On quantitative convergence to quasi-stationarity

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Plan of the talk

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A very simple example

For fixed $N \in \mathbb{N}$, consider the usual discrete time random walk $(X_t)_{t \in \mathbb{Z}_+}$ on $[\![0, N]\!]$, with holding at 0 and N. The invariant (and reversible) probability measure is the uniform distribution η.

Qualitative result: whatever the initial condition $\mathcal{L}(X_0)$, $\mathcal{L}(X_t)$ converges to η as $t \in \mathbb{Z}_+$ goes to infinity.

Quantitative result:

$$\|\mathcal{L}(X_t) - \eta\|_{\mathrm{tv}} \quad \leqslant \quad \sqrt{2\exp(-s)}$$

for

$$t \quad \geqslant \quad \frac{1}{4}(N+1)^2(1+s)$$

Total variation

Let us recall the definition of the **total variation** norm: for any signed measure m on a discrete space,

$$\begin{aligned} \|m\|_{\mathrm{tv}} &:= 2 \sup_{A \subset S} |m(A)| \\ &= \sup_{f \in \mathcal{F},\, \|f\|_\infty \leqslant 1} m(f) \\ &= \sum_{x \in S} |m(x)| \end{aligned}$$

Mixing time associated to $(\mathcal{L}(X_t))_{t \in \mathbb{Z}_+}$:

$$T_{\mathrm{mix}} := \sup_{\mathcal{L}(X_0)} \inf\{t \in \mathbb{Z}_+ : \|\mathcal{L}(X_t) - \eta\|_{\mathrm{tv}} \leqslant 1\}$$

Absorption

In the previous example: $T_{\mathrm{mix}} \leqslant (N+1)^2(1+\ln(2))/4$. This is a refined bound using the whole spectrum. If just the spectral gap is used: $T_{\mathrm{mix}} \leqslant (N+1)^2(1+\ln(N+1))/4$.

What happens if 0 is absorbing?
Then $\mathcal{L}(X_t) \to \delta_0$. This can be quantified also, using the first Dirichlet eigenvalue. Our interest here:

$$
\begin{aligned}
\tau &:= \inf\{t \in \mathbb{Z}_+ \; : \; X_t = 0\} \\
\mu_t &:= \mathcal{L}(X_t | \tau > t)
\end{aligned}
$$

Then (if $\mathcal{L}(X_0) \neq \delta_0$!), qualitative result:

$$
\lim_{t \to +\infty} \mu_t \quad = \quad \nu
$$

Quasi-stationary measure

where the probability ν on $[\![1, N]\!]$ is called the **quasi-stationary distribution** and is given by

$$\nu(x) \quad := \quad Z^{-1} \cos\left(\frac{(2N+1-2x)\pi}{2(2N+1)}\right)$$

with $Z^{-1} := 2\tan\left(\frac{\pi}{2(2N+1)}\right)$, the normalizing constant.
If $\mathcal{L}(X_0) = \nu$, then $\mu_t = \nu$ for all $t \geqslant 0$ and

$$\mathbb{P}[\tau > t] \quad = \quad \left(\cos\left(\frac{\pi}{2N+1}\right)\right)^t$$

(geometric distribution)

Our purpose: to obtain corresponding quantitative results, for instance $T_{\mathrm{quasi-mix}} \leqslant \text{cst } N^2 \ln(N)$ in the above case.
We will rather work in the continuous time setting, where results are simpler to state.

The framework

The whole finite state space is $\bar{S} := S \sqcup \{0\}$, where 0 is the absorbing point. It means that $\bar{S}$ is endowed with a Markov generator matrix $\bar{L} := (\bar{L}(x,y))_{x,y\in\bar{S}}$ whose restriction to $S \times S$ is irreducible and which is such that

$$\begin{array}{rrcl} \forall\ x \in \bar{S}, & \bar{L}(0,x) & = & 0 \\ \exists\ x \in S\ : & \bar{L}(x,0) & > & 0 \end{array}$$

If μ_0 is an initial distribution on S, we can associate a Markov process $X := (X_t)_{t\geqslant 0}$. The absorbing time τ and conditional distribution μ_t, for $t \geqslant 0$, are constructed as before. There exists a quasi-stationary distribution ν such that

$$\lim_{t\to+\infty} \mu_t \quad = \quad \nu$$

We want to quantify this convergence.

Notations

Let K be the $S \times S$ minor of $\overline{L}$. It can be written under the Schrödinger form $L - V$ where $V(x) = \overline{L}(x, 0)$, L an irreducible Markov generator on S. Let η be the invariant probability for L. Perron-Frobenius theory: there are $\lambda_0 > 0$ and a positive function φ such that

$$K[\varphi] \quad = \quad -\lambda_0 \varphi$$

Dual L^* of L in $\mathbb{L}^2(\eta)$: still a Markov generator given by

$$\forall\, x, y \in S, \qquad L^*(x, y) \quad = \quad \frac{\eta(y)}{\eta(x)} L(y, x)$$

$K^* = L^* - V$ and there is a positive function φ^* such that

$$K^*[\varphi^*] \quad = \quad -\lambda_0 \varphi^*$$

Different normalizations: $\eta[\varphi^*] = 1$ and $\eta[\varphi\varphi^*] = 1$.
We have $\nu = \varphi^* \cdot \eta$: it can be easily checked that for any function f on S, $\nu[K[f]] = -\lambda_0 \nu[f]$. As a consequence, for any $t \geqslant 0$,

$$\nu \exp(tK) \quad = \quad \exp(-\lambda_0 t)\nu$$

namely, if $\mathcal{L}(X_0) = \nu$, then

$$\mathcal{L}(X_t) = \exp(-\lambda_0 t)\nu + (1 - \exp(-\lambda_0 t))\delta_0$$

In particular

$$\mathbb{P}_\nu[\tau > t] = \mathbb{P}_\nu[X_t \in S] = \exp(-\lambda_0 t)$$

(exponential distribution).

Doob transform

Consider the Markovian operator $\widetilde{L}$ on S defined by its off-diagonal entries via

$$\forall\ x \neq y \in S, \qquad \widetilde{L}(x,y) \ := \ L(x,y)\frac{\varphi(y)}{\varphi(x)}$$

(the diagonal entries are such the row sums vanish).
It is called the **Doob transform** of L through φ. We have for any function f,

$$\widetilde{L}[f] \ = \ \frac{1}{\varphi}(L - V + \lambda_0 I)[\varphi f]$$

$\widetilde{L}$ is irreducible and its invariant measure $\widetilde{\eta}$ is given by

$$\forall\ x \in S, \qquad \widetilde{\eta}(x) \ = \ \varphi(x)\varphi^*(x)\eta(x)$$

Indeed:

$$\begin{aligned} \widetilde{\eta}[\varphi^{-1}(L - V + \lambda_0)[\varphi f]] &= \eta[\varphi^*(L - V + \lambda_0)[\varphi f]] \\ &= \eta[\varphi f(L^* - V + \lambda_0)[\varphi^*]] \\ &= 0 \end{aligned}$$

Reduction to ergodicity

Let $(\widetilde{P}_t)_{t\geqslant 0}$ the ergodic semi-group generated by $\widetilde{L}$. Its interest:

Theorem

For any probability measure μ_0 on S and for any $t \geqslant 0$, we have

$$\frac{\varphi_\wedge}{2\varphi_\vee}\left\|\widetilde{\mu}_0\widetilde{P}_t - \widetilde{\eta}\right\|_{\mathrm{tv}} \;\leqslant\; \|\mu_t - \nu\|_{\mathrm{tv}} \;\leqslant\; 2\frac{\varphi_\vee}{\varphi_\wedge}\left\|\widetilde{\mu}_0\widetilde{P}_t - \widetilde{\eta}\right\|_{\mathrm{tv}}$$

where $\widetilde{\mu}_0$ is the probability on S whose density with respect to μ_0 is proportional to φ. In particular the asymptotic exponential rate of convergence of $\|\mu_t - \nu\|_{\mathrm{tv}}$ and $\left\|\widetilde{\mu}_0\widetilde{P}_t - \widetilde{\eta}\right\|_{\mathrm{tv}}$ are the same.

We used the notation

$$\varphi_\vee := \max_{x\in S}\varphi(x) \qquad \varphi_\wedge := \min_{x\in S}\varphi(x)$$

Amplitude

Thus it appears that the first Dirichlet eigenfunction φ is the main ingredient needed to reduce the quantitative study of the convergence to quasi-stationarity to that of the convergence to equilibrium. A crucial quantity seems to be the **amplitude** of φ:

$$a_\varphi \quad := \quad \frac{\varphi_\vee}{\varphi_\wedge}$$

The convergence to equilibrium has been intensively investigated, through various approaches: Lyapounov functions, coupling, strong stationary times, isoperimetry, spectral theory, functional inequalities... The above bounds enable to recycle them for convergence to quasi-stationarity.

The simplest of these methods: the $\mathbb{L}^2$ approach.
Let $\widehat{L}$ be the **additive symmetrization** of $\widetilde{L}$ in $\mathbb{L}^2(\widetilde{\eta})$: it is $(\widetilde{L}+\widetilde{L}^*)/2$, where $\widetilde{L}^*$ is the adjoint operator of $\widetilde{L}$ in $\mathbb{L}^2(\widetilde{\eta})$. By self-adjointness, $\widehat{L}$ is diagonalizable in $\mathbb{R}$. Let $\widehat{\lambda} > 0$ stand for the smallest non-zero eigenvalue (spectral gap) of $-\widehat{L}$.

Theorem

For any $t \geqslant 0$, we have

$$\sup_{\mu_0 \in \mathcal{P}} \|\mu_t - \nu\|_{\mathrm{tv}} \quad \leqslant \quad \sqrt{\frac{1}{(\varphi\varphi^*\eta)_\wedge}\frac{\varphi_\vee}{\varphi_\wedge}}\exp(-\widehat{\lambda}t)$$

where $\mathcal{P}$ stands for the set of probability measures on S.

Logarithmic Sobolev inequality

It is possible to improve the pre-exponential factor in the above result, but at the expense of the rate $\widehat{\lambda}$, via the **logarithmic Sobolev inequalities** associated to $\widehat{L}$.
Let $\widehat{\alpha} > 0$ be the largest constant such that for all $f \in \mathcal{F}$,

$$\begin{aligned} \widehat{\alpha} \sum_{x \in S} f^2(x) \ln\left(\frac{f^2(x)}{\widetilde{\eta}[f^2]}\right) \varphi^*(x)\varphi(x)\eta(x) \\ \leqslant \quad \sum_{x,y \in S} (f(y) - f(x))^2 \varphi^*(x)\varphi(y)\eta(x)L(x,y) \end{aligned}$$

Then we have

$$\sup_{\mu_0 \in \mathcal{P}} \|\mu_t - \nu\|_{\mathrm{tv}} \quad \leqslant \quad \sqrt{2 \ln\left(\frac{\eta[\varphi\varphi^*]}{(\varphi\varphi^*\eta)_\wedge}\right) \frac{\varphi_\vee}{\varphi_\wedge}} \exp(-(\widehat{\alpha}/2)t)$$

This is interesting for not too large t and when $\widetilde{\alpha}$ can be computed (e.g. by tensorization).

Reversible case

Assume that η is reversible for L:

$$\forall\ x, y \in S, \quad \eta(x)L(x,y) \quad = \quad \eta(y)L(y,x)$$

Then $-K = V - L$ is diagonalizable in $\mathbb{R}$, we have already met its smallest eigenvalue λ_0. Let $\lambda_1 > \lambda_0$ be the second eigenvalue. The spectral gap bound can be rewritten:

Theorem

Under the reversibility assumption, for any $t \geqslant 0$, we have

$$\begin{aligned}\sup_{\mu_0 \in \mathcal{P}} \|\mu_t - \nu\|_{\mathrm{tv}} &\leqslant \sqrt{\frac{1}{(\varphi^2\eta)_\wedge}\frac{\varphi_\vee}{\varphi_\wedge}}\exp(-(\lambda_1-\lambda_0)t) \\ &\leqslant \sqrt{\frac{1}{\eta_\wedge}\left(\frac{\varphi_\vee}{\varphi_\wedge}\right)^2}\exp(-(\lambda_1-\lambda_0)t)\end{aligned}$$

(In this situation $\varphi^* = \varphi$).

A finite birth and death example with $\lambda_0 \sim \lambda_1 - \lambda_0$ (1)

The 3 first examples are **birth and death processes** on $\bar{S} := [\![0, N]\!]$, with $N \in \mathbb{N}$, absorbing at 0. So L gives positive rates only to the oriented edges $(x, x+1)$ and $(x+1, x)$ where $x \in [\![1, N-1]\!]$ and admits a reversible probability η. Assume that the killing rate at 1 is 1, namely $V(1) = \bar{L}(1,0) = 1$. The other values of V are taken to be zero.

Specifically for the first example, we choose

$$\begin{array}{rcl} \forall\ x \in [\![1, N-2]\!], \qquad L(x, x+1) & := & L(x+1, x) \ := \ 1 \\ L(N-1, N) \ = \ 1 \ \text{ and } \ L(N, N-1) & = & 2 \end{array}$$

The reversible probability η is almost the uniform distribution on $S = [\![1, N]\!]$ (N has a weight divided by 2). The function φ is defined by

$$\forall\ x \in S, \qquad \varphi(x) \ := \ \frac{1}{Z}\sin(\pi x/(2N))$$

where Z is the renormalization constant such that $\eta[\varphi^2] = 1$.

A finite birth and death example with $\lambda_0 \sim \lambda_1 - \lambda_0$ (2)

We compute that λ_0 and $\lambda_1 - \lambda_0$ are of the same order (as $N \to \infty$), meaning that absorption and convergence to quasi-stationarity happen at similar rates:

$$\begin{aligned}\lambda_0 &= \frac{\pi^2}{4N^2}(1 + \mathcal{O}(N^{-2})) \\ \lambda_1 - \lambda_0 &= 2\frac{\pi^2}{N^2}(1 + \mathcal{O}(N^{-2}))\end{aligned}$$

Furthermore, we have

$$a_\varphi = \frac{2N}{\pi}(1 + \mathcal{O}(N^{-2}))$$

It follows from the previous bound that for any given $s > 0$, if

$$t = \frac{5}{4\pi^2}N^2 \ln(N) + \frac{s}{2\pi^2}N^2$$

then

$$\sup_{\mu_0 \in \mathcal{P}} \|\mu_t - \nu\|_{\mathrm{tv}} \leqslant \frac{4}{\pi^2}(1 + \mathcal{O}(N^{-1})) \exp(-s)$$

A finite birth and death example with $\lambda_0 \ll \lambda_1 - \lambda_0$ (1)

It is similar to the previous example, except that for some $r > 1$, we take

$$\forall\ x \in [\![1, N-2]\!], \qquad \begin{cases} L(x, x+1) & := & r \\ L(x+1, x) & := & 1 \end{cases}$$
$$L(N-1, N) \;=\; r \;\text{ and }\; L(N, N-1) \;=\; 1+r$$

The reversible probability η is given by

$$\forall\ x \in S, \qquad \eta(x) \;=\; \begin{cases} \frac{r^2-1}{2r^N-r-1} r^{x-1} & ,\ \text{if } x \in [\![N-1]\!] \\ \frac{r-1}{2r^N-r-1} r^{N-1} & ,\ \text{if } x = N \end{cases}$$

More involved computations are needed to get information on the eigenvalues and eigenfunctions, but finally we get, for large N,

$$\lambda_0 \;\sim\; \frac{1}{2}(r+1)(r-1)^2 \frac{1}{r^{N+1}}$$
$$a_\varphi \;=\; \frac{r}{r-1}(1 + \mathcal{O}(r^{-N}))$$

A finite birth and death example with $\lambda_0 \ll \lambda_1 - \lambda_0$ (2)

$$\lambda_1 \quad > \quad (1-\sqrt{r})^2$$

It implies that

$$\lambda_1 - \lambda_0 \ \sim \ \lambda_1 \ \gg \ \lambda_0$$

meaning that convergence to quasi-stationarity happens at a much faster rate than absorption. It follows that for any fixed $s \geqslant 0$, if for N large enough we consider the time

$$t \quad := \quad \frac{1}{2(1-\sqrt{r})^2}\,(\ln(r)N + 2s)$$

then

$$\sup_{\mu_0 \in \mathcal{P}} \|\mu_t - \nu\|_{\mathrm{tv}} \quad \leqslant \quad \frac{r^2}{(r-1)^{5/2}}\,(1+\circ(1))\exp(-s)$$

It can be shown that the relaxation time to quasi-stationarity is at least of order N, so the order is optimal here.

A finite birth and death example with $\lambda_0 \gg \lambda_1 - \lambda_0$ (1)

The setting is as in the previous example, except that now $r < 1$. But the behavior of our quantities of interest are very different for large N:

$$\begin{aligned} a_\varphi &\leqslant \frac{2N}{(1-r)r^{(N-1)/2}}(1+\circ(1)) \\ \lambda_0 &\sim (1-\sqrt{r})^2 \end{aligned}$$

and

$$\frac{(1-r)^2\sqrt{r}}{2N^2}(1+\circ(1)) \leqslant \lambda_1 - \lambda_0 \leqslant \frac{16\pi^2-(1-r)^2}{4N^2}\sqrt{r}(1+\circ(1))$$

In particular absorption happens at a much faster rate than convergence to quasi-stationarity, since $\lambda_0 \gg \lambda_1 - \lambda_0$.

A finite birth and death example with $\lambda_0 \gg \lambda_1 - \lambda_0$ (2)

Taking furthermore into account that $\eta_\wedge \sim (1-r)r^{N-1}$, we get

$$\sup_{\mu_0\in\mathcal{P}} \|\mu_t - \nu\|_{\mathrm{tv}} \leqslant \frac{4\sqrt{1+r}N^2}{(1-r)^{5/2}r^{3(N-1)/2}}(1+\circ(1)) \exp\left(-\frac{(1-r)^2\sqrt{r}}{2N^2}(1+\circ(1))t\right)$$

In particular, for any given $\epsilon > 0$, if we consider

$$t_N := 4(1+\epsilon)\frac{N^2\ln(N)}{(1-r)^2\sqrt{r}}$$

then

$$\lim_{N\to\infty}\sup_{\mu_0\in\mathcal{P}} \|\mu_{t_N} - \nu\|_{\mathrm{tv}} = 0$$

A non-reversible example

For fixed $N \in \mathbb{N}$, consider $S = \mathbb{Z}_N$ endowed with the (turning) generator L

$$\forall\, x, y \in \mathbb{Z}_N, \qquad L(x,y) \;\; := \;\; \begin{cases} 1 & \text{, if } y = x+1 \\ -1 & \text{, if } y = x \\ 0 & \text{, otherwise} \end{cases}$$

whose invariant probability measure η is the uniform distribution.
The potential V takes the value 1 at 0 and 0 otherwise.
We can show that

$$\sup_{\mu_0 \in \mathcal{P}} \|\mu_t - \nu\|_{\mathrm{tv}} \;\; \leqslant \;\; 2\sqrt{N}(1+\circ(1)) \exp\left(\frac{2\pi^2}{N^2}(1+\circ(1))t\right)$$

In particular, for any given $\epsilon > 0$, if we consider

$$t_N \;\; := \;\; (1+\epsilon)\frac{N^2 \ln(N)}{4\pi^2}$$

then

$$\lim_{N\to\infty} \sup_{\mu_0 \in \mathcal{P}} \|\mu_{t_N} - \nu\|_{\mathrm{tv}} \;\; = \;\; 0$$

Probabilistic interpretation of a_φ

How to estimate a_φ in practice?
First resort to the following probabilistic interpretation due to **Jacka and Roberts** [1995]. For any $x, y \in \bar{S}$, denote by τ_y^x the reaching time of y by X^x, the Markov process starting from x:

$$\tau_y^x \quad := \quad \inf\{t \geqslant 0 \, : \, X_t^x = y\} \, \in \, \mathbb{R}_+ \sqcup \{+\infty\}$$

Proposition

For any $x, y \in S$, we have

$$\frac{\varphi(x)}{\varphi(y)} \quad = \quad \mathbb{E}[\exp(\lambda_0 \tau_y^x) \mathbf{1}_{\tau_y^x < \tau_0^x}]$$

In particular, with $O := \{x \in S \, : \, \bar{L}(x, 0) > 0\}$, we have

$$a_\varphi \quad = \quad \max_{x \in S, \, y \in O} \mathbb{E}[\exp(\lambda_0 \tau_y^x) \mathbf{1}_{\tau_y^x < \tau_0^x}]$$

A path method

This result leads to two methods of estimating a_φ.
The first one is through a path argument.
If $\gamma = (\gamma_0, \gamma_1, ..., \gamma_l)$ is a path in S, denote

$$P(\gamma) \quad := \quad \prod_{k \in [\![0, l-1]\!]} \frac{\bar{L}(\gamma_k, \gamma_{k+1})}{|\bar{L}(\gamma_k, \gamma_k)| - \lambda_0}$$

Proposition

Assume that for any $y \in O$ and $x \in S$, we are given a path $\gamma_{y,x}$ going from y to x. Then we have

$$a_\varphi \quad \leqslant \quad \left(\min_{y \in O,\, x \in S} P(\gamma_{y,x}) \right)^{-1}$$

An example for the path method

Let G be the oriented graph induced by L on S, denote by d its maximum outgoing degree and by D its "oriented diameter". Let $0 < \rho_* \leqslant \rho^*$ be such that

$$\forall\ x \neq y \in \bar{S}, \qquad \bar{L}(x,y) \quad \in \quad \{0\} \sqcup [\rho_*, \rho^*]$$

Then we get

$$a_\varphi \quad \leqslant \quad \left(\frac{\rho^* d}{\rho_*}\right)^D$$

In the previous finite birth and death examples, it gives a bound on a_φ exploding exponentially in $D = N$, which is the true behavior only if $r > 1$. The second method based on spectral estimates enables to recover the fact that φ_N explodes linearly in N for $r = 1$ and is bounded if $0 < r < 1$.

Spectral estimates

Assume that η is reversible for L. The operator $-K$ is then diagonalizable, denote $\lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{N-1}$ its eigenvalues ($N = \mathrm{card}(S)$). For any $x \in S$, let $\lambda_0(S \backslash \{x\})$ be the first eigenvalue of the $(S \backslash \{x\}) \times (S \backslash \{x\})$ minor of $-K$. Finally, consider

$$\lambda_0' \quad := \quad \min_{x \in O} \lambda_0(S \backslash \{x\})$$

Proposition

Under the reversibility assumption, we have

$$a_\varphi \quad \leqslant \quad \left(\left(1 - \frac{\lambda_0}{\lambda_0'}\right) \prod_{k \in [\![N-1]\!]} \left(1 - \frac{\lambda_0}{\lambda_k}\right) \right)^{-1}$$

Under appropriate assumptions, it can be extended to denumerable state spaces.

Denumerable birth and death (1)

Consider $S := \mathbb{N}$ and $\bar{S} := \mathbb{Z}_+$, endowed with a **birth and death generator** $\bar{L}$: namely of the form

$$\forall\; x \neq y \in \bar{S}, \qquad \bar{L}(x,y) \;=\; \begin{cases} b_x & \text{, if } y = x+1 \\ d_x & \text{, if } y = x-1 \\ -d_x - b_x & \text{, if } y = x \\ 0 & \text{, otherwise} \end{cases}$$

where $(b_x)_{x\in\mathbb{Z}_+}$ and $(d_x)_{x\in\mathbb{N}}$ are the positive birth and death rates, except that $b_0 = 0$: 0 is the absorbing state and the restriction of $\bar{L}$ to $\mathbb{N}$ is irreducible.

The boundary point ∞ is said to be an **entrance boundary** for $\bar{L}$ if the following conditions are met:

$$\sum_{x=1}^{\infty} \frac{1}{\pi_x b_x} \sum_{y=1}^{x} \pi_y \;=\; +\infty \tag{1}$$

$$\sum_{x=1}^{\infty} \frac{1}{\pi_x b_x} \sum_{y=x+1}^{\infty} \pi_y \;<\; +\infty \tag{2}$$

where

$$\forall\ x \in \mathbb{N}, \qquad \pi_x \quad := \quad \begin{cases} 1 & , \text{ if } x = 1 \\ \frac{b_1 b_2 \cdots b_{x-1}}{d_2 d_3 \cdots d_x} & , \text{ if } x \geqslant 2 \end{cases}$$

The probabilistic meanings: (1): for $x \in \mathbb{Z}_+$, X^x, does not explode to ∞ in finite time, (2): these processes "go down from infinity". One consequence of (2): $\sum_{x\in\mathbb{N}} \pi_x < +\infty$ and η is the normalization of π into a probability measure.
(1) and (2) imply that the operator $-K$ has only eigenvalues of multiplicity 1, say the $(\lambda_n)_{n\in\mathbb{Z}_+}$ in increasing order, and **Gong, Mao and Zhang** [2012] have shown that they are well approximated by the eigenvalues of the Neumann restriction of $\bar{L}$ to $[\![0, N]\!]$ for large $N \in \mathbb{N}$. Finally, define

$$\lambda_0' \quad := \quad \lambda_0(\mathbb{N}\backslash\{1\})$$

Extension of the spectral estimate

Theorem

Under the assumptions (1) *and* (2), *we have*

$$\left(1-\frac{\lambda_0}{\lambda_0'}\right)\prod_{n\in\mathbb{N}}\left(1-\frac{\lambda_0}{\lambda_n}\right) \quad > \quad 0$$

The eigenvector φ *is bounded and its amplitude satisfies:*

$$\begin{aligned}\frac{\sup_{x\in\mathbb{N}}\varphi(x)}{\inf_{y\in\mathbb{N}}\varphi(y)} \quad &= \quad \frac{\lim_{x\to\infty}\varphi(x)}{\varphi(1)}\\ &\leqslant \quad \left(\left(1-\frac{\lambda_0}{\lambda_0'}\right)\prod_{n\in\mathbb{N}}\left(1-\frac{\lambda_0}{\lambda_n}\right)\right)^{-1}\end{aligned}$$

There is a somewhat converse statement, via the Lyapounov function approach.

There is a huge literature on quasi-stationarity, with recent surveys provided by **Méléard and Villemonais** [2012], **Van Doorn and Pollett** [2013] or by the book of **Collet, Martínez and San Martín** [2013]. All of these review the history (**Yaglom**, **Bartlett**, **Darroch-Seneta**, ...). An annotated online bibliography is kept up to date by **Pollett** at

`http://www.maths.uf.edu.au/~pkp/papers/qsds.html`.

But the quantitative aspect was not fully investigated, usually only the asymptotical rate $\lambda_1 - \lambda_0$ was identified, but without the pre-exponential factor. See nevertheless **Van Doorn and Pollett** [2013], **Barbour and Pollett** [2010, 2012] or recent preprints of **Cloez and Thai** and of **Champagnat and Villemonais**.
The amplitude a_φ was used by **Jacka and Roberts** [1995] to investigate the process conditioned to have never been absorbed.