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Modified logarithmic Sobolev inequalities and transportation inequalities

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Abstract. We present a class of modified logarithmic Sobolev inequality, interpolating between Poincaré and logarithmic Sobolev inequalities, suitable for measures of the type $\exp(-|x|^\alpha)$ or $\exp(-|x|^\alpha \log^\beta(2+|x|))$ ($\alpha \in]1,2[$ and $\beta \in \mathbb{R}$) which lead to new concentration inequalities. These modified inequalities share common properties with usual logarithmic Sobolev inequalities, as tensorisation or perturbation, and imply as well Poincaré inequality. We also study the link between these new modified logarithmic Sobolev inequalities and transportation inequalities.

1. Introduction

A probability measure μ on $\mathbb{R}^n$ satisfies a logarithmic Sobolev inequality if there exists $C < \infty$ such that, for every smooth enough functions f on $\mathbb{R}^n$,

$$\mathbf{Ent}_\mu\left(f^2\right) \leqslant C \int |\nabla f|^2 d\mu, \tag{1}$$

where

$$\mathbf{Ent}_\mu\left(f^2\right) = \int f^2 \log f^2 d\mu - \int f^2 d\mu \log \int f^2 d\mu$$

and where $|\nabla f|$ is the Euclidean length of the gradient ∇f of f.

Gross in [18] defines this inequality and shows that the canonical Gaussian measure with density $(2\pi)^{-n/2} e^{-|x|^2/2}$ with respect to the Lebesgue measure on $\mathbb{R}^n$ is the basic example of measure μ satisfying (1) with the optimal constant $C = 2$. Since then, many results have presented measures satisfying such an inequality, among them the famous Bakry-Emery Γ_2 criterion, we refer to Bakry [2] and Ledoux [19] for further references and details on various applications of these inequalities.

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Let $\alpha \geqslant 1$ and define the probability measure μ_α on $\mathbb{R}$ by

$$\mu_\alpha(dx) = \frac{1}{Z_\alpha} e^{-|x|^\alpha} dx, \tag{2}$$

where $Z_\alpha = \int e^{-|x|^\alpha} dx$. It is well-known that the probability measure μ_α satisfies a logarithmic Sobolev inequality (1) if and only if $\alpha \geqslant 2$. But for $\alpha \in [1, 2[$, even if the measure μ_α does not satisfy (1), it satisfies a Poincaré inequality (or spectral gap inequality) which is for every smooth enough function f,

$$\mathbf{Var}_{\mu_\alpha}(f) \leqslant C \int |\nabla f|^2 d\mu_\alpha, \tag{3}$$

where $\mathbf{Var}_{\mu_\alpha}(f) = \int f^2 d\mu_\alpha - \left(\int f d\mu_\alpha\right)^2$ and $C < \infty$.

Recall, see for example Section 1.2.6 of [1], that if a probability measure on $\mathbb{R}^n$ satisfies a logarithmic Sobolev inequality with constant C then it satisfies a Poincaré inequality with a constant no greater than $C/2$.

The problem is then to interpolate between logarithmic Sobolev and Poincaré inequalities, which will help us to study further properties, such as concentration, of measures $\mu_\alpha^{\otimes n}$ for $\alpha \in [1, 2]$ and $n \in \mathbb{N}^*$.

A first answer was brought by Latała-Oleszkiewicz in [21] and recently extended by Barthe-Roberto in [10]. Let μ be a probability measure on $\mathbb{R}^n$, μ satisfies inequality $I_\mu(a)$ (for $a \in [0, 1]$) with constant $0 \leqslant C < \infty$ if for all $p \in [1, 2[$ and f be a measurable, square integrable non-negative function on $\mathbb{R}^n$,

$$\int f^2 d\mu - \left(\int f^p d\mu\right)^{2/p} \leqslant C(2-p)^a \int |\nabla f|^2 d\mu. \tag{4}$$

Inequality (4) was introduced by Beckner in [5], it interpolates Poincaré and logarithmic Sobolev inequalities for the Gaussian measure. In [21], the authors prove that the measure μ_α (for $\alpha \in [1, 2]$) satisfies such an inequality for a constant $C < \infty$ and with $a = 2(\alpha - 1)/\alpha$. And in [10] the authors present a simple proof of Latała-Oleszkiewicz's results and describe measures on the line which enjoy the same inequality.

Our main purpose here will be to establish another type of interpolation between logarithmic Sobolev and Poincaré inequalities, more directly linked to the structure of the usual logarithmic Sobolev inequalities, i.e. an inequality "entropy-energy" where we will modify the energy to enable us to consider μ_α measure. Note that this point of view was the one used by Bobkov-Ledoux in [8] when considering double sided exponential measure. Let us describe further these modified logarithmic Sobolev inequalities.

Let $\alpha \in [1, 2]$, $a > 0$ and β satisfying $1/\alpha + 1/\beta = 1$ ($\beta \geqslant 2$), we note

$$H_{a,\alpha}(x) = \begin{cases} \dfrac{x^2}{2} & \text{if } |x| \leqslant a \\ a^{2-\beta}\dfrac{|x|^\beta}{\beta} + a^2\dfrac{\beta-2}{2\beta} & \text{if } |x| \geqslant a \text{ and } \alpha \neq 1 \\ +\infty & \text{if } |x| \geqslant a \text{ and } \alpha = 1. \end{cases}$$

In Section 2 we give definition and general properties of the following inequality

$$\mathbf{Ent}_\mu\Big(f^2\Big) \leqslant C \int H_{a,\alpha}\Big(\frac{\nabla f}{f}\Big) f^2 d\mu. \qquad (LSI_{a,\alpha}(C))$$

In particular we prove that inequality $LSI_{a,\alpha}$ satisfies some of the properties shared by Poincaré or Gross logarithmic Sobolev inequalities ((1) or (3)), namely tensorisation and perturbation. Note that in the case $\alpha = 1$, we find inequalities used by Bobkov-Ledoux in [8] and for $\alpha = 2$ inequality $LSI_{a,\alpha}(C)$ is exactly the Gross logarithmic Sobolev inequality.

We present also a concentration property which is adapted to this inequality. More precisely, if a measure μ satisfies the inequality $LSI_{a,\alpha}(C)$, we have that if f is a Lipschitz function on $\mathbb{R}$ with $\|f\|_{Lip} \leqslant 1$ then, there is $B > 0$ such that for every $\lambda > 0$ one has

$$\mu_\alpha\Big(f - \int f d\mu_\alpha \geqslant \lambda\Big) \leqslant \exp\Big(-B \min\Big(\lambda^\alpha, \lambda^2\Big)\Big). \qquad (5)$$

This inequality was proved by Talagrand in [26, 27] and also described by Maurey with the so called property (τ) in [23], Bobkov-Ledoux in [8] study the particular case ($\alpha = 1$). Let us note that the cases $\alpha \geqslant 2$ are studied by Bobkov-Ledoux in [9], relying mainly on Brunn-Minkowski inequalities, and by Bobkov-Zegarlinski in [12] which refine the results presenting, via Hardy's inequality, some necessary and sufficient condition for measures on the real line. Let us note to finish that they use, for the case $\alpha \geqslant 2$, $H_\beta(x) = |x|^\beta$ with $1/\alpha + 1/\beta = 1$.

In Section 2.2, we extend Otto-Villani's theorem (see [25]) for the relation with logarithmic Sobolev inequality and transportation inequality. Let us define $L_{a,\alpha}$ by $L_{a,\alpha} = H^*_{a,\alpha}$, the Fenchel-Legendre transform of $H_{a,\alpha}$. We prove that if a probability measure μ on $\mathbb{R}^n$ satisfies the inequality $LSI_{a,\alpha}(C)$ then there are $a' > 0$ and $D > 0$ such that it satisfies also a transportation inequality: for all function F on $\mathbb{R}^n$, density of probability with respect to μ,

$$T_{L_{a',\alpha}}(F d\mu, d\mu) \leqslant D\mathbf{Ent}_\mu(F)\,, \qquad (T_{a',\alpha}(D))$$

where

$$T_{L_{a',\alpha}}(F d\mu, d\mu) = \inf\left\{\iint L_{a',\alpha}(x-y) d\pi(x,y)\right\},$$

where the infimum is taken over the set of probabilities measures π on $\mathbb{R}^n \times \mathbb{R}^n$ such that π has two margins $F d\mu$ and $d\mu$. This inequality was introduced by Talagrand in [28] for the case $\alpha = 2$ and $\alpha = 1$. Let us note that the case $\alpha = 1$ was also studied in [7] with exactly this form and the case $\alpha \geqslant 2$ was studied in [16].

In Section 3 we prove, as in [21], that the measure μ_α defined in (2) satisfies the inequality $LSI_{a,\alpha}(C)$. More precisely we prove that there is $A, B > 0$ such that μ_α satisfies for all smooth function such that $f \geqslant 0$ and $\int f^2 d\mu_\alpha = 1$,

$$\mathbf{Ent}_{\mu_\alpha}\Big(f^2\Big) \leqslant A\mathbf{Var}_{\mu_\alpha}(f) + B \int_{f \geqslant 2} \left|\frac{f'}{f}\right|^\beta f^2 d\mu_\alpha.$$

Due to the fact that μ_α enjoys Poincaré inequality, μ_α satisfies also inequality $LSI_{a,\alpha}(C)$ for some constants $C > 0$ and $a > 0$.

Our method relies crucially on Hardy's inequality we recall now: let μ, ν be two finite Borel measures on $\mathbb{R}^+$. Then the best constant A so that every smooth function f satisfies

$$\int_0^\infty (f(x) - f(0))^2 d\mu(x) \leqslant A \int_0^\infty f'^2 d\nu \tag{6}$$

is finite if and only if

$$B = \sup_{x \geqslant 0} \mu([x, \infty[) \int_0^x \left(\frac{d\nu^{ac}}{dt} \right)^{-1} dt \tag{7}$$

is finite, where ν^{ac} is the absolutely continuous part of ν with respect to μ. Moreover, when A is finite we have

$$B \leqslant A \leqslant 4B.$$

This inequality was proved by Muckenhoupt [24], one can see also [1, 10] for interesting review and application of this result.

Finally in Section 4 we will present some inequalities satisfied by other measures. More precisely, let φ be twice continuously differentiable and note the probability measure μ_φ by,

$$\mu_\varphi(dx) = \frac{1}{Z} e^{-\varphi(x)} dx. \tag{8}$$

Among them is considered

$$\varphi(x) = |x|^\alpha (\log(2 + |x|))^\beta, \text{ with } \alpha \in]1, 2[, \ \beta \in \mathbb{R},$$

which exhibits a modified logarithmic Sobolev inequality of function H (different in nature from $H_{a,\alpha}$), and which is not covered by Latała-Oleskiewickz inequality. We also present examples which are unbounded perturbation of μ_α. We then derive new concentration inequalities in the spirit of Talagrand and Maurey or Bobkov-Ledoux

Let us finally comment the case of general convex potential φ and $\mu(dx) = e^{-\varphi(x)} dx$. The natural extension of our modified logarithmic Sobolev inequality would be to consider a function H behaving quadratically near the origin and like φ^* (the Legendre transform of φ) for large values. The extension is however by no way trivial and requires appropriate technique currently under study.

2. Modified logarithmic Sobolev inequalities: definition and general properties

2.1. Definitions and classical properties

Let $\alpha \in [1, 2]$ and $\beta \geqslant 2$ satisfying $1/\alpha + 1/\beta = 1$ and let $a > 0$. Let define the functions $L_{a,\alpha}$ and $H_{a,\alpha}$.

If $\alpha \in]1,2]$ we note

$$L_{a,\alpha}(x) = \begin{cases} \dfrac{x^2}{2} & \text{if } |x| \leqslant a \\ a^{2-\alpha}\dfrac{|x|^\alpha}{\alpha} + a^2\dfrac{\alpha-2}{2\alpha} & \text{if } |x| \geqslant a \end{cases}$$

and

$$H_{a,\alpha}(x) = \begin{cases} \dfrac{x^2}{2} & \text{if } |x| \leqslant a \\ a^{2-\beta}\dfrac{|x|^\beta}{\beta} + a^2\dfrac{\beta-2}{2\beta} & \text{if } |x| \geqslant a \end{cases}$$

If $\alpha = 1$ we note

$$L_{a,1}(x) = \begin{cases} \dfrac{x^2}{2} & \text{if } |x| \leqslant a \\ a|x| - \dfrac{a^2}{2} & \text{if } |x| \geqslant a \end{cases} \text{ and } H_{a,1}(x) = \begin{cases} \dfrac{x^2}{2} & \text{if } |x| \leqslant a \\ \infty & \text{if } |x| > a \end{cases}$$

Let $n \in \mathbb{N}^*$ and $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, we note

$$L^{(n)}_{a,\alpha}(x) = \sum_{i=1}^n L_{a,\alpha}(x_i) \text{ and } H^{(n)}_{a,\alpha}(x) = \sum_{i=1}^n H_{a,\alpha}(x_i).$$

Note that when there is no ambiguity we will drop the dependence in n and note $L_{a,\alpha}$ instead of $L^{(n)}_{a,\alpha}$.

Let us define the logarithmic Sobolev inequality of function $H_{a,\alpha}$.

Definition 2.1. *Let μ be a probability measure on $\mathbb{R}^n$, μ satisfies a logarithmic Sobolev inequality of function $H_{a,\alpha}$ with constant C, noted $LSI_{a,\alpha}(C)$, if for every C^1 function $f > 0$ such that every integrals exists one has*

$$\mathbf{Ent}_\mu\left(f^2\right) \leqslant C \int H_{a,\alpha}\left(\frac{\nabla f}{f}\right) f^2 d\mu, \qquad (LSI_{a,\alpha}(C))$$

where

$$H_{a,\alpha}\left(\frac{\nabla f}{f}\right) = \sum_{i=1}^n H_{a,\alpha}\left(\frac{\partial f}{\partial x_i}\frac{1}{f}\right).$$

We detail some properties of $L_{a,\alpha}$ and $H_{a,\alpha}$ in the following lemma.

Lemma 2.2. *Functions $L_{a,\alpha}$ and $H_{a,\alpha}$ satisfies:*

i: If $\alpha \in]1,2]$, $L_{a,\alpha}$ and $H_{a,\alpha}$ are C^1 on $\mathbb{R}$.
ii: $L^*_{a,\alpha} = H_{a,\alpha}$, where $L^*_{a,\alpha}$ is the Fenchel-Legendre transform of $L_{a,\alpha}$. Of course we have also $H^*_{a,\alpha} = L_{a,\alpha}$.
iii: For all $t > 0$ one has for all $x \in \mathbb{R}$

$$L_{a,\alpha}(tx) = t^2 L_{\frac{a}{t},\alpha}(x), \quad H_{a,\alpha}(tx) = t^2 H_{\frac{a}{t},\alpha}(x).$$

iv: Let $0 \leqslant a \leqslant a'$*, one has for all* $x \in \mathbb{R}$

$$L_{a,\alpha}(x) \leqslant L_{a',\alpha}(x), \quad H_{a',\alpha}(x) \leqslant H_{a,\alpha}(x).$$

v: If $\alpha \in]1,2]$, $L_{a,\alpha}$ *and* $H_{a,\alpha}$ *are strictly convex and satisfies*

$$\lim_{|x|\to\infty} \frac{H_{a,\alpha}(x)}{x} = \lim_{|x|\to\infty} \frac{L_{a,\alpha}(x)}{x} = \infty.$$

The assumptions given on α and β are significant only for condition iv, and condition v is significant for Brenier-McCann-Gangbo's theorem, which is crucial for the study of the link between modified logarithmic Sobolev inequalities and transportation inequalities of the next section.

Here are some properties of the inequality $LSI_{a,\alpha}(C)$.

Proposition 2.3. *1. This property is known under the name of tensorisation.*
Let μ_1 *and* μ_2 *two probability measures on* $\mathbb{R}^{n_1}$ *and* $\mathbb{R}^{n_2}$*. Suppose that* μ_1 *(resp.* μ_2*) satisfies the inequality* $LSI_{a,\alpha}(C_1)$ *(resp.* $LSI_{a,\alpha}(C_2)$*) then the probability* $\mu_1 \otimes \mu_2$ *on* $\mathbb{R}^{n_1+n_2}$*, satisfies inequality* $LSI_{a,\alpha}(D)$*, where* $D = \max\{C_1, C_2\}$*.*

2. This property is known under the name of perturbation.
Let μ *a measure on* $\mathbb{R}^n$ *satisfying* $LSI_{a,\alpha}(C)$*. Let h a bounded function on* $\mathbb{R}^n$ *and defined* $\tilde{\mu}$ *as*

$$d\tilde{\mu} = \frac{e^h}{Z} d\mu,$$

where $Z = \int e^h d\mu$*.*
Then the measure $\tilde{\mu}$ *satisfies the inequality* $LSI_{a,\alpha}(D)$ *with* $D = Ce^{2osc(h)}$*, where* $osc(h) = \sup(h) - \inf(h)$*.*

3. Link between $LSI_{a,\alpha}(C)$ *inequality with Poincaré inequality.*
Let μ *a measure on* $\mathbb{R}^n$*. If* μ *satisfies* $LSI_{a,\alpha}(C)$*, then* μ *satisfies a Poincaré inequality with the constant* $C/2$*. Let us recall that* μ *satisfies a Poincaré inequality with constant* $C/2$ *if*

$$\mathbf{Var}_\mu(f) \leqslant \frac{C}{2} \int |\nabla f|^2 d\mu, \tag{9}$$

for all smooth function f*.*

Proof.
$\triangleleft$ One can find the details of the proof of the properties of tensorisation and perturbation and the implication of the Poincaré inequality in chapters 1 and 3 of [1] (Section 1.2.6., Theorem 3.2.1 and Theorem 3.4.3). $\triangleright$

Note that the tensorisation of the Entropy is well known property discussed by Lieb in [20].

Remark 2.4. We may of course define logarithmic Sobolev inequality of function H, where $H(x)$ is quadratic for small values of $|x|$ and with convex, faster than quadratic, growth for large $|x|$. See Section 4 for such examples. Note that Proposition 2.3 is of course still valid for this kind of inequality. These inequalities are also studied in a general case in [19] in Proposition 2.9.

As in [21, 8], by using Herbst's argument, one can give precise estimates about concentration.

Proposition 2.5. *Assume that the probability measure μ on $\mathbb{R}$ satisfies the inequality $LSI_{a,\alpha}(C)$. Let F be a function on $\mathbb{R}^n$ such that $\forall i$, $\|\partial_i F\| \leq \zeta$, then we get for $\lambda \geqslant 0$,*

$$\mu^{\otimes n}(\left|F - \mu^{\otimes n}(F)\right| \geqslant \lambda) \leqslant \begin{cases} 2\exp\left(-\frac{K_\alpha}{n^{\alpha-1}\zeta^\alpha}(\lambda - aCn\zeta(2-\alpha))^\alpha - a^2\frac{2-\alpha}{2\alpha}\right) & \textit{if } \lambda \geqslant \frac{aCn\zeta}{2}, \\ 2\exp\left(-\frac{2\lambda^2}{nC\zeta^2}\right) & \textit{otherwise,} \end{cases}$$

where $K_\alpha = \dfrac{2^\alpha(\alpha-1)^{1-\alpha}a^{2-\alpha}}{\alpha C^{\alpha-1}}$.

Proof.
◁ Let us first present the proof when $n = 1$. Assume, without loss of generality, that $\int F d\mu = 0$. Let us recall briefly Herbst's argument (see [1] for more details). Denote $\Phi(t) = \int e^{tF} d\mu$, and remark that $LSI_{a,\alpha}(C)$ applied to $f^2 = e^{tF}$, using basic properties of $H_{a,\alpha}$, yields to

$$t\Phi'(t) - \Phi(t)\log\Phi(t) \leq CH_{a,\alpha}\left(\frac{t\zeta}{2}\right)\Phi(t) \tag{10}$$

which, denoting $K(t) = (1/t)\log\Phi(t)$, entails

$$K'(t) \leq \frac{C}{t^2}H_{a,\alpha}\left(\frac{t\zeta}{2}\right).$$

Then, integrating, and using $K(0) = \int F d\mu = 0$, we obtain

$$\Phi(t) \leq \exp\left(Ct\int_0^t \frac{1}{s^2}H_{a,\alpha}\left(\frac{s\zeta}{2}\right)ds\right). \tag{11}$$

The Laplace transform of F is then bounded by

$$\Phi(t) \leqslant \begin{cases} \exp\left(Ct^\beta\zeta^\beta\frac{a^{2-\beta}}{2^\beta\beta(\beta-1)} + Cta\zeta\frac{\beta-2}{2(\beta-1)} - Ca^2\frac{\beta-2}{2\beta}\right) & \text{if } t \geqslant \frac{2a}{\zeta}, \\ \exp\left(C\frac{\zeta^2t^2}{8}\right) & \text{if } 0 \leqslant t \leqslant \frac{2a}{\zeta}. \end{cases}$$

For the n-dimensional extension, use the tensorisation property of $LSI_{a,\alpha}$ and

$$\sum_{i=1}^n H_{a,\alpha}\left(\frac{t}{2}\partial_i F\right) \leqslant nH_{a,\alpha}\left(\frac{t\zeta}{2}\right).$$

Then we can use the case of dimension 1 with the constant C replaced by Cn. ▷

Remark 2.6. Let us present a simple application of the preceding Proposition to deviation inequality of the empirical mean of a function. Consider the real valued function f, with $|f'| \leq 1$, and $F(x_1, \dots, x_n) = \frac{1}{n}\sum_{k=1}^n f(x_k)$, which inherits the property that $|\partial_i F| \leq 1/n$, we thus get the following Hoeffding type inequality, $(X_i)_{1\leq i\leq n}$ being independent and identically distributed according to μ_α,

$$\mathbb{P}\left(\frac{1}{n}\left|\sum_{k=1}^n f(X_k) - \mu_\alpha(f)\right| > \lambda\right) \leqslant \begin{cases} 2\exp\left(-nK_\alpha(\lambda - aC(2-\alpha))^\alpha - a^2\dfrac{2-\alpha}{2\alpha}\right) & \text{if } \lambda \geqslant \frac{aC}{2}, \\ 2\exp\left(-n\dfrac{2\lambda^2}{C}\right) & \text{otherwise.} \end{cases}$$

Remark 2.7. Note that the obtained form $\min(\lambda^\alpha, \lambda^2)$ is natural in regard to Gaussian approximation. Indeed, consider, for example, $F(x_1, ..., x_n) = \frac{1}{\sqrt{n}}\sum_{k=1}^n f(x_i)$ where $|f'| \leqslant 1$ we have $|\partial_i F| \leqslant n^{-1/2}$ which enables us to recover the Gaussian concentration

$$\mathbb{P}\left(\frac{1}{\sqrt{n}}\left|\sum_{k=1}^n f(X_k) - \mu_\alpha(f)\right| > \lambda\right) \leq e^{-2\lambda^2/C},$$

for all $n \geqslant 4\lambda^2/(ac)^2$.

Remark 2.8. For general logarithmic Sobolev of function H, we may obtain crude estimation of the concentration, at least for large λ. Indeed, using inequality (11), we have directly that the concentration behavior is given by the Fenchel-Legendre transform of H for large values, see Section 4 for more details.

2.2. Link between inequality $LSI_{a,\alpha}(C)$ and transportation inequality

Definition 2.9. *Let μ be a probability measure on $\mathbb{R}^n$, μ satisfies a transportation inequality of function $L_{a,\alpha}$ with constant C, noted $T_{a,\alpha}(C)$, if for every function F, density of probability with respect to μ, one has*

$$T_{L_{a,\alpha}}(Fd\mu, d\mu) \leqslant C\mathbf{Ent}_\mu(F)\,, \qquad (T_{a,\alpha}(C))$$

where

$$T_{L_{a,\alpha}}(Fd\mu, \mu) = \inf\left\{\int L_{a,\alpha}(x-y)d\pi(x,y)\right\},$$

where the infimum is taken over the set of probabilities measures π on $\mathbb{R}^n \times \mathbb{R}^n$ such that π has two margins $Fd\mu$ and μ.

Otto and Villani proved that a logarithmic Sobolev inequality implies a transportation inequality with a quadratic cost (this is the case $\alpha = \beta = 2$), see [25, 7]. They prove that if μ satisfies the inequality $LSI_{\cdot,2}(C)$, (when $\alpha = 2$ the constant a is not any more a parameter in this case), then μ satisfies the inequality $T_{\cdot,2}(4C)$. In [7] another case is studied, when $\alpha = 1$ and $\beta = \infty$. In this first theorem we give an extension for the other cases, where $\alpha \in [1, 2]$.

Theorem 2.10. *Let μ be a probability measure on $\mathbb{R}^n$ and suppose that μ satisfies the inequality $LSI_{a,\alpha}(C)$.*

Then μ satisfies the transportation inequality $T_{\frac{aC}{2},\alpha}(C/4)$.

Proof.

$\lhd$ As in [7], we use Hamilton-Jacobi equations. Let f be a Lipschitz bounded function on $\mathbb{R}^n$, and set

$$Q_t f(x) = \inf_{y\in\mathbb{R}} \left\{ f(y) + tL_{\frac{aC}{2},\alpha}\left(\frac{x-y}{t}\right)\right\}, \ t > 0, \ x \in \mathbb{R}^n, \qquad (12)$$

and $Q_0 f = f$. The function $Q_t f$ is known as the Hopf-Lax solution of the Hamilton-Jacobi equation

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) = H_{\frac{aC}{2},\alpha}(\nabla v)(t,x), \ t > 0, \ x \in \mathbb{R}^n, \\ v(0,x) = f(x), \ x \in \mathbb{R}^n, \end{cases}$$

see for example [3, 15].

For $t \geqslant 0$, define the function ψ by

$$\psi(t) = \int e^{\frac{4t}{C} Q_t f} d\mu.$$

Since f is Lipschitz and bounded function one can prove that $Q_t f$ is also a Lipschitz and bounded function on t for almost every $x \in \mathbb{R}^n$, then ψ is a C^1 function on $\mathbb{R}^+$. One gets

$$\begin{aligned}\psi'(t) &= \int \frac{4}{C} Q_t f e^{\frac{4t}{C} Q_t f} d\mu - \int \frac{4t}{C} H_{\frac{aC}{2},\alpha}(\nabla Q_t f) e^{\frac{4t}{C} Q_t f} d\mu \\ &= \frac{1}{t}\mathbf{Ent}_\mu\left(e^{\frac{4t}{C} Q_t f}\right) + \frac{1}{t}\psi(t)\log\psi(t) - \int \frac{4t}{C} H_{\frac{aC}{2},\alpha}(\nabla Q_t f) e^{\frac{4t}{C} Q_t f} d\mu\end{aligned}$$

Let use inequality $LSI_{a,\alpha}(C)$ to the function $\exp\left(\frac{2t}{C} Q_t f\right)$ to get

$$\begin{aligned}\psi'(t) \leqslant \frac{1}{t}\psi(t)\log\psi(t) + \frac{C}{t}\Bigg(\int H_{a,\alpha}\left(\frac{2t}{C}\nabla Q_t f\right) e^{\frac{4t}{C} Q_t f} d\mu \\ - \int \frac{4t^2}{C^2} H_{\frac{aC}{2},\alpha}(\nabla Q_t f) e^{\frac{4t}{C} Q_t f} d\mu\Bigg).\end{aligned}$$

Due to the property of $H_{a,\alpha}$ (see Lemma 2.2),

$$H_{a,\alpha}\left(\frac{2t}{C}\nabla Q_t f\right) = \frac{4t^2}{C^2} H_{\frac{aC}{2t},\alpha}(\nabla Q_t f).$$

Then for all $t \in [0, 1]$, one has

$$H_{a,\alpha}\left(\frac{2t}{C}\nabla Q_t f\right) \leqslant \frac{4t^2}{C^2} H_{\frac{aC}{2},\alpha}(\nabla Q_t f).$$

Then

$$\forall t \in [0,1], \quad t\psi'(t) - \psi(t)\log\psi(t) \leqslant 0$$

After integration on [0, 1], we have

$$\psi(1) \leqslant \exp\frac{\psi'(0)}{\psi(0)},$$

from where

$$\int e^{\frac{4}{C}Q_1 f} d\mu \leqslant e^{\int \frac{4}{C} f d\mu}. \tag{13}$$

Since

$$\mathbf{Ent}_\mu(F) = \sup\left\{\int Fg d\mu, \ \int e^g d\mu \leqslant 1\right\},$$

we have with $g = \frac{4}{C}Q_1 f - \int \frac{4}{C} f d\mu$,

$$\int F\Big(Q_1 f - \int f d\mu\Big) d\mu \leqslant \frac{C}{4}\mathbf{Ent}_\mu(F).$$

Let take the supremum on the set of Lipschitz function f, the Kantorovich-Rubinstein's theorem applied to the distance $T_{L_{a,\alpha}}(Fd\mu, d\mu)$, see [29], implies that

$$T_{L_{\frac{aC}{2},\alpha}}(Fd\mu, d\mu) \leqslant \frac{C}{4}\mathbf{Ent}_\mu(F).$$

▷

As it is also the case in quadratic case, when the measure is log-concave one can prove that a transportation inequality implies a logarithmic Sobolev inequality.

For the next theorem we suppose that the function of transport given by the theorem of Brenier-Gangbo-McCann is a $\mathcal{C}^2$ function. Such a regularity result is outside the scope of this paper and we refer to Villani [29] for further discussions around this problem. However we show here, that once this result assumed, the methodology presented in Bobkov-Gentil-Ledoux [7], for the exponential measure, still works.

Theorem 2.11. *Let μ be a probability measure on $\mathbb{R}^n$. Assume that*

$$\mu(dx) = e^{-\varphi(x)}dx$$

where φ is a convex function on $\mathbb{R}^n$.

If μ satisfies the inequality $T_{a,\alpha}(C)$ then for all $\lambda > C$, μ satisfies the logarithmic Sobolev inequality $LSI_{\frac{a}{2\lambda},\alpha}\left(\frac{4\lambda^2}{\lambda - C}\right)$.

Proof.
◁ Let note F density of probability with respect to μ. Assume that F is $\mathcal{C}^2$, the general case can result by density.

By the Brenier-Gangbo-McCann's theorem, see [11, 17], there exists a function Φ such that

$$S = \mathrm{Id} - \nabla H_{a,\alpha} \circ \nabla \Phi,$$

transports $F d\mu$ to the measure μ, for every measurable bounded function g

$$\int g(S) F d\mu = \int g d\mu.$$

The function Φ is a $L_{a,\alpha}$-concave function and if Φ is $\mathcal{C}^2$, a classical argument of convexity (see chapter 2 of [29]), one has $D\left[\nabla H_{a,\alpha} \circ \nabla \Phi(x)\right]$ is diagonalizable with real eigenvalues, all less than 1.

According to the assumption made on function Φ, one can assume that S is sufficiently smooth and we obtain for $x \in \mathbb{R}^n$,

$$F(x) e^{-\varphi(x)} = e^{-\varphi \circ S(x)} \det\left(\nabla S(x)\right). \tag{14}$$

Moreover this function gives the optimal transport, i.e.

$$T_{L_{a,\alpha}}(F d\mu, d\mu) = \int L_{a,\alpha}\big(\nabla H_{a,\alpha} \circ \nabla \Phi\big) F d\mu.$$

Then by (14), one has for $x \in \mathbb{R}^n$,

$$\log F(x) = \varphi(x) - \varphi(x - \nabla H_{a,\alpha} \circ \nabla \Phi(x)) + \log\det\Big(\mathrm{Id} - D\left[\nabla H_{a,\alpha} \circ \nabla \Phi(x)\right]\Big).$$

Then since $D\left[\nabla H_{a,\alpha} \circ \nabla \Phi(x)\right]$ is diagonalizable with real eigenvalues, all less than 1, we get

$$\log\det\Big(\mathrm{Id} - D\left[\nabla H_{a,\alpha} \circ \nabla \Phi(x)\right]\Big) \leqslant -\mathrm{div}\big(\nabla H_{a,\alpha} \circ \nabla \Phi(x)\big).$$

Since φ is convex we have $\varphi(x) - \varphi(x - \nabla H_{a,\alpha} \circ \nabla \Phi(x)) \leqslant \nabla H_{a,\alpha} \circ \nabla \Phi(x) \cdot \nabla \varphi(x)$ and we obtain

$$\mathbf{Ent}_\mu(F) \leqslant \int \left\{\nabla H_{a,\alpha} \circ \nabla \Phi(x) \cdot \nabla \varphi(x) - \mathrm{div}\left(\nabla H_{a,\alpha} \circ \nabla \Phi(x)\right)\right\} F(x) d\mu(x),$$

after integration by parts

$$\mathbf{Ent}_\mu(F) \leqslant \int \nabla F \cdot \nabla H_{a,\alpha} \circ \nabla \Phi d\mu.$$

Let $\lambda > 0$ and let use Young inequality for the combined functions $L_{a,\alpha}$ and $H_{a,\alpha}$

$$\lambda \frac{\nabla F}{F} \cdot \nabla H_{a,\alpha} \circ \nabla \Phi \leqslant H_{a,\alpha}\left(\lambda \frac{\nabla F}{F}\right) + L_{a,\alpha}\big(\nabla H_{a,\alpha} \circ \nabla \Phi\big).$$

Thus

$$\begin{aligned} \mathbf{Ent}_\mu(F) &\leqslant \frac{1}{\lambda} \int H_{a,\alpha}\left(\lambda \frac{\nabla F}{F}\right) F d\mu + \frac{1}{\lambda} \int L_{a,\alpha}\big(\nabla H_{a,\alpha} \circ \nabla \Phi\big) F d\mu \\ &\leqslant \lambda \int H_{\frac{a}{\lambda},\alpha}\left(\frac{\nabla F}{F}\right) F d\mu + \frac{1}{\lambda} T_{L_{a,\alpha}}(F d\mu, d\mu). \end{aligned}$$

Thus if μ satisfies the inequality $T_{a,\alpha}(C)$ we get for all $\lambda > C$

$$\mathbf{Ent}_\mu(F) \leqslant \frac{\lambda^2}{\lambda - C} \int H_{\frac{a}{\lambda},\alpha}\left(\frac{\nabla F}{F}\right) F d\mu.$$

Let us note now $f^2 = F$, we get

$$\begin{aligned}\mathbf{Ent}_\mu\left(f^2\right) &\leqslant \frac{\lambda^2}{\lambda - C} \int H_{\frac{a}{\lambda},\alpha}\left(2\frac{\nabla f}{f}\right) f^2 d\mu \\ &\leqslant \frac{4\lambda^2}{\lambda - C} \int H_{\frac{a}{2\lambda},\alpha}\left(\frac{\nabla f}{f}\right) f^2 d\mu.\end{aligned}$$

Then μ satisfies, for all $\lambda > C$ inequality $LSI_{\frac{a}{2\lambda},\alpha}\left(\frac{4\lambda^2}{\lambda - C}\right)$. ▷

Remark 2.12. One can summarizes Theorem 2.10 and 2.11 by the following diagram (under assumption of Theorem 2.11):

$$\begin{aligned} LSI_{a,\alpha}(C) &\to T_{\frac{aC}{2},\alpha}(C/4) \\ T_{a,\alpha}(C) &\to \left\{ LSI_{\frac{a}{2\lambda},\alpha}\left(\frac{4\lambda^2}{\lambda - C}\right)\right\}_{\lambda > C}. \end{aligned}$$

Notice, as it is the case for the traditional logarithmic Sobolev inequality, that there is a loss at the level of the constants in the direction transportation inequality implies logarithmic Sobolev inequality. When $\alpha = \beta = 2$, we get as in [25], $T_{\cdot,2}(C) \to LSI_{\cdot,2}(16C)$. As in [25], Theorem 2.11 can be modified in the case $\mathrm{Hess}(\varphi) \geqslant \lambda \mathrm{Id}$, where $\lambda \in \mathbb{R}$.

Also let us notice that as in the quadratic case we do not know if these two inequalities are equivalent.

As in Proposition 2.3, here are some properties of the inequality $T_{L_{a,\alpha}}(C)$.

Proposition 2.13. *1. Concentration inequality.*
Assume that μ satisfies a transportation inequality $T_{L_{a,\alpha}}(C)$ then μ satisfies the following concentration inequality

$$\forall A \subset R^n, \quad with \quad \mu(A) \geqslant \frac{1}{2}, \quad \mu((A_r)^c) \leqslant 2e^{\left(-\frac{1}{C}L_{a,\alpha}(r)\right)},$$

where $(A_r)^c = \{x \in \mathbb{R}^n,\ d(A,x) \geqslant r\}$.

2. *As in Proposition 2.3, the properties of tensorisation are also valid for transportation inequality $T_{a,\alpha}(C)$.*
Let μ_1 and μ_2 be two probability measures on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$. Suppose that μ_1 (resp. μ_2) satisfies the inequality $T_{a,\alpha}(C_1)$ (resp. $T_{a,\alpha}(C_2)$) then the probability $\mu_1 \otimes \mu_2$ on $\mathbb{R}^{n_1+n_2}$, satisfies inequality $T_{a,\alpha}(D)$, where $D = \max\{C_1, C_2\}$.
3. *If the measure μ verifies $T_{a,\alpha}(C)$, then μ satisfies a Poincaré inequality* (9) *with the constant C.*

Proof.
◁ The demonstration of *1*, *2* of these results is a simple adaptation of the traditional case introduce by Marton in [22], We return to the references for proofs (for example chapters 3, 7 and 8 of [1]).

The proof of *3* is an adaptation of the quadratic case. Suppose that μ satisfies a $T_{a,\alpha}(C)$. By a classical argument of Bobkov-Götze, the measure μ satisfies the dual form of $T_{a,\alpha}(C)$ which is the inequality (13),

$$\int e^{\frac{1}{C}Q_1 f} d\mu \leqslant e^{\int \frac{1}{C} f d\mu}, \tag{15}$$

where $Q_1 f$ is defined as in (12) with the function $L_{a,\alpha}$.

Let note $f = \epsilon g$ with $g, \mathcal{C}^1$, bounded and with ∇g also bounded, we get

$$\begin{aligned} Q_1 f(x) = Q_1(\epsilon g)(x) &= \epsilon \inf_{z\in\mathbb{R}^n} \left\{ g(x-\epsilon z) + \epsilon L_{\frac{a}{\epsilon},\alpha}(z) \right\} \\ &= \epsilon g(x) - \frac{\epsilon^2}{2}|\nabla g|^2 + o(\epsilon^2) \end{aligned}$$

Then we obtain by (15),

$$\begin{aligned} &1 + \frac{\epsilon}{C}\int g d\mu - \frac{\epsilon^2}{2C}\int |\nabla g|^2 d\mu + \frac{\epsilon^2}{2C^2}\int g^2 d\mu + o(\epsilon^2) \\ &\leqslant 1 + \frac{\epsilon}{C}\int g d\mu + \frac{\epsilon^2}{2C^2}\left(\int g d\mu\right)^2 + o(\epsilon^2), \end{aligned}$$

imply that

$$\mathbf{Var}_\mu(g) \leqslant C \int |\nabla g|^2 d\mu.$$

▷

Unfortunately, as in the traditional case of the transportation inequality, we do not know if this one has property of perturbation as for inequality $LSI_{a,\alpha}(C)$ (Proposition 2.3).

3. An important example on $\mathbb{R}$, the measure μ_α

Let $\alpha \geqslant 1$ and define the probability measure μ_α on $\mathbb{R}$ by

$$\mu_\alpha(dx) = \frac{1}{Z_\alpha} e^{-|x|^\alpha} dx,$$

where $Z_\alpha = \int e^{-|x|^\alpha} dx$.

For Section 3 and 4 we will note by *smooth function* a locally absolutely continuous function on $\mathbb{R}$.

Theorem 3.1. *Let* $\alpha \in]1,2]$. *There exists* $0 \leqslant A, B < \infty$ *such that the measure* μ_α *satisfies the following modified logarithmic Sobolev inequality, for any smooth function* f *on* $\mathbb{R}$ *such that* $f \geqslant 0$ *and* $\int f^2 d\mu_\alpha = 1$ *we have*

$$\mathbf{Ent}_{\mu_\alpha}\left(f^2\right) \leqslant A\mathbf{Var}_{\mu_\alpha}(f) + B\int_{f\geqslant 2}\left|\frac{f'}{f}\right|^\beta f^2 d\mu_\alpha, \tag{16}$$

where $1/\alpha + 1/\beta = 1$.

In the extreme case, $\alpha = 1$, *there exists* $\lambda > 0$ *and* $0 \leqslant A' < \infty$ *such that we obtain the following inequality: for all* f *smooth such that* $f \geqslant 0$, $\int f^2 d\mu_1 = 1$ *and* $|f'| \leq \lambda$,

$$\mathbf{Ent}_{\mu_1}\left(f^2\right) \leqslant A'\mathbf{Var}_{\mu_1}(f)\,. \tag{17}$$

Corollary 3.2. *Let* $\alpha \in]1, 2]$ *and assume that* f *is a smooth function on* $\mathbb{R}$. *Then we obtain the following estimation*

$$\mathbf{Ent}_{\mu_\alpha}\left(f^2\right) \leqslant A\mathbf{Var}_{\mu_\alpha}(f) + B\int_{\Omega}\left|\frac{f'}{f}\right|^\beta f^2 d\mu_\alpha, \tag{18}$$

where

$$\Omega = \left\{ f_+ \geqslant 2\sqrt{\int f_+^2 d\mu_\alpha} \right\} \cup \left\{ f_- \geqslant 2\sqrt{\int f_-^2 d\mu_\alpha} \right\},$$

$f_+ = \max(f, 0)$ *and* $f_- = \max(-f, 0)$.

Proof.
$\lhd$ We have $f^2 = f_+^2 + f_-^2$. Then

$$\begin{aligned}\mathbf{Ent}_{\mu_\alpha}\left(f^2\right) &= \sup\left\{\int f^2 g d\mu_\alpha \text{ with } \int e^g d\mu_\alpha \leqslant 1\right\} \\ &= \sup\left\{\int f_+^2 g d\mu_\alpha + \int f_-^2 g d\mu_\alpha \text{ with } \int e^g d\mu_\alpha \leqslant 1\right\} \\ &\leqslant \mathbf{Ent}_{\mu_\alpha}\left(f_+^2\right) + \mathbf{Ent}_{\mu_\alpha}\left(f_-^2\right).\end{aligned}$$

By Theorem 3.1 there exists $A, B > 0$ independent of f such that

$$\mathbf{Ent}_{\mu_\alpha}\left(f_+^2\right) \leqslant A\mathbf{Var}_{\mu_\alpha}(f_+) + B\int_{\Omega_+}\left|\frac{f_+'}{f_+}\right|^\beta f_+^2 d\mu_\alpha,$$

$$\mathbf{Ent}_{\mu_\alpha}\left(f_-^2\right) \leqslant A\mathbf{Var}_{\mu_\alpha}(f_-) + B\int_{\Omega_-}\left|\frac{f_-'}{f_-}\right|^\beta f_-^2 d\mu_\alpha,$$

where $\Omega_+ = \left\{ f_+ \geqslant 2\sqrt{\int f_+^2 d\mu_\alpha} \right\}$ and $\Omega_- = \left\{ f_- \geqslant 2\sqrt{\int f_-^2 d\mu_\alpha} \right\}$.

To conclude, it is enough to notice that

$$\begin{aligned}\mathbf{Var}_{\mu_\alpha}(f_+) + \mathbf{Var}_{\mu_\alpha}(f_-) &= \int f^2 d\mu_\alpha - \left(\left(\int f_+ d\mu_\alpha\right)^2 + \left(\int f_- d\mu_\alpha\right)^2\right) \\ &\leqslant \mathbf{Var}_{\mu_\alpha}(f)\,,\end{aligned}$$

and

$$\int_{\Omega_+} \left|\frac{f'_+}{f_+}\right|^\beta f_+^2 d\mu_\alpha + \int_{\Omega_-} \left|\frac{f'_-}{f_-}\right|^\beta f_-^2 d\mu_\alpha = \int_\Omega \left|\frac{f'}{f}\right|^\beta f^2 d\mu_\alpha.$$

$\triangleright$

It implies the existence of $a_\alpha > 0$ and $0 \leqslant C_\alpha < \infty$, such that μ_α satisfies a logarithmic Sobolev inequality of function $H_{a_\alpha,\alpha}$ with constant C_α. Indeed, this is clear that μ_α satisfies a Poincaré inequality, (see chapter 6 of [1]), with constant $0 \leqslant \lambda_\alpha < \infty$,

$$\mathbf{Var}_{\mu_\alpha}(f) \leqslant \lambda_\alpha \int f'^2 d\mu_\alpha.$$

Then, by inequality (16), we obtain for any smooth function $f > 0$ on $\mathbb{R}$,

$$\mathbf{Ent}_{\mu_\alpha}\left(f^2\right) \leqslant A\lambda_\alpha \int f'^2 d\mu_\alpha + B \int \left|\frac{f'}{f}\right|^\beta f^2 d\mu_\alpha.$$

Let us give a few hint on the proof of the Theorem 3.1, which will enable us to present key auxiliary lemmas. We first use the following inequality for $f \geqslant 0$ such that $\int f^2 d\mu_\alpha = 1$,

$$\int f^2 \log f^2 d\mu_\alpha \leqslant 5 \int (f-1)^2 d\mu_\alpha + \int (f-2)_+^2 \log(f-2)_+^2 d\mu_\alpha \qquad (19)$$

where it is obvious that truncation arguments are crucial. We will then need the following lemma:

Lemma 3.3. *Let μ be a probability measure on $\mathbb{R}$ and let $f \geqslant 0$ such that $\int f^2 d\mu = 1$ then we obtain*

i: $\displaystyle\int (f-1)^2 d\mu \leqslant 2\mathbf{Var}_\mu(f)$.

ii: $\displaystyle\int_{f\geq 2} f^2 d\mu \leqslant 8\mathbf{Var}_\mu(f)$.

iii: $\displaystyle\int_{f\geqslant 2} f^2 \log f^2 d\mu \leqslant \frac{\log 4}{\log 4 - 1}\mathbf{Ent}_\mu\left(f^2\right) < 4\mathbf{Ent}_\mu\left(f^2\right)$.

Proof.
$\triangleleft$ *i.* We have $\int (f-1)^2 d\mu = \mathbf{Var}_\mu(f) + \left(1 - \int f d\mu\right)^2$. Since $\int f^2 d\mu = 1$ and $f \geqslant 0$ we obtain that $0 \leqslant \int f d\mu \leqslant 1$, then $\left(\int f d\mu\right)^2 \leqslant \int f d\mu$. Then

$$\int (f-1)^2 d\mu \leqslant \mathbf{Var}_\mu(f) + (\mathbf{Var}_\mu(f))^2,$$

but since $\int f^2 d\mu = 1$, $\mathbf{Var}_\mu(f) \leqslant 1$, then $\int (f-1)^2 d\mu \leqslant 2\mathbf{Var}_\mu(f)$.

ii. One verifies trivially that when $x \geq 2$, $x^2 \leq 4(x-1)^2$ and apply *i.*

iii. Let us give the proof given in [13]. If $x > 0$ we have $x \log x + 1 - x \geqslant 0$ which yields

$$\int_{f \leqslant 2} f^2 \log f^2 d\mu + \mu(f \leqslant 2) - \int_{f \leqslant 2} f^2 d\mu \geqslant 0,$$

hence $\mathbf{Ent}_\mu\big(f^2\big) \geqslant \int_{f \geqslant 2} f^2 \log f^2 d\mu - \int_{f \geqslant 2} f^2 d\mu$. Since

$$\int_{f \geqslant 2} f^2 d\mu \leqslant \frac{1}{\log 4} \int_{f \geqslant 2} f^2 \log f^2 d\mu,$$

we obtain $\mathbf{Ent}_\mu\big(f^2\big) \geqslant \left(1 - \frac{1}{\log 4}\right) \int_{f \geqslant 2} f^2 \log f^2 d\mu$. $\rhd$

Recall the Hardy's inequality presented in the introduction. Let μ, ν be Borel measures on $\mathbb{R}^+$, the best constant A so that every smooth function f such that integrals are well defined, satisfies

$$\int_0^\infty (f(x) - f(0))^2 d\mu(x) \leqslant A \int_0^\infty f'^2 d\nu, \tag{20}$$

is finite if and only if

$$B = \sup_{x \geqslant 0} \mu([x, \infty[) \int_0^x \left(\frac{d\nu^{ac}}{d\mu}\right)^{-1} dt \tag{21}$$

is finite. And when A is finite we have this estimation

$$B \leqslant A \leqslant 4B.$$

A direct proof of this inequality with these properties of regularities for f can be found for example in Theorem 6.2.1 of [1].

We then present different proof of the desired inequality, starting from (19), according to the value of $\mathbf{Ent}_\mu\big(f^2\big)$, in which Hardy's inequality plays a crucial role. First, when the entropy is large we will need

Lemma 3.4. *Let h defined as follow,*

$$h(x) = \begin{cases} 1 & \textit{if } |x| \leqslant 1, \\ |x|^{2-\alpha} & \textit{if } |x| \geqslant 1. \end{cases}$$

Then there exists $0 \leqslant C_h < \infty$ *such that for every smooth function g we have*

$$\mathbf{Ent}_{\mu_\alpha}\Big(g^2\Big) \leqslant C_h \int g'^2 h d\mu_\alpha. \tag{22}$$

Proof.
$\lhd$ We use Theorem 3 of [10] which is a refinement of the criterion of a Bobkov-Götze theorem (see Theorem 5.3 of [6]).

The constant C_h satisfies $\max(b_-, b_+) \leqslant C_h \leqslant \max(B_-, B_+)$ where

$$b_+ = \sup_{x \geqslant 0} \mu_\alpha([x, +\infty[) \log\left(1 + \frac{1}{2\mu_\alpha([x, +\infty[)}\right) \int_0^x Z_\alpha \frac{e^{|t|^\alpha}}{h(t)} dt,$$

$$b_- = \sup_{x \leqslant 0} \mu_\alpha(]-\infty, x]) \log\left(1 + \frac{1}{2\mu_\alpha(]-\infty, x])}\right) \int_x^0 Z_\alpha \frac{e^{|t|^\alpha}}{h(t)} dt,$$

$$B_+ = \sup_{x \geqslant 0} \mu_\alpha([x, +\infty[) \log\left(1 + \frac{e^2}{\mu_\alpha([x, +\infty[)}\right) \int_0^x Z_\alpha \frac{e^{|t|^\alpha}}{h(t)} dt,$$

$$B_- = \sup_{x \leqslant 0} \mu_\alpha(]-\infty, x]) \log\left(1 + \frac{e^2}{\mu_\alpha([-\infty, x[)}\right) \int_x^0 Z_\alpha \frac{e^{|t|^\alpha}}{h(t)} dt.$$

An easy approximation prove that for large positive x

$$\mu_\alpha([x, \infty[) = \int_x^\infty \frac{1}{Z_\alpha} e^{-|t|^\alpha} dt \sim_\infty \frac{1}{Z_\alpha \alpha x^{\alpha-1}} e^{-x^\alpha} \tag{23}$$

$$\int_0^x Z_\alpha \frac{e^{|t|^\alpha}}{h(t)} dt \sim_\infty \frac{Z_\alpha}{\alpha x} e^{x^\alpha},$$

and one may prove same equivalent for negative x. A simple calculation then yields that constants b_+, b_-, B_+ and B_- are finite and the lemma is proved. □ ▷

Note that the function h is the smallest function such that the constant C_h in the inequality (22) is finite. More precisely, if $\tilde{h}$ satisfy

$$\lim_{x \to \infty} \frac{\tilde{h}(x)}{h(x)} = \infty$$

then the constant $C_{\tilde{h}} = \infty$.

In the case of small entropy, we will use so-called Φ-Sobolev inequalities (even if our context is less general), see Chafaï [14] for a comprehensive review, and Barthe-Cattiaux-Roberto [4] for a general approach in the case of measure μ_α.

Lemma 3.5. *Let $T_1 < T_2$, $T \in [T_1, T_2]$ and g be a smooth function defined on $[T, \infty[$. Assume that*

$$g(T) = 2, \;\; g \geqslant 2 \text{ and } \int_T^\infty g^2 d\mu_\alpha \leqslant 13,$$

then

$$\int_T^\infty (g-2)^2 \Phi(g^2) d\mu_\alpha \leqslant C_g \int_{[T,\infty[} g'^2 d\mu_\alpha, \tag{24}$$

where $\Phi(x) = \log^{\frac{2(\alpha-1)}{\alpha}}(x)$. The constant C_g depend on the measure μ_α but does not depend on the value of $T \in [T_1, T_2]$.

Proof.
◁ Let use Hardy's inequality as explained in the introduction. We have $g(T) = 2$. We apply inequality (20) on $[T, \infty[$ and with the function $f = g - 2$ (note that $f(T) = 0$) and the following measures

$$d\mu = \left(\log g^2\right)^{\frac{2(\alpha-1)}{\alpha}} d\mu_\alpha \text{ and } \nu = \mu_\alpha.$$

Then the constant C in inequality (24) is finite if and only if

$$B = \sup_{x \geqslant T} \int_T^x Z_\alpha e^{|t|^\alpha} dt \int_x^\infty \left(\log g^2\right)^{\frac{2(\alpha-1)}{\alpha}} d\mu_\alpha,$$

is finite.

Since $2(\alpha - 1)/\alpha < 1$ the function $x \to (\log x)^{\frac{2(\alpha-1)}{\alpha}}$ is concave on $[4, \infty[$. By Jensen inequality we obtain for all $x \geqslant T$,

$$\int_x^\infty \left(\log g^2\right)^{\frac{2(\alpha-1)}{\alpha}} d\mu_\alpha \leqslant \log^{\frac{2(\alpha-1)}{\alpha}} \left(\frac{\int_x^\infty g^2 d\mu_\alpha}{\mu_\alpha([x, \infty[)}\right) \mu_\alpha([x, \infty[).$$

Then by the property of g we have

$$\begin{aligned} B &\leqslant \sup_{x \geqslant T} \int_T^x Z_\alpha e^{|t|^\alpha} dt \log^{\frac{2(\alpha-1)}{\alpha}} \left(\frac{13}{\mu_\alpha([x, \infty[)}\right) \mu_\alpha([x, \infty[) \\ &\leqslant \sup_{x \geqslant T_1} \int_{T_1}^x Z_\alpha e^{|t|^\alpha} dt \log^{\frac{2(\alpha-1)}{\alpha}} \left(\frac{13}{\mu_\alpha([x, \infty[)}\right) \mu_\alpha([x, \infty[). \end{aligned}$$

Using the approximation

$$\int_0^x Z_\alpha e^{|t|^\alpha} dt \sim_\infty \frac{Z_\alpha}{\alpha x^{\alpha-1}} e^{x^\alpha},$$

and that given in equality (23) we prove that B is finite, bounded by a constant C_g which does not depend on T. ▷

We divide the proof of Theorem 3.1 in two parts: large and small entropy, both in the case of positive function. Let us now present the proof in the case of large entropy.
Large entropy case:

Proposition 3.6. *Suppose that* $\alpha \in]1, 2]$. *There exists* $0 \leqslant A_L, B_L < \infty$ *such that for any smooth functions* $f \geqslant 0$ *satisfying*

$$\int f^2 d\mu_\alpha = 1 \text{ and } \mathbf{Ent}_{\mu_\alpha}\left(f^2\right) \geqslant 1, \tag{25}$$

we have

$$\mathbf{Ent}_{\mu_\alpha}\left(f^2\right) \leqslant A_L \mathbf{Var}_{\mu_\alpha}(f) + B_L \int_{f \geqslant 2} \left|\frac{f'}{f}\right|^\beta f^2 d\mu_\alpha. \tag{26}$$

If $\alpha = 1$*, there exists* $0 \leqslant A'_L < \infty$ *and* $\lambda > 0$ *such that for every smooth function* $f \geqslant 0$ *with* $\int f^2 d\mu_1 = 1$*, when* $|f'| \leq \lambda$*, then we get*

$$\mathbf{Ent}_{\mu_1}\left(f^2\right) \leqslant A'_L \mathbf{Var}_{\mu_1}(f).$$

Proof of Proposition 3.6

$\lhd$ Let f be a smooth function satisfying $f \geqslant 0$, $\int f^2 d\mu_\alpha = 1$ and $\int f^4 d\mu_\alpha < \infty$.

A careful study on $\mathbb{R}^+$ of this function

$$x \to -x^2 \log x^2 + 5(x-1)^2 + x^2 - 1 + (x-2)_+^2 \log(x-2)_+^2$$

proves that for every $x \geqslant 0$

$$x^2 \log x^2 \leqslant 5(x-1)^2 + x^2 - 1 + (x-2)_+^2 \log(x-2)_+^2.$$

Then we obtain by Lemma 3.3.i, recalling that $\int f^2 d\mu_\alpha = 1$ and $f \geqslant 0$,

$$\begin{aligned}\int f^2 \log f^2 d\mu_\alpha &\leqslant 5\int (f-1)^2 d\mu_\alpha + \int (f^2-1) d\mu_\alpha \\ &\quad + \int (f-2)_+^2 \log(f-2)_+^2 d\mu_\alpha \\ &\leqslant 10\mathbf{Var}_{\mu_\alpha}(f) + \int (f-2)_+^2 \log(f-2)_+^2 d\mu_\alpha\end{aligned}$$

which is the announced starting point inequality (19).

Since $\int f^2 d\mu_\alpha = 1$, one can easily prove that

$$\int (f-2)_+^2 d\mu_\alpha \leqslant 1,$$

then $\int (f-2)_+^2 \log(f-2)_+^2 d\mu_\alpha \leqslant \mathbf{Ent}_{\mu_\alpha}\left((f-2)_+^2\right)$, and

$$\mathbf{Ent}_{\mu_\alpha}\left(f^2\right) \leqslant 10\mathbf{Var}_{\mu_\alpha}(f) + \mathbf{Ent}_{\mu_\alpha}\left((f-2)_+^2\right). \tag{27}$$

Hardy's inequality of Lemma 3.4 with $g = (f-2)_+$ gives

$$\mathbf{Ent}_{\mu_\alpha}\left((f-2)_+^2\right) \leqslant C_h \int (f-2)_+'^2 h d\mu_\alpha = C_h \int_{f \geqslant 2} f'^2 h d\mu_\alpha. \tag{28}$$

For $p, q > 1$ such that and $1/p + 1/q = 1$ we have for every $x, y > 0$ by Young inequality,

$$xy \leqslant \frac{x^p}{p} + \frac{y^q}{q}. \tag{29}$$

Consider then $\alpha \in]1, 2]$ and $\beta = \alpha/(\alpha - 1)$. Let $p = \beta/2$ and $q = \beta/(\beta - 2)$. Let $\varepsilon > 0$ and let apply inequality (29) to the right term of (28), we obtain on $\{f \geqslant 2\}$,

$$\frac{1}{\varepsilon^{(\beta-2)/\beta}}\left(\frac{f'}{f}\right)^2 \varepsilon^{(\beta-2)/\beta} h \leqslant \frac{2}{\beta \varepsilon^{(\beta-2)/2}}\left|\frac{f'}{f}\right|^\beta + \frac{\beta-2}{\beta}\varepsilon h^{\beta/(\beta-2)},$$

then

$$\mathbf{Ent}_{\mu_\alpha}\Big((f-2)_+{}^2\Big) \leqslant \frac{2C_h}{\beta\varepsilon^{(\beta-2)/2}}\int_{f\geqslant 2}\left|\frac{f'}{f}\right|^\beta f^2 d\mu_\alpha + \frac{\beta-2}{\beta}C_h\varepsilon\int_{f\geqslant 2} h^{\beta/(\beta-2)} f^2 d\mu_\alpha$$

Let μ a probability measure, then we have for every function f such that $\int f^2 d\mu = 1$ and for every measurable function g such that $\int f^2 g d\mu$ exists we get

$$\int f^2 g d\mu \leqslant \mathbf{Ent}_\mu\Big(f^2\Big) + \log\int e^g d\mu.$$

This inequality is also true for all function $g \geqslant 0$ even integrals are infinite.

Let $\eta > 0$ and we apply the previous inequality with $g = \eta h^{\beta/(\beta-2)}$ $(g \geqslant 0)$, we obtain then

$$\mathbf{Ent}_{\mu_\alpha}\Big((f-2)_+{}^2\Big) \leqslant \frac{2C_h}{\beta\varepsilon^{(\beta-2)/2}}\int_{f\geqslant 2}\left|\frac{f'}{f}\right|^\beta f^2 d\mu_\alpha + \frac{(\beta-2)C_h\varepsilon}{\beta\eta}\left(\mathbf{Ent}_{\mu_\alpha}\Big(f^2\Big) + \log\int \exp\Big(\eta h^{\beta/(\beta-2)}\Big)d\mu_\alpha\right). \quad (30)$$

Since $\beta = \alpha/(\alpha-1)$, $h(x)^{\beta/(\beta-2)} = x^\alpha$ if $|x| \geqslant 1$, then we fix $\eta = 1/2$. And note

$$\Delta = \log\int \exp\left(\frac{1}{2}h^{\beta/(\beta-2)}\right)d\mu_\alpha < \infty.$$

Fix now $\varepsilon = \inf\{\beta/(\Delta(\beta-2)4C_h), \beta/((\beta-2)4C_h)\}$ and note $\kappa = C_h/\varepsilon^{(\beta-2)/2}$ We obtain

$$\mathbf{Ent}_{\mu_\alpha}\Big((f-2)_+{}^2\Big) \leqslant \kappa\int_{f\geqslant 2}\left|\frac{f'}{f}\right|^\beta f^2 d\mu_\alpha + \frac{1}{4}\mathbf{Ent}_{\mu_\alpha}\Big(f^2\Big) + \frac{1}{4}.$$

As $\mathbf{Ent}_{\mu_\alpha}(f^2) \geqslant 1$, inequality (27) implies

$$\mathbf{Ent}_{\mu_\alpha}\Big(f^2\Big) \leqslant 20\mathbf{Var}_{\mu_\alpha}(f) + 2\kappa\int_{f\geqslant 2}\left(\frac{f'}{f}\right)^\beta f^2 d\mu_\alpha,$$

which proves inequality (26) with $A_L = 20$ and $B_L = 2\kappa$.

Assume now that $\alpha = 1$ and take f such that $|f'| \leq \lambda$. We apply the limit case of Young inequality to get on $\{f \geqslant 2\}$,

$$\left(\frac{f'}{f}\right)^2 hf^2 \leqslant \left(\frac{\lambda}{2}\right)^2 hf^2.$$

Then, with the same computation as in the case $\alpha \in]1,2]$, on can found $\lambda > 0$ and $0 \leqslant A'_L < \infty$ such that for all smooth function f satisfying hypothesis on (25) and $|f'| \leqslant \lambda$,

$$\mathbf{Ent}_{\mu_1}\Big(f^2\Big) \leqslant A'_L\mathbf{Var}_{\mu_1}(f).$$

▷

Remark 3.7. With the same method as developed in Proposition 3.6 we can prove the inequality (26) without $\mathbf{Var}_{\mu_\alpha}(f)$. Suppose that $\alpha \in]1,2]$. There exists $A > 0$ such that for any functions $f > 0$ satisfying

$$\int f^2 d\mu_\alpha = 1 \text{ and } \mathbf{Ent}_{\mu_\alpha}\left(f^2\right) \geqslant 1$$

we have

$$\mathbf{Ent}_{\mu_\alpha}\left(f^2\right) \leqslant A \int \left|\frac{f'}{f}\right|^\beta f^2 d\mu_\alpha.$$

Small entropy case:

Proposition 3.8. *Let* $\alpha \in]1,2]$. *There exists* $A_S, B_S > 0$ *such that for any functions* $f \geqslant 0$ *satisfying*

$$\int f^2 d\mu_\alpha = 1 \textit{ and } \mathbf{Ent}_{\mu_\alpha}\left(f^2\right) \leqslant 1,$$

we have

$$\mathbf{Ent}_{\mu_\alpha}\left(f^2\right) \leqslant A_S \mathbf{Var}_{\mu_\alpha}(f) + B_S \int_{f \geqslant 2} \left|\frac{f'}{f}\right|^\beta f^2 d\mu_\alpha. \tag{31}$$

If $\alpha = 1$, *there exists* $0 \leqslant A'_S < \infty$ *such that,*

$$\mathbf{Ent}_{\mu_1}\left(f^2\right) \leqslant A'_S \mathbf{Var}_{\mu_1}(f).$$

for all f *such that* $\left|f'\right| \leqslant 1$.

Proof of Proposition 3.8
$\lhd$ Let $f \geqslant 0$ satisfying $\int f^2 d\mu_\alpha = 1$. As in Proposition 3.6, we start with inequality (19), which readily implies

$$\mathbf{Ent}_{\mu_\alpha}\left(f^2\right) = \int f^2 \log f^2 d\mu_\alpha \leqslant 10 \mathbf{Var}_{\mu_\alpha}(f) + \int (f-2)_+^2 \log f^2 d\mu_\alpha. \tag{32}$$

We will now control the second term of the right hand side of this last inequality via the use of Φ-Sobolev inequalities, namely Lemma 3.5.

Therefore we have to construct a function g defined on $[T, \infty[$ (for a well chosen T) with $g \geqslant 2$ and $g(T) = 2$ and which satisfies,

(g1) $\displaystyle\int_T^\infty g^2 d\mu_\alpha \leq 13;$
(g2) $\displaystyle\int_T^\infty (g-2)^2 \Phi(g^2) d\mu_\alpha \geq C \int_T^\infty (f-2)_+^2 \log f^2 d\mu_\alpha;$
(g3) $\displaystyle\int_T^\infty g'^2 d\mu_\alpha \leq C \int_{[T,\infty[\cap\{f \geq 2\}} \Psi\left(\left|\frac{f'}{f}\right|\right) f^2 d\mu_\alpha + D\, \mathbf{Ent}_{\mu_\alpha}\left(f^2\right),$

with $\Phi(x) = \log^{\frac{2(\alpha-1)}{\alpha}}(x)$, $0 < D \leqslant 1/2$ and $\Psi(x) = x^{\beta}$.

Let now define $T_1 < 0$ and $T_2 > 0$ such that

$$\mu_\alpha(]-\infty, T_1]) = \frac{3}{8},\ \ \mu_\alpha([T_1, T_2]) = \frac{1}{4} \text{ and } \mu_\alpha([T_2, +\infty[) = \frac{3}{8}.$$

Since $\int f^2 d\mu_\alpha = 1$ there exists $T \in [T_1, T_2]$ such that $f(T) \leqslant 2$.

Introduce now g on $[T, \infty]$ as follow

$$g = 2 + (f-2)_+ \log^{\gamma} f^2,$$

where $\gamma = (2-\alpha)/(2\alpha)$.

Due to the fact that f is a smooth function (locally absolutely continuous function) then g is also a smooth function. Moreover g satisfies $g(T) = 2$ and $g(x) \geqslant 2$ for all $x \geqslant T$. Let now compute $\int_T^\infty g^2 d\mu_\alpha$. We get

$$\begin{aligned}\int_T^\infty g^2 d\mu_\alpha &\leqslant 2\int_{T_1}^\infty 4 d\mu_\alpha + 2\int_{T_1}^\infty (f-2)_+^2 \log^{2\gamma} f^2 d\mu_\alpha \\ &\leqslant 5 + 2\int_{[T_1,\infty[\cap\{f\geqslant 2\}} f^2 \log^{2\gamma} f^2 d\mu_\alpha.\end{aligned}$$

Since $2\gamma \in [0,1]$ we have $\log^{2\gamma} f^2 \leqslant \log f^2$ on $\{f \geqslant 2\}$. Then we obtain by Lemma 3.3.iii

$$\begin{aligned}\int_T^\infty g^2 d\mu_\alpha &\leqslant 5 + 2\int_{f\geqslant 2} f^2 \log^{2\gamma} f^2 d\mu_\alpha \\ &\leqslant 5 + 8\mathbf{Ent}_{\mu_\alpha}\left(f^2\right) \\ &\leqslant 13,\end{aligned}$$

since $\mathbf{Ent}_{\mu_\alpha}\left(f^2\right) \leqslant 1$.

Assumptions on Lemma 3.5 are satisfied, we obtain by inequality (24)

$$\int_T^\infty (g-2)_+^2 \log^{\frac{2(\alpha-1)}{\alpha}} g^2 d\mu_\alpha \leqslant C_g \int_{[T,\infty[\cap\{g\geqslant 2\}} g'^2 d\mu_\alpha.$$

Let us compare the various terms now.

First, denote $u = 2(\alpha-1)/\alpha$, we thus obtain

$$(g-2)_+^2 \log^u g^2 = (f-2)_+^2 \log^{2\gamma} f^2 \log^u \left(2 + (f-2)_+ \log^{\gamma} f^2\right)^2.$$

On $\{f \geqslant 2\}$, we have $2 + (f-2)_+ \log^\gamma f^2 \geqslant 2 + (f-2)_+ K$, where $K = \log^\gamma 4$. Since $K \geqslant 1$ and $u + 2\gamma = 1$, one has

$$(g-2)_+^2 \log^u g^2 \geqslant (f-2)_+^2 \log^{2\gamma+u} f^2 = (f-2)_+^2 \log f^2.$$

Then we obtain

$$\int_T^\infty (f-2)_+^2 \log f^2 d\mu_\alpha \leqslant \int_T^\infty (g-2)_+^2 \log^{\frac{2(\alpha-1)}{\alpha}} g^2 d\mu_\alpha. \tag{33}$$

Secondly one has on $\{f \geqslant 2\}$,

$$g' = f' \log^{\gamma} f^2 \left(1 + \gamma 2^{\gamma} \frac{f-2}{f \log f^2}\right),$$

then using $\log f^2 \geqslant \log 4$ on $\{f \geqslant 2\}$ one obtain

$$|g'|^2 \leqslant |f'|^2 \log^{2\gamma} f^2 \left(1 + \frac{\gamma 2^{\gamma}}{łog 4}\right)^2.$$

Denoting $D = (1 + \gamma 2^{\gamma}/\log 4)^2$, one has

$$\int_{[T,\infty[\cap\{f\geqslant 2\}} g'^2 d\mu_\alpha \leqslant D \int_{[T,\infty[\cap\{f\geqslant 2\}} f'^2 \log^{2\gamma} f^2 d\mu_\alpha, \tag{34}$$

on $[T,\infty[\cap\{f \geqslant 2\}$.

Then, using inequalities (33) and (34), there exists $C \geqslant 0$ (independent of $T \in [T_1, T_2]$), such that

$$\int_T^{\infty} (f-2)_+^2 \log f^2 d\mu_\alpha \leqslant C \int_{[T,\infty[\cap\{f\geqslant 2\}} f'^2 \log^{2\gamma} f^2 d\mu_\alpha. \tag{35}$$

When $\alpha \in]1,2]$, we apply Inequality (29) with $q = \alpha/(2-\alpha)$ and $p = \alpha/(2(\alpha-1))$. We obtain for every $\varepsilon > 0$,

$$\begin{aligned}\int_{[T,\infty[\cap\{f\geqslant 2\}} \left(\frac{f'}{f}\right)^2 \left(\log^{2\gamma} f^2\right) f^2 d\mu_\alpha \leqslant & \frac{2(\alpha-1)}{\alpha \varepsilon^{\frac{2-\alpha}{2(\alpha-1)}}} \int_{[T,\infty[\cap\{f\geqslant 2\}} \left|\frac{f'}{f}\right|^{\beta} f^2 d\mu_\alpha \\ & + \varepsilon \frac{2-\alpha}{\alpha} \int_{[T,\infty[\cap\{f\geqslant 2\}} f^2 \log f^2 d\mu_\alpha.\end{aligned}$$

Fix ε such that $\varepsilon C \frac{2-\alpha}{\alpha} < 1/16$, then there exists $A > 0$ such that

$$\begin{aligned}\int_T^{\infty} (f-2)_+^2 \log f^2 d\mu_\alpha \leqslant & A \int_{[T,\infty[\cap\{f\geqslant 2\}} \left|\frac{f'}{f}\right|^{\beta} f^2 d\mu_\alpha \\ & + \frac{1}{16} \int_{[T,\infty[\cap\{f\geqslant 2\}} f^2 \log f^2 d\mu_\alpha.\end{aligned}$$

Using Lemma 3.3.iii we have,

$$\begin{aligned}& \int_T^{\infty} (f-2)_+^2 \log f^2 d\mu_\alpha \\ & \leqslant A \int_{[T,\infty[\cap\{f\geqslant 2\}} \left|\frac{f'}{f}\right|^{\beta} f^2 d\mu_\alpha + \frac{1}{4} \mathbf{Ent}_{\mu_\alpha}\left(f^2\right).\end{aligned}$$

The same method can be used on $]-\infty, T]$ and then we get

$$\begin{aligned}& \int_{-\infty}^{T} (f-2)_+^2 \log f^2 d\mu_\alpha \\ & \leqslant A \int_{[-\infty,T]\cap\{f\geqslant 2\}} \left|\frac{f'}{f}\right|^{\beta} f^2 d\mu_\alpha + \frac{1}{4} \mathbf{Ent}_{\mu_\alpha}\left(f^2\right).\end{aligned}$$

And then, we get

$$\int (f-2)_+^2 \log f^2 d\mu_\alpha \leqslant 2A \int_{f \geqslant 2} \left| \frac{f'}{f} \right|^\beta f^2 d\mu_\alpha + \frac{1}{2} \mathbf{Ent}_{\mu_\alpha}\left(f^2\right).$$

Note that the constant A does not depend on T.

Then, by inequality (32), inequality (31) is proved for $\alpha \in]1, 2]$, with $A_S = 34$ and $B_S = 4A$.

Assume now that $\alpha = 1$. In this case $2\gamma = 1$, then using inequality (35) we obtain

$$\int_T^\infty (f-2)_+^2 \log f^2 d\mu_1 \leqslant D \int_{[T,\infty[\cap\{f\geqslant 2\}} \log f^2 d\mu_1 \leqslant 8D \int_{[T,\infty[\cap\{f\geqslant 2\}} f^2 d\mu_1,$$

for some constant C'. Then by Lemma 3.3.ii, we obtain the result which concludes the proof in this case with $A'_S = 10 + 8D$. $\triangleright$

Let us give now a proof of the theorem.

Proof of Theorem 3.1

$\triangleleft$ The proof of the theorem is a simple consequence of Propositions 3.6 and 3.8. For $\alpha \in]1, 2]$, we get inequality (16) with $A = \max\{A_L, A_S\}$ and $B = \max\{B_L, B_S\}$ and for $\alpha = 1$ one find $\lambda > 0$ and $0 \leqslant A' = \max\{A'_S, A'_L\} < \infty$ such that inequality (17) is true. $\triangleright$

4. Extension to other measures

We will present in this section modified logarithmic Sobolev inequality of function H for more general measure than μ_α which can be derived using the proof carried on in Section 3: the large entropy case where the optimal Hardy function h is identified and used to derive the optimal H, and the small entropy case where Φ and g (used on the proof of Proposition 3.7) have to be identified leading to the same H function.

Let us first consider the following probability measure $\mu_{\alpha,\beta}$ for $\alpha \in [1, 2]$ and $\beta \in \mathbb{R}$ defined by

$$\mu_{\alpha,\beta}(dx) = \frac{1}{Z} e^{-\varphi(x)} dx \text{ where } \varphi(x) = |x|^\alpha (\log |x|)^\beta \text{ for } |x| \geq 1$$

and φ twice continuously differentiable.

Theorem 4.1. *There exists $0 \leqslant A, B < \infty$ such that the measure $\mu_{\alpha,\beta}$ satisfies the following logarithmic Sobolev inequality: for any smooth f on $\mathbb{R}$ such that $\int f^2 d\mu_{\alpha\beta} = 1$ and $f \geqslant 0$, we have*

$$\mathbf{Ent}_{\mu_{\alpha,\beta}}\left(f^2\right) \leqslant A\mathbf{Var}_{\mu_{\alpha,\beta}}(f) + B \int_{f\geqslant 2} H\left(\left|\frac{f'}{f}\right|\right) f^2 d\mu_{\alpha,\beta}, \qquad (36)$$

where H is positive smooth and given for $x \geq 2$ by

$$\begin{cases} H(x) = \dfrac{x^{\frac{\alpha}{\alpha-1}}}{\log^{\frac{\beta}{\alpha-1}} x} \text{ if } \alpha \in]1,2[, \beta \in \mathbb{R}, \\ H(x) = x^2 e^{x^{1/\beta}} \text{ if } \alpha = 1, \beta \in \mathbb{R}^+, \\ H(x) = x^2 \log^{-\beta}(x) \text{ if } \alpha = 2, \beta \in \mathbb{R}^-. \end{cases}$$

Proof.
$\lhd$ We will mimic closely the proof given in the μ_α case, considering large and small entropy case. We will not present all the calculus but give the essential arguments.

Let now treat the case $\alpha \in]1,2[$.

Large entropy. We will first apply Lemma 3.4 to measure $\mu_{\alpha,\beta}$, one has then that b_+, b_-, B_+, B_- are finite if one take h positive smooth

$$h(x) = \frac{x^{2-\alpha}}{\log^\beta x} \qquad |x| \geq 2.$$

One has then to determine H to construct ψ such that there exists $\eta > 0$ with $\eta\psi(h)$ exponentially integrable with respect to $\mu_{\alpha,\beta}$ and $H = \psi^*(x^2)$ where ψ^* is the Fenchel-Legendre transform of ψ.

Considering the exponential integrability condition leads us to consider $\psi(x)$ behaving asymptotically as $x^{\frac{\alpha}{2-\alpha}} \log^{\frac{2\beta}{2-\alpha}} x$. One may thus derive the asymptotic behavior of ψ^* and finally H.

Small entropy. One desires here to apply Lemma 3.5, evaluating Φ and then build the function g satisfying conditions (g1), (g2) and (g3). By Hardy's inequality and arguments in the proof of Lemma 3.5, one may choose Φ for x large enough as

$$\Phi(x) = \log^{2\frac{\alpha-1}{\alpha}}(x)\,(\log\log x)^{\frac{2\beta}{\alpha}}.$$

Setting then

$$g = 2 + (f-2)_+ \log^{\frac{2-\alpha}{2\alpha}} f^2 (\log\log f^2)^{-\frac{\beta}{\alpha}},$$

one may then verify (g1), (g2) and (g3) with $\Psi = H$ defined in the large entropy step.

Now if $\alpha = 1$ and $\beta \geqslant 0$, then the same arguments gives that for large enough x

$$\psi(x) = x\log^{2\beta} x, \quad \psi^*(x) = xe^{x^{1/(2\beta)}} \text{ and } H(x) = x^2 e^{x^{1/\beta}}.$$

If $\alpha = 2$ and $\beta \leqslant 0$, we have for large enough x.

$$\psi(x) = \frac{1}{x} e^{2x^{-1/\beta}}, \quad \psi^*(x) = x\log^{-\beta} x \text{ and } H(x) = x^2 \log^{-\beta} x.$$

$\rhd$

Remark 4.2. 1. Using once again Herbst's argument, we may derive concentration properties for the measure $\mu_{\alpha,\beta}$ of desired order, for every function F with $|F'| \leq 1$, there exists $C > 0$ such that, for all $\lambda > 0$,

$$\mu_{\alpha,\beta}\left(\left|F - \mu_{\alpha,\beta}(F)\right| \geq \lambda\right) \leq 2e^{-C\min(\lambda^\alpha \log^\beta \lambda, \lambda^2)}.$$

The extension to greater dimension being handled as in the previous case

2. Note that the Latała-Oleszkiewicz inequalities $I(r)$ (see [21]) are not well adapted for the family of measures $\mu_{\alpha,\beta}$. Indeed, using Hardy's characterization of this inequalities obtained by Barthe-Roberto [10, Th. 13 and Prop. 15], one may show that $\mu_{\alpha,\beta}$ satisfies an $I(\alpha/2)$ inequality if $\beta \geq 0$ and an $I(\alpha/2 - \epsilon)$ (ϵ being arbitrary small) for $\beta < 0$, which entails consequently not optimal concentration properties.
3. By the characterization of the spectral gap property on $\mathbb{R}$, one obtains that each measure $\mu_{\alpha,\beta}$ satisfies a Poincaré inequality and thus a modified logarithmic Sobolev inequality.

Following the previous proof, we may generalize the family $\mu_{\alpha,\beta}$ adding an explicit multiplicative term to the potential $|x|^\alpha \log^\beta |x|$, as for example $\log\log^\gamma |x|$ which will give us new modified logarithmic Sobolev inequality, but each of this new measure has to be considered "one-by-one" (we hope some general results for φ convex). We may now state a result enabling us to get the stability of these modified logarithmic Sobolev inequality by addition of an unbounded perturbation: consider the measures

$$d\tau_\alpha(x) = \exp\left(-|x|^\alpha - |x|^{\alpha-1}\cos(x)\right)\frac{dx}{Z_\alpha}, \qquad \alpha \in]1,2],$$

$$d\gamma_{\alpha,b}(x) = (1+x)^b e^{-x^\alpha}\frac{dx}{Z_{\alpha,b}} 1_{x \geqslant 0}, \qquad \alpha \in]1,2], b \in \mathbb{R}.$$

Proposition 4.3. *There exists $a > 0$ such that the measures τ_α and $\gamma_{\alpha,b}$ satisfy a logarithmic Sobolev inequality of function $H_{a,\alpha}$.*

Proof.
$\lhd$ Following the proof given in Section 3 , one sees that the result hold true once one may verify that the Hardy's inequalities of Lemma 3.4 and Lemma 3.5 hold with the h and Φ obtained for the case of μ_α. It is easily checked once remarked that

$$\log\frac{d\tau_\alpha(x)}{dx} \sim_\infty -|x|^\alpha \qquad \text{and} \qquad \left(\log\frac{d\tau_\alpha(x)}{dx}\right)' \sim_\infty -(\alpha-1)|x|^{\alpha-1}$$

and the same for $\gamma_{\alpha,b}$. $\rhd$

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