An absorbing eigentime identity

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Abstract

Consider a finite irreducible Markov process X. Sampling two points x and y independently according to the invariant measure, the eigentime identity states that the expected time for X to go from x to y is equal to the sum of the inverses of the non-zero eigenvalues of the (opposite of the) underlying generator. This short paper gives a simple proof of this equality and propose a new extension to the finite absorbing irreducible Markov framework, in continuous and discrete times.

Keywords: finite ergodic/absorbing Markov process, eigentime identity, invariant probability, quasi-invariant probability, first Dirichlet eigenvalue/eigenvector, algebraic spectrum.

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1 Introduction

Consider a finite reversible and irreducible Markov process: if two points from the state space are chosen independently according to the invariant probability, the expected time to go from one to the other is equal to the sum of the inverses of the non-zero eigenvalues of the (opposite of the) underlying generator. If it had not been known earlier, this property is due to Broder and Karlin (Theorem 15 of [2]), who rather worked in the discrete time framework, and was later called an eigentime identity by Aldous and Fill in their unpublished book [1]. Mao and his co-authors have extensively studied such identities and in particular have extended them to a lot of settings: transient and absorbing processes, non-reversible processes, jump processes on denumerable state spaces, diffusion processes, see [8, 9, 5, 3]. In this paper we are only interested in finite state spaces and we begin by presenting an alternative proof for ergodic Markov processes. Extending this approach to absorbing Markov processes (which are irreducible outside the absorbing point), we get a new absorbing eigentime identity, which converges to the ergodic one when the asymptotic rate of absorption goes to zero.

We begin by recalling the finite ergodic framework. Let L be an irreducible Markov generator on a finite state space S: it is represented by a matrix $(L(x,y))_{x,y\in S}$ whose off-diagonal entries are non-negative, whose diagonal entries are such that the row sums vanish and which is such that all the entries of $\exp(L)$ are positive. An associated continuous-time Markov process $X \coloneqq (X_t)_{t\geqslant 0}$ is a Markov process whose trajectories are S-valued and right-continuous and whose jump rates are given by the off-diagonal entries of L. Once the initial law of X_0 is given, the law of X is uniquely determined. If for some fixed $x \in S$, $X_0 = x$, we will denote by $\mathbb{P}_x$ and $\mathbb{E}_x$ the probability and the expectation relative to X.

A probability distribution μ on S is said to be invariant for L, if taking X_0 distributed according to μ, then for any $t \geqslant 0$, X_t is equally distributed according to μ. The Perron-Frobenius theorem ensures the existence and the uniqueness of the invariant probability μ.

Let Λ be the multi-set of complex eigenvalues of $-L$, repeated with their algebraic multiplicities. Thus the cardinality of Λ is N, the cardinality of S. We have that $0 \in \Lambda$ and by irreducibility its algebraic multiplicity is 1, the associated eigenspace being the set of constant functions. We will denote $\Lambda_0 = \Lambda\backslash\{0\}$.

Finally define for any $y \in S$,

$$\tau_y \quad \coloneqq \quad \inf\{t \geqslant 0 \,:\, X_t = y\} \tag{1}$$

the reaching time of y by X (by convention $\inf \varnothing = +\infty$, but the irreducibility condition is equivalent to the fact that the event $\{\tau_y = +\infty\}$ is negligible whatever the initial condition).

The following result is due to Aldous and Fill [1] if L is furthermore assumed to be reversible (namely $\mu(x)L(x,y) = \mu(y)L(y,x)$ for any $x, y \in S$) and to Cui and Mao [5] in the general case.

Theorem 1 *For any $x \in S$, we have*

$$\sum_{y\in S} \mathbb{E}_x[\tau_y]\,\mu(y) \quad = \quad \sum_{\lambda\in\Lambda_0} \frac{1}{\lambda}$$

By integration with respect to μ, it implies the eigentime identity

$$\sum_{x,y\in S} \mathbb{E}_x[\tau_y]\,\mu(x)\mu(y) \quad = \quad \sum_{\lambda\in\Lambda_0} \frac{1}{\lambda}$$

As it was noticed by Cui and Mao in Remark 1.2 of [5], it can be seen directly that the above r.h.s. is positive, because Λ_0 is stable by complex conjugation and all its element have positive real parts. In the reversible case, it is even more immediate, since all elements of Λ_0 are positive real numbers.

Our first goal is to give a simple functional proof of Theorem 1, which has the advantage to extend to the absorbing setting to give a new absorbing eigentime identity.

In the finite absorbing framework, we are given a strictly subMarkovian irreducible generator L on the finite set S. The only difference with the above setting is that the row sums are now non-positive and one of them is negative. It is customary to add to S a cemetery point $\infty \notin S$ to get the extended state space $\bar{S} \coloneqq S \sqcup \{\infty\}$. The matrix L is extended to $\bar{S}$ by taking

$$\forall\, x, y \in \bar{S}, \qquad \bar{L}(x,y) \;\coloneqq\; \begin{cases} L(x,y) & \text{, if } x, y \in S \\ -\sum_{z\in S} L(x,z) & \text{, if } x \in S \text{ and } y = \infty \\ 0 & \text{, if } x = \infty \end{cases}$$

The matrix $\bar{L}$ is Markovian on $\bar{S}$ and we can associate to it a $\bar{S}$-valued Markov process $X \coloneqq (X_t)_{t\geqslant 0}$ as above. It is absorbing at ∞, in the sense that when X reaches ∞, it stays there forever afterward. By our assumptions, whatever the initial distribution on $\bar{S}$, X a.s. reaches ∞, namely the absorption time τ_∞ (extending the notation (1)) is a.s. finite.

The Perron-Frobenius theorem can also be applied in this situation and it gives the following informations (see for instance the book [4] of Collet, Martínez and San Martín):

• The algebraic spectrum Λ of $-L$ contains a real element λ_0 which is strictly less than the real parts of the other elements of Λ. We denote $\Lambda_0 \coloneqq \Lambda \backslash \{\lambda_0\}$.

• There exists a unique probability ν, called the quasi-stationary distribution of L, such that if X_0 is distributed according to ν, then for any $t \geqslant 0$, the restriction to S of law of X_t is proportional to ν (next it follows more precisely that the law of X_t is $(1-\exp(-\lambda_0 t))\delta_\infty + \exp(-\lambda_0 t)\nu)$. The probability ν is the unique normalized positive eigenmeasure associated to L.

• If φ is an eigenfunction associated to the eigenvalue $-\lambda_0$ of L, then φ has a fixed sign. In the sequel we assume that φ is positive and normalized so that $\nu[\varphi] = 1$. We denote μ the probability admitting φ as density with respect to ν.

Our main result is the following extension of the eigentime identity to the absorbing setting:

Theorem 2 *We have*

$$\sum_{x,y\in S} \mathbb{E}_x\left[\int_0^{\tau_{y,\infty}} \varphi(X_s)\exp(\lambda_0 s)\,ds\right]\nu(x)\mu(y) \;=\; \sum_{\lambda\in\Lambda_0} \frac{1}{\lambda-\lambda_0}$$

where for any $y \in S$,

$$\tau_{y,\infty} \;\coloneqq\; \inf\{t \geqslant 0 \,:\, X_t \in \{y,\infty\}\}$$

The observation following the statement of Theorem 1 is equally valid here, since Λ_0 is stable by complex conjugation and the real parts of its elements are strictly larger than λ_0.

Cui and Mao [5] have also proposed an extension of the eigentime identity to absorbing Markov processes, but it is of a different nature. It says that

$$\sum_{\lambda\in\Lambda} \frac{1}{\lambda} \;=\; \sum_{x\in S} \frac{1}{|L(x,x)|\,\mathbb{P}_x[\tau_x^+ = +\infty]}$$

where for $x \in S$, τ_x^+ is the first time X jumps to x from another position.

Remark 3 When the chain is indeed ergodic, we have $\lambda_0 = 0$, $\varphi = 1$ and $\nu = \mu$, so that Theorem 2, and the following Proposition 4 and Corollary 5, are all reduced to Theorem 1. In some sense, Theorem 2 could be called an intrinsic eigentime identity, different from that obtained by Cui and Mao [5]. It may also be informative to see how Theorem 1 is "strongly approximated" by Theorem 2. Consider the following academic example.
Let L be an ergodic Markov generator as in Theorem 1. For $\epsilon > 0$, consider the subMarkovian

generator $L^{(\epsilon)} := L - \epsilon I$, where I is the $S \times S$-identity matrix. The invariant measure μ of L is also the quasi-stationary measure $\nu^{(\epsilon)}$ of $L^{(\epsilon)}$. The eigenfunction $\varphi^{(\epsilon)}$ is exactly $\mathbb{1}$, so that the probability $\mu^{(\epsilon)}$ is also equal to μ. The algebraic spectrum $\Lambda^{(\epsilon)}$ of $-L^{(\epsilon)}$ is $\Lambda - \epsilon$, where Λ is the algebraic spectrum of $-L$. In particular $\lambda_0^{(\epsilon)} = \epsilon$ and thus

$$\sum_{\lambda \in \Lambda_0^{(\epsilon)}} \frac{1}{\lambda - \lambda_0^{(\epsilon)}} = \sum_{\lambda \in \Lambda_0} \frac{1}{\lambda}$$

Furthermore, an absorbing Markov process $X^{(\epsilon)}$ associated to $L^{(\epsilon)}$ can be constructed from a Markov process X associated to L with the same initial condition via

$$\forall\, t \geqslant 0, \qquad X_t^{(\epsilon)} := \begin{cases} X_t & \text{, if } t < T^{(\epsilon)} \\ \infty & \text{, if } t \geqslant T^{(\epsilon)} \end{cases}$$

where $T^{(\epsilon)}$ is a exponential random variable of parameter ϵ independent of X. In particular, we have $\tau_{y,\infty}^{(\epsilon)} = \tau_y \wedge T^{(\epsilon)}$, where τ_y is as in (1). We thus get that

$$\sum_{x,y \in S} \mathbb{E}_x\left[\int_0^{\tau_{y,\infty}^{(\epsilon)}} \varphi^{(\epsilon)}(X_s) \exp(\lambda_0 s)\, ds\right] \nu^{(\epsilon)}(x)\mu^{(\epsilon)}(y) = \sum_{x,y \in S} \mathbb{E}_x\left[\frac{\exp(\lambda_0(\tau_y \wedge T^{(\epsilon)})) - 1}{\lambda_0}\right] \mu(x)\mu(y)$$

An elementary computation shows that

$$\forall\, t \geqslant 0, \qquad \mathbb{E}\left[\frac{\exp(\lambda_0(t \wedge T^{(\epsilon)})) - 1}{\lambda_0}\right] = t$$

Indeed, we have

$$\begin{aligned} \mathbb{E}\left[\exp(\lambda_0(t \wedge T^{(\epsilon)}))\right]/\lambda_0 &= \int_0^{+\infty} \exp(\lambda_0(t \wedge s))\, \exp(-\lambda_0 s) ds \\ &= \int_0^t ds + \int_t^{+\infty} \exp(\lambda_0(t-s))\, ds \\ &= t + 1/\lambda_0 \end{aligned}$$

It follows that for any $\epsilon > 0$ and $x \in S$,

$$\mathbb{E}_x\left[\frac{\exp(\lambda_0(\tau_y \wedge T^{(\epsilon)})) - 1}{\lambda_0}\right] = \mathbb{E}_x[\tau_y]$$

Thus, for any fixed $\epsilon > 0$, the eigentime identity of Theorem 1 for L is "equivalent" to the eigentime identity of Theorem 2 for $L^{(\epsilon)}$.

□

We come back to the general framework of finite irreducible absorbing Markov processes. The following result is an analogous property of the first part of Theorem 1.

Proposition 4 *For any $x \in S$, we have*

$$\varphi(x) = \left(\sum_{\lambda \in \Lambda_0} \frac{1}{\lambda - \lambda_0}\right)^{-1} \sum_{y \in S} \mathbb{E}_x\left[\int_0^{\tau_{y,\infty}} \varphi(X_s) \exp(\lambda_0 s)\, ds\right] \mu(y)$$

Let us give some immediate consequences of Theorem 2 and Proposition 4 which make them more similar to Theorem 1. It is convenient to consider the amplitude of φ, introduced and studied in [6]:

$$a_\varphi \;\coloneqq\; \frac{\max_S \varphi}{\min_S \varphi} \;\geqslant\; \max_S \varphi \tag{2}$$

In particular it was seen in [6] that under the reversibility assumption (there exists a probability measure η on S such that $\eta(x)L(x,y)$ is symmetrical with respect to the couple (x,y)), we have

$$a_\varphi \;\leqslant\; \left(\left(1-\frac{\lambda_0}{\lambda_0'}\right)\prod_{\lambda\in\Lambda_0}\left(1-\frac{\lambda_0}{\lambda}\right)\right)^{-1}$$

where

$$\lambda_0' \;\coloneqq\; \min_{x\in S\,:\,\bar{L}(x,\infty)>0} \lambda_0(S\backslash\{x\})$$

and $\lambda_0(S\backslash\{x\})$ is the Perron-Frobenius smallest eigenvalue of the restriction of $-L$ to $S\backslash\{x\}$.

Corollary 5 *For all $x\in S$, we have*

$$a_\varphi^{-2}\sum_{\lambda\in\Lambda_0}\frac{1}{\lambda-\lambda_0} \;\leqslant\; \sum_{y\in S}\mathbb{E}_x\left[\frac{\exp(\lambda_0\tau_{y,\infty})-1}{\lambda_0}\right]\nu(y) \;\leqslant\; a_\varphi^2\sum_{\lambda\in\Lambda_0}\frac{1}{\lambda-\lambda_0}$$

and in particular

$$\sum_{y\in S}\mathbb{E}_x\left[\tau_{y,\infty}\right]\nu(y) \;\leqslant\; a_\varphi^2\sum_{\lambda\in\Lambda_0}\frac{1}{\lambda-\lambda_0}$$

By integration of these bounds with respect to ν, one gets eigentime inequalities where the starting and ending points x and y play symmetrical roles.

Probably that under appropriate assumptions, the previous results can be extended to denumerable infinite state spaces. The extension to diffusion processes seems more challenging. But these frameworks are left for future investigations.

The plan of the paper is as follows: in next section we recover Theorem 1 with a simple functional proof. The situation of absorbing Markov processes is treated in Section 3. The last section extends the previous considerations to the setting of discrete time.

2 The ergodic case

The known results concerning the ergodic eigentime identity presented in Theorem 1 are recovered here via an elementary approach.

The underlying simple linear algebra principle is as follows:

Lemma 6 *Let $\mathcal{F}$ be an Euclidean space of dimension $N\in\mathbb{N}$ whose scalar product is denoted by $\langle\cdot,\cdot\rangle$. Consider $\mathcal{F}_0$ a subspace of $\mathcal{F}$ and G an endomorphism of $\mathcal{F}_0$. If $(e_n)_{n\in[\![N]\!]}$ is an orthonormal basis of $\mathcal{F}$, consider for any $n\in[\![N]\!]$, g_n the orthogonal projection of e_n on $\mathcal{F}_0$. Then the trace of G is given by*

$$\mathrm{tr}(G) \;=\; \sum_{n\in[\![N]\!]}\langle g_n, Gg_n\rangle$$

Proof

This result is well-known if $\mathcal{F}_0 = \mathcal{F}$: then the diagonal of the matrix of G expressed in the basis $(e_n)_{n\in[\![N]\!]}$ is $(\langle e_n, Ge_n\rangle)_{n\in[\![N]\!]}$. In the general case, consider the extension $\bar{G}$ of G on $\mathcal{F}$ which vanishes on the orthogonal complement of $\mathcal{F}_0$. It appears that

$$\begin{aligned}\mathrm{tr}(G) &= \mathrm{tr}(\bar{G})\\ &= \sum_{n\in[\![N]\!]} \langle e_n, \bar{G}e_n\rangle\\ &= \sum_{n\in[\![N]\!]} \langle g_n, Gg_n\rangle\end{aligned}$$

■

This result is applied with $\mathcal{F} \coloneqq \mathbb{L}^2(\mu)$, endowed with its natural scalar product and with the orthonormal basis $(\delta_y/\sqrt{\mu(y)})_{y\in S}$, where δ_y is the function from $\mathcal{F}$ taking the value 1 at y and 0 elsewhere. For the subspace $\mathcal{F}_0$, consider

$$\mathcal{F}_0 \;\coloneqq\; \{f \in \mathcal{F} \,:\, \mu[f] = 0\} \tag{3}$$

The orthogonal projection of the basis $(\delta_y/\sqrt{\mu(y)})_{y\in S}$ on $\mathcal{F}_0$ is $(g_y)_{y\in S}$, where

$$\forall\, y \in S, \qquad g_y \;\coloneqq\; \frac{\delta_y}{\sqrt{\mu(y)}} - \sqrt{\mu(y)} \tag{4}$$

The invariance of μ is equivalent to the property that

$$\forall\, f \in \mathcal{F}, \qquad \mu[L[f]] \;=\; 0$$

namely the image of $\mathcal{F}$ by L is included into $\mathcal{F}_0$. By irreducibility, we know that the kernel of L is the set of constant functions, thus the restriction of $-L$ to $\mathcal{F}_0$ is a bijective endomorphism, whose algebraic spectrum is Λ_0. Consider G its inverse operator, which is an endomorphism of $\mathcal{F}_0$ whose algebraic spectrum is

$$\left\{\frac{1}{\lambda} \,:\, \lambda \in \Lambda_0\right\}$$

(write $-L_{|\mathcal{F}_0}$ in a basis where the associated matrix is upper diagonal).

From the previous lemma, we get

$$\sum_{y\in S} \langle g_y, G[g_y]\rangle \;=\; \sum_{\lambda\in\Lambda_0} \frac{1}{\lambda}$$

The following result enables to conclude to the ergodic eigentime identity.

Proposition 7 *For any fixed $y \in S$, we have*

$$\langle g_y, G[g_y]\rangle \;=\; \mu(y) \sum_{x\in S} \mathbb{E}_x[\tau_y]\, \mu(x)$$

Proof

For given $y \in S$, consider the mapping $f_y \in \mathcal{F}$ defined by

$$\forall\, x \in S, \qquad f_y(x) \;\coloneqq\; \mathbb{E}_x[\tau_y] \tag{5}$$

It is well-known that f_y is characterized by the fact that

$$\left\{\begin{array}{rcl} L[f_y] & = & -1 \text{ on } S\backslash\{y\} \\ f_y(y) & = & 0 \end{array}\right. \tag{6}$$

Since $L[f_y]$ belongs to $\mathcal{F}_0$, we get that $L[f_y](y) = (1-\mu(y))/\mu(y)$ and it appears that

$$L[f_y] \quad = \quad \frac{g_y}{\sqrt{\mu(y)}}$$

In particular we deduce that

$$G[g_y] \quad = \quad -\sqrt{\mu(y)}\widetilde{f}_y \tag{7}$$

where $\widetilde{f}_y \coloneqq f_y - \mu[f_y] \in \mathcal{F}_0$.

From (6) we get that

$$\begin{aligned} \langle f_y, L[f_y]\rangle &= -\sum_{x\neq y} f_y(x)\,\mu(x) \\ &= -\sum_{x\in S} f_y(x)\,\mu(x) \end{aligned}$$

By invariance and since L vanishes on constant functions,

$$\left\langle \widetilde{f}_y, L[\widetilde{f}_y]\right\rangle \quad = \quad \langle f_y, L[f_y]\rangle$$

It remains to use (7) to conclude to the wanted identity.

■

To be self-contained, let us give a probabilistic proof of the characterization (6). The same approach will also be useful in the absorbing case. It consists in exchanging the roles of the known and unknown functions and f_y and $g_y/\sqrt{\mu(y)}$ in the Poisson equation

$$\left\{\begin{array}{rcl} L[f_y] & = & \frac{g_y}{\sqrt{\mu(y)}} \\ f_y(y) & = & 0 \end{array}\right. \tag{8}$$

Since $L_{|\mathcal{F}_0}$ is bijective, there is a unique function $\widetilde{f}_y \in \mathcal{F}_0$ such that $L[\widetilde{f}_y] = g_y/\sqrt{\mu(y)}$. Thus we know a priori there is a unique solution f_y to (8), it is given by $f_y = \widetilde{f}_y - \widetilde{f}_y(y)$.

Lemma 8 *The unique solution f_y of (8) is given by (5).*

Proof

Recall that the law of a Markov process X associated to L is a solution to the following martingale problem: for any $f \in \mathcal{F}$, the process

$$\forall\ t \geqslant 0, \qquad M_t[f] \quad \coloneqq \quad f(X_t) - f(X_0) - \int_0^t L[f](X_s)\,ds$$

is a martingale. In particular, the process $(M_{t\wedge\tau_y}[f])_{t\geqslant 0}$ is a bounded martingale, so we get for any fixed $x \in S$ and $t \geqslant 0$,

$$\mathbb{E}_x\left[f(X_{t\wedge\tau_y}) - f(X_0) - \int_0^{t\wedge\tau_y} L[f](X_s)\,ds\right] \quad = \quad 0$$

i.e.

$$f(x) \quad = \quad \mathbb{E}_x\left[f(X_{t\wedge\tau_y}) - \int_0^{t\wedge\tau_y} L[f](X_s)\,ds\right]$$

Replace f by f_y, to see that

$$f(x) \quad = \quad \mathbb{E}_x\left[f(X_{t\wedge\tau_y}) + t\wedge\tau_y\right]$$

and letting t go to infinity it appears that

$$f(x) \quad = \quad \mathbb{E}_x[\tau_y]$$

as announced.

■

It remains to show the first part of Theorem 1, namely that the function $f := \sum_{y\in S} f_y\,\mu(y)$ is constant. It follows immediately by applying L:

$$\begin{aligned} L[f] &= \sum_{y\in S} L[f_y]\,\mu(y) \\ &= \sum_{y\in S} \frac{g_y}{\sqrt{\mu(y)}}\mu(y) \\ &= \sum_{y\in S} \delta_y - \mu(y) \;=\; 1-1 \;=\; 0 \end{aligned}$$

3 The absorbing case

It is seen here how the arguments of the previous section can be extended to the absorbing situation to prove Theorem 2 and Proposition 4

Again we apply Lemma 6 to the Euclidean space $\mathcal{F} := \mathbb{L}^2(\mu)$, where μ has density φ with respect to ν, the quasi-stationary distribution. We equally consider the basis $(\delta_y/\sqrt{\mu(y)})_{y\in S}$ and the subspace $\mathcal{F}_0$ defined in (3). Thus the family of functions $(g_y)_{y\in S}$, introduced in (4), will play an important role.

The main difference with Section 2 is that the generator L is replaced by the operator $\widetilde{L}$ acting on $\mathcal{F}$ by

$$\forall\, f\in\mathcal{F}, \qquad \widetilde{L}[f] \quad := \quad \frac{1}{\varphi}(L+\lambda_0)[\varphi f]$$

It is quite natural, since $\widetilde{L}$ is an ergodic Markov generator whose convergence to equilibrium is strongly related to the absorption for L, as it was seen in [7]. Note that the invariant probability of $\widetilde{L}$ is μ. Indeed, the quasi-invariance of ν is equivalent to the property that

$$\forall\, f\in\mathcal{F}, \qquad \nu[L[f]] \quad = \quad -\lambda_0\nu[f]$$

so that

$$\forall\, f\in\mathcal{F}, \qquad \mu[\widetilde{L}[f]] \;=\; \nu[(L+\lambda_0)[\varphi f]] \;=\; 0$$

In particular the image of $\mathcal{F}$ by $\widetilde{L}$ is included into $\mathcal{F}_0$. By the Perron-Frobenius theorem, the kernel of $L+\lambda_0$ is of dimension 1 and generated by the positive function φ. It follows that the kernel of $\widetilde{L}$ consists of the constant functions (as it should be for an ergodic Markovian generator) and thus the restriction of $-\widetilde{L}$ to $\mathcal{F}_0$ is an bijective endomorphism of $\mathcal{F}_0$. Denote by G its inverse. Its algebraic spectrum is

$$\left\{\frac{1}{\lambda-\lambda_0}\ :\ \lambda\in\Lambda_0\right\}$$

From Lemma 6, we deduce that

$$\sum_{y\in S}\langle g_y, G[g_y]\rangle \quad=\quad \sum_{\lambda\in\Lambda_0}\frac{1}{\lambda-\lambda_0}$$

It leads to consider, for any fixed $y\in S$, the solution f_y of the equation

$$\left\{\begin{array}{lcl}\widetilde{L}[f_y] &=& \frac{g_y}{\sqrt{\mu(y)}}\\ f_y(y) &=& 0\end{array}\right. \tag{9}$$

Indeed, it is given by $f_y = -(G[g_y]-G[g_y](y)))/\sqrt{\mu(y)}$, or equivalently $G[g_y] = -\sqrt{\mu(y)}(f_y - \mu[f_y])$. The proof of Proposition 7, where L is replaced by $\widetilde{L}$, shows that

$$\begin{aligned}\langle g_y, G[g_y]\rangle &= -\mu(y)\left\langle\widetilde{L}[f_y-\mu[f_y]], f_y-\mu[f_y]\right\rangle\\ &= -\mu(y)\left\langle f_y, \widetilde{L}[f_y]\right\rangle\\ &= \mu(y)\sum_{x\in S} f_y(x)\,\mu(x)\end{aligned}$$

The next result is analogous to Lemma 8 in the identification of f_y. It ends the proof of Theorem 2, since $\mu(x)/\varphi(x)=\nu(x)$ for all $x\in S$.

Lemma 9 *For any fixed $y\in S$, the mapping f_y is given by*

$$\forall\ x\in S,\qquad f_y(x) \quad=\quad \frac{1}{\varphi(x)}\mathbb{E}_x\left[\int_0^{\tau_{y,\infty}}\varphi(X_s)\exp(\lambda_0 s)\,ds\right]$$

Proof

Let X be the absorbing process associated to L. The martingale problem solved by its law can be extended into a time-space version, sometimes called Dynkin's lemma: For any function $f:\mathbb{R}_+\times S\to\mathbb{R}$ which is $\mathcal{C}^1$ with respect to the first (time) variable, the process

$$\forall\ t\geqslant 0,\qquad M_t[h] \quad:=\quad h(t,X_t)-h(0,X_0)-\int_0^t(\partial_s h(s,\cdot)+L[h(s,\cdot)])(X_s)\,ds$$

is a martingale (by convention, all the functions defined on S are extended to $\bar{S}$ by making them vanish at ∞). In particular, the process $(M_{t\wedge\tau_{y,\infty}}[h])_{t\geqslant 0}$ is a bounded martingale, so we get for any fixed $x\in S$ and $t\geqslant 0$,

$$\mathbb{E}_x\left[h(t\wedge\tau_{y,\infty}, X_{t\wedge\tau_{y,\infty}})-h(0,X_0)-\int_0^{t\wedge\tau_{y,\infty}}(\partial_s h(s,\cdot)+L[h(s,\cdot)])(X_s)\,ds\right] \quad=\quad 0$$

i.e.

$$h(0,x) \quad=\quad \mathbb{E}_x\left[h(t\wedge\tau_{y,\infty}, X_{t\wedge\tau_{y,\infty}})-\int_0^{t\wedge\tau_{y,\infty}}(\partial_s h(s,\cdot)+L[h(s,\cdot)])(X_s)\,ds\right]$$

Consider the function f defined by

$$\forall\ s\geqslant 0,\ z\in S,\qquad h(s,z) \quad:=\quad \exp(\lambda_0 t)\varphi(z)f_y(z)$$

where $y \in S$ is fixed as in the statement of the lemma. Taking into account that

$$(L+\lambda_0)[\varphi f_y] \quad = \quad \varphi \frac{g_y}{\sqrt{\mu(y)}}$$

we get

$$\forall\ s \geqslant 0,\ \forall\ z \in S\backslash\{y\}, \qquad (\partial_s h(s,\cdot) + L[h(s,\cdot)])(z) \quad = \quad -\exp(\lambda_0 s)\varphi(z) \tag{10}$$

so that for any $x \in S$ and $t \geqslant 0$,

$$\varphi(x)f_y(x) \quad = \quad \mathbb{E}_x\left[\exp(\lambda_0(t\wedge\tau_{y,\infty}))f_y(X_{t\wedge\tau_{y,\infty}}) + \int_0^{t\wedge\tau_{y,\infty}} \exp(\lambda_0 s)\varphi(X_s)\,ds\right] \tag{11}$$

By monotone convergence, we have

$$\lim_{t\to+\infty} \mathbb{E}_x\left[\int_0^{t\wedge\tau_{y,\infty}} \exp(\lambda_0 s)\varphi(X_s)\,ds\right] \quad = \quad \mathbb{E}_x\left[\int_0^{\tau_{y,\infty}} \exp(\lambda_0 s)\varphi(X_s)\,ds\right]$$

To see that

$$\begin{aligned}\lim_{t\to+\infty} \mathbb{E}_x\left[\exp(\lambda_0(t\wedge\tau_{y,\infty}))f_y(X_{t\wedge\tau_{y,\infty}})\right] \quad &= \quad \mathbb{E}_x\left[\exp(\lambda_0\tau_{y,\infty})f_y(X_{\tau_{y,\infty}})\right] \\ &= \quad 0\end{aligned}$$

we would like to apply the dominated convergence theorem. Namely we wish that $\mathbb{E}_x\left[\exp(\lambda_0\tau_{y,\infty})\right] < +\infty$. It is true as a consequence of two well-known facts (for a proof, see e.g. Lemma 6 and Lemma 8 of [6]):

• For any $x \in S\backslash\{y\}$, $\mathbb{E}_x\left[\exp(l\tau_{y,\infty})\right] < +\infty$ if and only if $l < \lambda_0(S\backslash\{y\})$, where $\lambda_0(S\backslash\{y\})$ was defined just before Corollary 5.

• Due to the irreducibility of L, $\lambda_0(S\backslash\{y\}) > \lambda_0$, which means that the underlying process goes out from $S\backslash\{y\}$ with a (strictly) better asymptotical rate than from S.

We are thus allowed to let t go to infinity in (11) to obtain the announced formula for f_y.

■

The proof of Proposition 4 is similar to that of the first part of Theorem 1: considering the function $f \coloneqq \sum_{y\in S} f_y\mu(y)$, it appears that $\widetilde{L}[f] = 0$, so that f must be constant. The eigentime identity asserts that $\mu[f] = \sum_{\lambda\in\Lambda_0} 1/(\lambda-\lambda_0)$ and we deduce that

$$\forall\ x \in S, \qquad f(x) \quad = \quad \sum_{\lambda\in\Lambda_0} \frac{1}{\lambda-\lambda_0}$$

which can be rewritten under the form given in Proposition 4.

4 The discrete-time setting

All the previous considerations can be adapted to the setting of discrete time. After recalling it in the ergodic and absorbing cases, we state the corresponding results and present the slight modifications needed in the arguments.

In the ergodic case, we are given an irreducible Markov transition matrix $P \coloneqq (P(x,y))_{x,y\in S}$ on the finite state space S. The associated Markov chains $X \coloneqq (X_n)_{n\in\mathbb{Z}_+}$ are those whose transition probabilities are dictated by P. For $x \in S$, we denote $\mathbb{E}_x$ the expectation relative to X when $X_0 = x$. For any $y \in S$, τ_y stands for the reaching time of y by X and is defined formally as in (1). We denote by μ the invariant probability for P (i.e. satisfying $\mu P = \mu$) and by Θ the algebraic spectrum of P. The multiplicity of $1 \in \Theta$ is 1 and let $\Theta_0 \coloneqq \Theta\backslash\{1\}$.

The following result is again due to Aldous and Fill [1] for the reversible Markov chains and to Cui and Mao [5] in the general case.

Theorem 10 *For any $x \in S$, we have*

$$\sum_{y\in S} \mathbb{E}_x[\tau_y]\,\mu(y) \quad = \quad \sum_{\theta\in\Theta_0} \frac{1}{1-\theta}$$

By integration with respect to μ, it implies the eigentime identity

$$\sum_{x,y\in S} \mathbb{E}_x[\tau_y]\,\mu(x)\mu(y) \quad = \quad \sum_{\theta\in\Theta_0} \frac{1}{1-\theta}$$

In the absorbing situation, P is a strictly subMarkovian transition matrix on the finite S. As usual, it can be extended into a Markov transition matrix $\bar{P}$ by adding a cemetery point ∞ to S. This enables to consider the associated Markov chains $X := (X_n)_{n\in\mathbb{Z}_+}$, with the corresponding notions, $\mathbb{E}_x$, $\tau_{x,\infty}$, for $x \in S$, etc. Let Θ be the algebraic spectrum of P. The Perron-Frobenius theory enables to see that in Θ there is a real element θ_0 which is (strictly) larger than the real parts of all the other eigenvalues. Furthermore, there exists a quasi-stationary probability ν characterized by $\nu P = \theta_0\nu$, and let φ be the positive function satisfying $P\varphi = \theta_0\varphi$ with $\nu[\varphi] = 1$. As before, denote $\Theta_0 = \Theta\backslash\{\theta_0\}$ and μ the probability admitting the density φ with respect to ν. With these notations, we can state an absorbing discrete-time eigentime identity:

Theorem 11 *We have*

$$\sum_{x,y\in S} \mathbb{E}_x\left[\sum_{n=0}^{\tau_{y,\infty}-1} \theta_0^{-n-1}\varphi(X_n)\right]\nu(x)\mu(y) \quad = \quad \sum_{\theta\in\Theta_0} \frac{1}{\theta_0-\theta}$$

and more precisely, for any $x \in S$,

$$\varphi(x) \quad = \quad \left(\sum_{\theta\in\Theta_0} \frac{\theta_0}{\theta_0-\theta}\right)^{-1} \sum_{y\in S} \mathbb{E}_x\left[\sum_{n=0}^{\tau_{y,\infty}-1} \theta_0^{-n}\varphi(X_n)\right]\mu(y)$$

Considering $L := P - I$ (with I the $S \times S$-identity matrix), which is an irreducible Markov (respectively strict subMarkov) generator in the ergodic (resp. absorbing) case, the functional arguments are exactly the same as in the continuous time. The differences appear with the probabilistic interpretations, namely in the proofs of Lemmas 8 and 9. But they are quite minor. In the ergodic case, for $f \in \mathcal{F}$, define the discrete-time martingale $M[f]$ by

$$\forall\ n \in \mathbb{Z}_+, \qquad M_n[f] \quad := \quad f(X_n) - f(X_0) - \sum_{m=0}^{n-1} (P-I)[f](X_m)$$

and consider for fixed $y \in S$, the martingale $(M_{m\wedge\tau_y}[f_y])_{m\in\mathbb{Z}_+}$.

In the absorbing situation, one rather use the time-space martingales $M[h]$, where h is a mapping from $\mathbb{Z}_+ \times S$ to $\mathbb{R}$, defined by

$$\forall\ n \in \mathbb{Z}_+, \qquad M_n[h] \quad := \quad h_n(X_n) - h_0(X_0) - \sum_{m=0}^{n-1} (P-I)[h_{m+1}](X_m) + (h_{m+1}-h_m)(X_m)$$

More precisely one needs to stop them at $\tau_{y,\infty}$, for fixed $y \in S$, and relatively to the function

$$f\ :\ \mathbb{Z}_+ \times S \ni (m,z) \quad \mapsto \quad h_m(z) := \theta_0^m\varphi(z)f_y(z)$$

Its interest is that

$$\forall\ m \in \mathbb{Z}_+,\ \forall\ z \in S\backslash\{y\}, \qquad (P-I)[h_{m+1}](z) + (h_{m+1}-h_m)(z) \quad = \quad -\theta_0^{-m-1}\varphi(z)$$

which is the analogous property to (10). Furthermore the two points mentioned at the end of the proof of Lemma 9 are equally satisfied.

• For the first point, one has to take into account that $\tau_{y,\infty}$ is a geometric variable of parameter $\theta_0(S\backslash\{y\})$ (the Perron-Frobenius largest eigenvalue of the restriction of P to $S\backslash\{y\}$) when X_0 is started from the quasi-stationary distribution (instead of a exponential variable of parameter $\lambda_0(S\backslash\{y\})$).

• For the second point, $\theta_0(S\backslash\{y\}) < \theta_0$, it comes directly from the corresponding assertion for the associated subMarkovian generator L, it is indeed a result of functional nature.

Finally, there is an immediate equivalent of Corollary 5, where the amplitude a_φ is defined as in (2).

Corollary 12 *For all $x \in S$, we have*

$$a_\varphi^{-2} \sum_{\theta\in\Theta_0} \frac{\theta_0}{\theta_0 - \theta} \ \leqslant\ \sum_{y\in S} \mathbb{E}_x\left[\frac{\theta_0^{-\tau_{y,\infty}} - 1}{\theta_0^{-1} - 1}\right] \nu(y) \ \leqslant\ a_\varphi^2 \sum_{\theta\in\Theta_0} \frac{\theta_0}{\theta_0 - \theta}$$

and in particular

$$\sum_{y\in S} \mathbb{E}_x\left[\tau_{y,\infty}\right] \nu(y) \ \leqslant\ a_\varphi^2 \sum_{\theta\in\Theta_0} \frac{\theta_0}{\theta_0 - \theta}$$

By integration of these bounds with respect to ν, one gets discrete time eigentime inequalities where the starting and ending points x and y play symmetrical roles.

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