e(f) \quad \coloneqq \quad \sum_{\{\mathfrak{v},\mathfrak{u}\}\subset S_e} \mu(\mathfrak{u})L(\mathfrak{u},\mathfrak{v})(f(\mathfrak{v})-f(\mathfrak{u}))^2$$

for any $f \in \mathcal{D}(\mathcal{E}^e)$, which is the space of functions from $\mathbb{L}^2(\mu_e)$ such that the above sum is finite. The remaining definitions and arguments are again the same, except that Lemma 14 is now trivial and that the equivalence

$$\sum_{k\geqslant 1} \frac{1}{\lambda_k^e} \;<\; +\infty \quad \Leftrightarrow \quad J_k < +\infty$$

is due to Mao [8].

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