Poincaré Inequality for Dirichlet Distributions and Infinite-Dimensional Generalizations *

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Abstract

For any $N \geq 2$ and $\alpha := (\alpha_1, \cdots, \alpha_{N+1}) \in (0, \infty)^{N+1}$, let $\mu_\alpha^{(N)}$ be the corresponding Dirichlet distribution on $\Delta := \left\{x = (x_i)_{1\leq i\leq N} \in [0,1]^N : \sum_{1\leq i\leq N} x_i \leq 1\right\}$. We prove the Poincaré inequality

$$\mu_\alpha^{(N)}(f^2) \leq \frac{1}{\alpha_{N+1}} \int_\Delta \left\{\Big(1 - \sum_{1\leq i\leq N} x_i\Big) \sum_{n=1}^N x_n(\partial_n f)^2\right\} \mu_\alpha^{(N)}(\mathrm{d}x) + \mu_\alpha^{(N)}(f)^2,\ f \in C^1(\Delta)$$

and show that the constant $\frac{1}{\alpha_{N+1}}$ is sharp. Consequently, the associated diffusion process on Δ converges to $\mu_\alpha^{(N)}$ in $L^2(\mu_\alpha^{(N)})$ at the exponentially rate α_{N+1}. The whole spectrum of the generator is also characterized. Moreover, the sharp Poincaré inequality is extended to the infinite-dimensional setting, and the spectral gap of the corresponding discrete model is derived.

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1 Introduction

Let $N \in \mathbb{N}$. For any $\alpha = (\alpha_1, \cdots, \alpha_{N+1}) \in (0,\infty)^{N+1}$, the Dirichlet distribution $\mu_\alpha^{(N)}$ with parameter α is a probability measure on the set

$$\Delta^{(N)} := \Big\{ x = (x_i)_{1\le i\le N} \in [0,1]^N : \sum_{1\le i\le N} x_i \le 1 \Big\}$$

with the density function

$$\rho(x_1, \cdots, x_N) := \frac{\Gamma(|\alpha|_1)}{\prod_{1\le i\le N+1}\Gamma(\alpha_i)}(1-|x|_1)^{\alpha_{N+1}-1}\prod_{1\le i\le N} x_i^{\alpha_i - 1}, \quad x \in \Delta^{(N)},$$

where $|x|_1 := \sum_{1\le i\le N}|x_i|$ for $x \in \mathbb{R}^N$. Obviously, $\mu_\alpha^{(N)}$ identifies to the distribution

$$\tilde{\mu}_\alpha^{(N+1)}(\mathrm{d}x, \mathrm{d}y) := \mu_\alpha^{(N)}(\mathrm{d}x)\delta_{1-|x|_1}(\mathrm{d}y)$$

on the space

$$\nabla^{(N+1)} := \Big\{ (x,y) \in [0,1]^{N+1} : \ y + |x|_1 = 1 \Big\}.$$

The Dirichlet distribution and its infinite-dimensional generalization arise naturally in Bayesian inference as conjugate priors for categorical distribution and infinite non-parametric discrete distributions respectively. They also arise in population genetics describing the distribution of allelic frequencies (see for instance [3, 13, 16]). In particular, for a population with $N+1$ allelic types, $x_i (1 \le i \le N+1)$ stands for the relative frequency of the i-th allele among $N+1$ ones.

The Dirichlet distribution possesses many nice properties. We will use the following partition (or aggregation) property of $\tilde{\mu}_\alpha^{(N+1)}$ for $\alpha \in (0,\infty)^{N+1}$. Let $(X_1, \ldots, X_{N+1})$ have law $\tilde{\mu}_\alpha^{(N+1)}$, let $A_1, A_2, \ldots, A_{k+1}$ be a partition of the set $\{1, 2, \ldots, N+1\}$, and set

$$Y_j = \sum_{r\in A_j} X_r, \ \beta_j = \sum_{r \in A_j} \alpha_r, \quad j = 1, \ldots, k+1.$$

Then $(Y_1, \ldots, Y_{k+1})$ has law $\tilde{\mu}_\beta^{(k+1)}$ with parameters $\beta := (\beta_1, \ldots, \beta_{k+1}) \in (0,\infty)^{k+1}$. We would also like to recall the neutral property of the Dirichlet distribution. For $(X_1, \cdots, X_N)$ having law $\mu_\alpha^{(N)}$, we define

$$U_1 = X_1, \ U_i = \frac{X_i}{1 - X_1 - \ldots - X_{i-1}}, \quad 2 \le i \le N.$$

Then U_i is a beta random variable with parameters $(\alpha_i, \alpha_{i+1} + \ldots + \alpha_{N+1})$ and $U_1, \ldots, U_N$ are independent. This leads to the following representation of the random variable with law $\mu_\alpha^{(N)}$:

$$(X_1, X_2, \ldots, X_N) = \Big(U_1, U_2(1-U_1), \ldots, U_N\prod_{i=1}^{N-1}(1-U_i)\Big).$$

A well known construction of the Dirichlet distribution is through a Pólya' urn scheme. More specifically, consider an urn containing $N+1$ balls of different colors labelled by $1, 2, \ldots, N+1$. The initial mass of the i-colored ball is α_i. Balls are drawn from the urn sequentially. The chance of a particular colored ball being selected is proportional to the total mass of that colored balls inside the urn. After each selection, the ball is returned with an additional ball of same color and mass one. The relative weight of different colored balls inside the urn will eventually converge to a Dirichlet vector $(X_1, X_2, \ldots, X_{N+1})$.

Several diffusion processes have been proposed and studied where the stationary distribution is the Dirichlet distribution. The Wright-Fisher diffusion (see [4, 14, 15, 17]) is a diffusion approximation to the Wright-Fisher Markov chain model in population genetics. It is reversible with respect to the Dirichlet distribution. The Infinite-dimensional generalizations of this model include the infinitely-many-neutral-alleles model ([5]) and the Fleming-Viot process with parent independent mutation ([10, 6]) .

Exploring the property of right neutrality, a GEM diffusion is introduced in [8] and studied further in [9]. This is a reversible diffusion with Dirichlet distribution as the reversible measure. The infinite-dimensional generalization of the model is also reversible and the reversible measure is the GEM distribution (see [7]).

In this paper, we focus on a diffusion process introduced in [11, (2.44)] (see also [1]), which solves the following SDE on $\Delta^{(N)}$:

E1 (1.1) $$\mathrm{d}X_i(t) = \big\{\alpha_i(1-|X(t)|) - \alpha_{N+1}X_i(t)\big\}\mathrm{d}t + \sqrt{2(1-|X(t)|_1)X_i(t)}\,\mathrm{d}B_i(t), \quad 1 \le i \le N,$$

where $B(t) := (B_1(t), \cdots, B_N(t))$ is the d-dimensional Brownian motion.

We will show that the Markov semigroup P_t^α associated to (1.1) is symmetric in $L^2(\mu_\alpha^{(N)})$; that is,

E2 (1.2) $$\int_{\Delta^{(N)}} f L_\alpha^{(N)} g \mathrm{d}\mu_\alpha^{(N)} = \int_\Delta g L_\alpha f \mathrm{d}\mu_\alpha^{(N)}, \quad f, g \in C^2(\mathbb{R}^N)$$

holds for

$$L_\alpha^{(N)}(x) := \sum_{1 \le n \le N} \Big(x_n(1-|x|_1)\partial_n^2 + \big\{\alpha_n(1-|x|_1) - \alpha_{N+1}x_n\big\}\partial_n\Big)$$

being the generator of P_t^α, where $\partial_n := \frac{\partial}{\partial x_n}$. So, $(L_\alpha, C^2(\Delta^{(N)}))$ is closable in $L^2(\mu_\alpha^{(N)})$ and its closure $(L_\alpha, \mathscr{D}(L_\alpha))$ is a negative definite self-adjoint operator.

Moreover, since

$$L_\alpha^{(N)}(fg)(x) = (fL_\alpha^{(N)}g + gL_\alpha^{(N)}f)(x) + 2(1-|x|_1)\sum_{n=1}^N x_n\{(\partial_n f)(\partial_n g)\}(x),$$

(1.2) implies the integration by parts formula

E2' (1.3) $$\begin{aligned} -\int_{\Delta^{(N)}} f L_\alpha^{(N)} g \mathrm{d}\mu_\alpha^{(N)} &= \int_{\Delta^{(N)}} \Big\{(1-|x|_1)\sum_{n=1}^N x_n\{(\partial_n f)(\partial_n g)\}(x)\Big\}\mu_\alpha^{(N)}(\mathrm{d}x) \\ &=: \mathscr{E}_\alpha^{(N)}(f,g), \quad f, g \in C^2(\Delta^{(N)}). \end{aligned}$$

Therefore, $(\mathscr{E}_\alpha^{(N)}, C^2(\Delta^{(N)}))$ is closable in $L^2(\mu_\alpha^{(N)})$ whose closure $(\mathscr{E}_\alpha^{(N)}, \mathscr{D}(\mathscr{E}_\alpha^{(N)}))$ is a symmetric Dirichlet form on $L^2(\mu_\alpha^{(N)})$, and it is easy to see that this Dirichlet form is associated to the Markov semigroup P_t^α.

Finally, the spectral gap of $L_\alpha^{(N)}$ is characterized as

$$\operatorname{gap}(L_\alpha^{(N)}) = \inf\Big\{\mathscr{E}_\alpha^{(N)}(f,f):\ f \in \mathscr{D}(\mathscr{E}_\alpha^{(N)}), \mu_\alpha^{(N)}(f) = 0, \mu_\alpha^{(N)}(f^2) = 1\Big\}.$$

It is known that when $N = 1$ we have $\operatorname{gap}(L_\alpha^{(N)}) = \alpha_1 + \alpha_2$, see e.g. [17]. So, in the following result we only consider $N \geq 2$.

T1.1 **Theorem 1.1.** *Let $N \geq 2$. Then P_t^α is symmetric in $L^2(\mu_\alpha^{(N)})$ and its generator has spectral gap $\operatorname{gap}(L_\alpha^{(N)}) = \alpha_{N+1}$. Consequently, P_t^α converge to $\mu_\alpha^{(N)}$ exponentially fast in $L^2(\mu_\alpha^{(N)})$:*

$$\|P_t^\alpha - \mu_\alpha^{(N)}\|_{L^2(\mu_\alpha^{(N)})} \leq \mathrm{e}^{-\alpha_{N+1}t}, \quad t \geq 0,$$

and the sharp Poincaré inequality for $(\mathscr{E}_\alpha^{(N)}, \mathscr{D}(\mathscr{E}_\alpha^{(N)}))$ is

$$\mu_\alpha^{(N)}(f^2) \leq \frac{1}{\alpha_{N+1}} \mathscr{E}_\alpha^{(N)}(f,f), \quad f \in \mathscr{D}(\mathscr{E}_\alpha^{(N)}), \mu_\alpha^{(N)}(f) = 0.$$

Next, we extend this result to the infinite-dimensional setting. Consider the infinite-dimensional simplex

$$\Delta^{(\infty)} := \Big\{x \in [0,1]^{\mathbb{N}}:\ |x|_1 = \sum_{i=1}^\infty x_i \leq 1\Big\},$$

which is equipped with the L^1-metric $|x-y|_1$. Let $\alpha \in (0,\infty)^{\mathbb{N}}$ with $|\alpha|_1 = \sum_{i=1}^\infty \alpha_i < \infty$, and let $\alpha_\infty > 0$ which refers to α_{N+1} in the finite-dimensional case as $N \to \infty$. Let

$$\alpha^{(n)} = \Big(\alpha_1, \cdots, \alpha_{n-1}, \sum_{i \geq n} \alpha_i, \alpha_\infty\Big) \in (0,\infty)^{n+1}, \quad n \geq 1.$$

Then for any $n \geq 1$,

$$\mu_{\alpha,\alpha_\infty}^{(n)}(\mathrm{d}x) := \mu_{\alpha^{(n)}}^{(n)}(\mathrm{d}x_1, \cdots, \mathrm{d}x_n) \prod_{i=n+1}^\infty \delta_0(\mathrm{d}x_i)$$

is a probability measure on $\Delta^{(\infty)}$. We will prove that when $n \to \infty$ these measures converges weakly to a probability measure $\mu_{\alpha,\alpha_\infty}^{(\infty)}$ on $\Delta^{(\infty)}$, which is the infinite-dimensional generalization of Dirichlet distribution with parameters (α, α_∞).

The following result extends Theorem 1.1 to the infinite-dimensional setting, for which we introduce the class of C^p-cylindrical functions for $p \geq 1$:

$$\mathscr{F}C^p := \big\{\Delta^{(\infty)} \ni x := (x_i)_{i\geq 1} \mapsto f(x_1, \cdots, x_n):\ n \geq 1, f \in C^p(\mathbb{R}^n)\big\}.$$

T1.2 **Theorem 1.2.** *Let $\alpha \in (0,\infty)^{\mathbb{N}}$ with $|\alpha|_1 < \infty$ and let $\alpha_\infty > 0$.*

(1) *The sequence* $\{\mu^{(n)}_{\alpha,\alpha_\infty}\}_{n\ge 1}$ *converges weakly to a probability measure* $\mu^{(\infty)}_{\alpha,\alpha_\infty}$ *on* $\Delta^{(\infty)}$.

(2) *The form*

$$\mathscr{E}^{(\infty)}_{\alpha,\alpha_\infty}(f,g) := \int_{\Delta^{(\infty)}} \Big\{(1-|x|_1)\sum_{n=1}^\infty x_n(\partial_n f)\partial_n g\Big\}(x)\mu^{(\infty)}_{\alpha,\alpha_\infty}(\mathrm{d}x),\quad f,g\in\mathscr{F}C^1$$

is closable in $L^2(\mu^{(\infty)}_{\alpha,\alpha_\infty})$ *whose closure is a symmetric Dirichlet form. The generator* $(L^{(\infty)}_{\alpha,\alpha_\infty},\mathscr{D}(L^{(\infty)}_{\alpha,\alpha_\infty}))$ *of the Dirichlet form satisfies* $\mathscr{F}C^2\subset\mathscr{D}(L^{(\infty)}_{\alpha,\alpha_\infty})$ *and*

$$L^{(\infty)}_{\alpha,\alpha_\infty}f(x) = \sum_{n=1}^\infty \Big(x_n(1-|x|_1)\partial_n^2 f(x) + \big\{\alpha_n(1-|x|_1)-\alpha_\infty x_n\big\}\partial_n f(x)\Big),\quad f\in\mathscr{F}C^2.$$

(3) *The generator* $L^{(\infty)}_{\alpha,\alpha_\infty}$ *has spectral gap* $\mathrm{gap}(L^{(\infty)}_{\alpha,\alpha_\infty})=\alpha_\infty$. *Consequently, the associated Markov semigroup* $P_t^{\alpha,\alpha_\infty}$ *converges to* $\mu^{(\infty)}_{\alpha,\alpha_\infty}$ *exponentially fast in* $L^2(\mu^{(\infty)}_{\alpha,\alpha_\infty})$:

$$\|P_t^{\alpha,\alpha_\infty}-\mu^{(\infty)}_{\alpha,\alpha_\infty}\|_{L^2(\mu^{(\infty)}_{\alpha,\alpha_\infty})}\le \mathrm{e}^{-\alpha_\infty t},\quad t\ge 0,$$

and the sharp Poincaré inequality is

$$\mu^{(\infty)}_{\alpha,\alpha_\infty}(f^2)\le \frac{1}{\alpha_\infty}\mathscr{E}^{(\infty)}_{\alpha,\alpha_\infty}(f,f),\quad f\in\mathscr{F}C^1,\mu^{(\infty)}_{\alpha,\alpha_\infty}(f)=0.$$

Finally, the next result shows that the diffusion process generated by $L^{(\infty)}_{\alpha,\alpha_\infty}$ is the weak limit of the $L^{(n)}_{\alpha,\alpha_\infty}$-diffusion process as $n\to\infty$, where

$$L^{(n)}_{\alpha,\alpha_\infty} := \sum_{i=1}^n \Big\{\Big[\alpha_i\Big(1-\sum_{i=1}^n x_i\Big)-\alpha_\infty x_i\Big]\partial_i + 2\Big(1-\sum_{i=1}^n x_i\Big)x_i\partial_i^2\Big\}.$$

For any $x\in\Delta^{(\infty)}$ and $T>0$, let $P^{(n)}_{x,T}$ be the distribution of the diffusion process generated by $L^{(n)}_{\alpha,\alpha_\infty}$ with initial point $x^{(n)} := \big(x_1,\cdots,x_{n-1},\sum_{j\ge n}x_j\big)$. Embedding $\Delta^{(n)}$ into $\Delta^{(\infty)}$ by setting $z_i=0$ for $z\in\Delta^{(n)}$ and $i\ge n+1$, we regard $P^{(n)}_{x,T}$ as a probability measure on $\Omega_T := C([0,T];\Delta_\infty)$ equipped with the uniform norm $\|\xi\|_{1,\infty} := \sup_{t\in[0,T]}|\xi(t)|_1$.

T1.3 **Theorem 1.3.** *For any* $x\in\Delta^{(\infty)}$ *and* $T>0$, $P^{(n)}_{x,T}$ *converges weakly to a probability measure* $P^{(\infty)}_{x,T}$ *on* Ω_T. *Moreover,* $P^{(\infty)}_{x,T}$ *solves the martingale problem of* $L^{(\infty)}_{\alpha,\alpha_\infty}$: *for any* $f\in\mathscr{F}C^2$, *the coordinate process* $X(t)(\omega):=\omega(t)$ *and the natural filtration* $\mathscr{F}_t:=\sigma(\omega_s:\ s\in[0,t])$,

$$f(X(t)) - \int_0^t L^{(\infty)}_{\alpha,\alpha_\infty}f(X(s))\mathrm{d}s,\quad t\in[0,T]$$

is a martingale under $P^{(\infty)}_{x,T}$.

We will prove Theorem 1.1 in Section 2 and characterize the whole spectrum of $L^{(N)}_\alpha$ in Section 3. The proofs of Theorems 1.2 and 1.3 are presented in Section 4. Finally, to understand the biology background of the study, we introduce in Section 5 a discrete model involving immigration, emigration and sampling, which approximates the diffusion process solving (1.1).

2 Proof of Theorem 1.1

We first prove (1.2) which implies the symmetry of P_t^α in $L^2(\mu_\alpha^{(N)})$. Since smooth functions on $\Delta^{(N)}$ are uniformly approximated by polynomials up to second order derivatives, it suffices to consider $f, g \in \mathscr{P}_\infty$, the set of all polynomials on $\Delta^{(N)}$. Let

$$A_\alpha^{(n)} = x_n(1-|x|_1)\partial_n^2 + \big\{\alpha_n(1-|x|_1) - \alpha_{N+1}x_n\big\}\partial_n, \quad 1 \le n \le N.$$

Then (1.2) follows from

E3 (2.1) $$\int_\Delta \Big(\prod_{1\le i\le N} x_i^{p_i}\Big) A_\alpha^{(n)}\Big(\prod_{1\le i\le N} x_i^{q_i}\Big)\mu_\alpha^{(N)}(\mathrm{d}x) = \int_\Delta \Big(\prod_{1\le i\le N} x_i^{q_i}\Big) A_\alpha^{(n)}\Big(\prod_{1\le i\le N} x_i^{p_i}\Big)\mu_\alpha^{(N)}(\mathrm{d}x)$$

for $p_i, q_i \in \mathbb{Z}_+, 1 \le i \le N$. Letting $p_{N+1} = q_{N+1} = 0$ and $C = \frac{\Gamma(|\alpha|_1)}{\prod_{1\le i\le N+1}\Gamma(\alpha_i)}$, and simply denote $x_{N+1} = 1 - |x|_1$, we have

$$\begin{aligned}
&\int_{\Delta^{(N)}} \Big(\prod_{1\le i\le N} x_i^{p_i}\Big) A_\alpha^{(n)}\Big(\prod_{1\le i\le N} x_i^{q_i}\Big)\mu_\alpha^{(N)}(\mathrm{d}x)\\
&= C\int_{\Delta^{(N)}} \Big(\prod_{1\le i\ne n\le N+1} x_i^{p_i+q_i+\alpha_i-1}\Big) x_n^{p_n+\alpha_n-1} A_\alpha^{(n)} x_n^{q_n}\mathrm{d}x\\
&= Cq_n\bigg\{(q_n+\alpha_n-1)\int_{\Delta^{(N)}}\Big(\prod_{1\le i\ne n\le N+1} x_i^{p_i+q_i+\alpha_i-1}\Big)x_{N+1}x_n^{p_n+q_n+\alpha_n-2}\mathrm{d}x\\
&\qquad\qquad - \alpha_{N+1}\int_{\Delta^{(N)}}\Big(\prod_{1\le i\le N+1} x_i^{p_i+q_i+\alpha_i-1}\Big)\mathrm{d}x\bigg\}\\
&= \frac{Cq_n\prod_{1\le i\ne n\le N+1}\Gamma(\alpha_i+p_i+q_i)}{\Gamma(\sum_{1\le i\le N+1}(\alpha_i+p_i+q_i))}\\
&\quad\times\Big((q_n+\alpha_n-1)\Gamma(\alpha_{N+1}+1)\Gamma(p_n+q_n+\alpha_n-1) - \alpha_{N+1}\Gamma(\alpha_{N+1})\Gamma(p_n+q_n+\alpha_n)\Big)\\
&= -\frac{C\Gamma(\alpha_{N+1}+1)\prod_{1\le i\ne n\le N+1}\Gamma(\alpha_i+p_i+q_i)}{\Gamma(\sum_{1\le i\le N+1}(\alpha_i+p_i+q_i))}p_nq_n\Gamma(p_n+q_n+\alpha_n-1),
\end{aligned}$$

where the last step is due to the identity $\Gamma(s+1) = s\Gamma(s), s > 0$. Since the result is symmetric in (p_n, q_n), it implies (2.1).

For any $d \in \mathbb{N}$, let $\mathscr{P}_d$ be the space of all polynomials in $\mathscr{P}_\infty$ whose total degrees are less than or equal to d. Let $\mathscr{P}_{0,d} = \{f \in \mathscr{P}_d : \mu_\alpha^{(N)}(f) = 0\}$. It is well known that $\mathscr{P}_\infty := \cup_{d\ge1}\mathscr{P}_d$ is dense in $C_b^1(\Delta^{(N)})$, so that $\mathscr{P}_{0,\infty} := \cup_{d\ge1}\mathscr{P}_{0,d}$ is dense in

$$\mathscr{D}_0 := \{f \in \mathscr{D}(\mathscr{E}_\alpha^{(N)}) : \mu_\alpha^{(N)}(f) = 0\}$$

under the Sobolev norm $\|f\|_{1,2} := \sqrt{\mu_\alpha^{(N)}(f^2) + \mathscr{E}_\alpha^{(N)}(f,f)}$.

To characterize $\mathrm{gap}(L_\alpha^{(N)})$, we make spectral decomposition of $L_\alpha^{(N)}$ in terms of the degree of polynomials. Obviously, every $\mathscr{P}_{0,d}$ is an invariant space of $L_\alpha^{(N)}$. Let $\mathscr{Q}_1 = \mathscr{P}_{0,1}$ and

$$\mathscr{Q}_d = \left\{ f \in \mathscr{P}_{0,d} : \mu_\alpha^{(N)}(fg) = 0 \text{ for all } g \in \mathscr{P}_{d-1} \right\}, \quad d \geq 2.$$

Then, by the symmetry of $L_\alpha^{(N)}$ in $L^2(\mu_\alpha^{(N)})$, every $\mathscr{Q}_d$ is an invariant space of $L_\alpha^{(N)}$ as well. Thus, letting $\pi_d : \mathscr{P}_\infty \to \mathscr{P}_d$ be the orthogonal projection with respect to the inner product in $L^2(\mu_\alpha^{(N)})$, we have

P1 (2.2)
$$L_\alpha^{(N)} \pi_d f = \pi_d L_\alpha^{(N)} f, \quad d \geq 1, f \in \mathscr{P}_\infty.$$

Therefore, to characterize the spectrum of $L_\alpha^{(N)}$ it suffices to consider that of $L_\alpha^{(N)}|_{\mathscr{Q}_i}$, the restriction of $L_\alpha^{(N)}$ on $\mathscr{Q}_i$, for every $i \geq 1$.

Let $d \geq 2$. To characterize the spectrum of $L_\alpha^{(N)}|_{\mathscr{Q}_d}$, let

$$K_d = \Big\{ k = (k_1, \cdots, k_N) \in \mathbb{Z}_+^N : \sum_{1 \leq i \leq N} k(i) = d \Big\}.$$

For any $k \in K_d$, let $x^k = \prod_{1 \leq i \leq N} x_i^{k_i}$. Then

Q (2.3)
$$\mathscr{Q}_d = \left\{ \sum_{k \in K_d} c_k x^k - \pi_{d-1} \sum_{k \in K_d} c_k x^k : \ c := (c_k)_{k \in K_d} \in \mathbb{R}^{K_d} \right\}.$$

We define the $K_d \times K_d$-matrix M_d by letting

$$M_d(k,k') = \begin{cases} d\alpha_{N+1} + \sum_{1 \leq n \leq N} (k_n + \alpha_n - 1) k_n, & \text{if } k = k', \\ (k_n + \alpha_n)(k_n + 1), & \text{if } k' = k + e_n - e_m, 1 \leq n \neq m \leq N, \\ 0, & \text{otherwise,} \end{cases}$$

where $\{e_n\}_{1 \leq n \leq N}$ is the canonical orthonormal basis on $\mathbb{R}^N$. We first identify eigenvalues of $L_\alpha^{(N)}|_{\mathscr{Q}_d}$ with those of M_d.

L1 **Lemma 2.1.** *For any $d \geq 2$, λ is an eigenvalue of $-L_\alpha^{(N)}|_{\mathscr{Q}_d}$ if and only if it is an eigenvalue of M_d. Consequently, $-L_\alpha^{(N)}|_{\mathscr{Q}_d} \geq (d\alpha_{N+1}) I_{\mathscr{Q}_d}$, where $I_{\mathscr{Q}_d}$ is the identity operator on $\mathscr{Q}_d$.*

Proof. (1) Let λ be an eigenvalue of $-L_\alpha^{(N)}$ on $\mathscr{Q}_d$. By (2.3) and (2.2), there exists $0 \neq c \in \mathbb{R}^{K_d}$ such that

W1 (2.4)
$$\sum_{k \in K_d} c_k (L_\alpha^{(N)} x^k - \pi_{d-1} L_\alpha^{(N)} x^k) = -\lambda \sum_{k \in K_d} c_k (x^k - \pi_{d-1} x^k).$$

Obviously,

$$
\begin{aligned}
&L_\alpha^{(N)} x^k - \sum_{1\le n\le N} (x_n\partial_n^2 + \alpha_n \partial_n) x^k \\
&= -\Big(\sum_{1\le n,m\le N} x_n x_m \partial_n^2 x^k + \sum_{1\le n,m\le N} x_m \alpha_n \partial_n x^k + \alpha_{N+1} \sum_{1\le n\le N} x_n \partial_n x^k \Big) \\
&= -\Big(\sum_{n,m\le N} k_n(k_n-1) x^{k-e_n+e_m} + \sum_{1\le n,m\le N} \alpha_n k_n x^{k-e_n+e_m} + \alpha_{N+1} \sum_{1\le n\le N} k_n x^k \Big) \\
&= -\Big(\sum_{1\le n,m\le N} k_n(k_n-1) x^{k-e_n+e_m} + \sum_{1\le n,m\le N} \alpha_n k_n x^{k-e_n+e_m} + d\alpha_{N+1} x^k \Big).
\end{aligned}
$$

By the change of variables $k' := k - e_n + e_m$, we obtain

$$
\begin{aligned}
&\sum_{k\in K_d} c_k \sum_{1\le n,m\le N} \alpha_n k_n x^{k-e_n+e_m} \\
&= \sum_{k\in K_d} c_k \sum_{1\le n\ne m\le N} \alpha_n k_n x^{k-e_n+e_m} + \sum_{k\in K_d} c_k \sum_{1\le n\le N} \alpha_n k_n x^k \\
&= \sum_{k\in K_d} \sum_{1\le n\ne m\le N} c_{k'+e_n-e_m} \alpha_n (k'+e_n-e_m)(n) x^{k'} + \sum_{k\in K_d} c_k \sum_{1\le n\le N} \alpha_n k_n x^k \\
&= \sum_{k\in K_d} \sum_{1\le n\ne m\le N} \alpha_n (k_n+1) c_{k+e_n-e_m} x^k + \sum_{k\in K_d} \sum_{1\le n\le N} \alpha_n k_n c_k x^k .
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\sum_{k\in K_d} c_k \sum_{1\le n,m\le N} k_n(k_n-1) x^{k-e_n+e_m} \\
&= \sum_{k\in K_d} \sum_{1\le n\ne m\le N} k_n(k_n+1) c_{k+e_n-e_m} x^k + \sum_{k\in K_d} c_k \sum_{1\le n\le N} k_n(k_n-1) x^k .
\end{aligned}
$$

Combining these together leads to

W2 (2.5)
$$
\sum_{k\in K_d} c_k L_\alpha^{(N)} x^k = \sum_{k\in K_d} c_k \sum_{1\le n\le N} (x_n\partial_n^2 + \delta_n\partial_n) x^k - \sum_{k,k'\in K_d} M_d(k,k') c_{k'} x^k . \tag{2.5}
$$

Substituting this into (2.4), we arrive at

$$
\sum_{k\in K_d} (M_d c)_k x^k = \lambda \sum_{k\in K_d} c_k x^k + \mathbf{p}_{d-1}(x)
$$

for some $\mathbf{p}_{d-1} \in \mathscr{P}_{d-1}$. Therefore, $M_d c = \lambda c$, i.e. λ is an eigenvalue of M_d.

(2) On the other hand, if λ is an eigenvalue of M_d, then there exists $c \in \mathbb{R}^{K_d} \setminus \{0\}$ such that $M_d c = \lambda c$. Let

$$
f(x) = \sum_{k\in K_d} c_k x^k - \pi_{d-1} \sum_{k\in K_d} c_k x^k .
$$

It follows from $M_d c = \lambda c$ and (2.5) that

$$L_\alpha^{(N)} f = \tilde{\mathbf{p}}_{d-1} - \lambda f$$

holds for some $\tilde{\mathbf{p}}_{d-1} \in \mathscr{P}_{d-1}$. Since $f \in \mathscr{Q}_d$ which is orthogonal to $\mathscr{P}_{d-1}$, this and (2.2) implies

$$L_\alpha^{(N)} f = (1 - \pi_{d-1}) L_\alpha^{(N)} f = -\lambda (1 - \pi_{d-1}) f = -\lambda f.$$

So, λ is an eigenvalue of $L_\alpha^{(N)}$ on $\mathscr{Q}_d$.

(3) Finally, since eigenvalues of $-L_\alpha^{(N)}$ are nonnegative, (2) implies that eigenvalues of $\tilde{M}_d := M_d - d\alpha_{N+1} I_{K_d \times K_d}$ is larger than or equal to $-d\alpha_{N+1}$. On the other hand, from the definition of M_d we see that $\tilde{M}_d$ does not depend on α_{N+1}. So, letting $\alpha_{N+1} \downarrow 0$ and noting that $M_d \geq 0$, we conclude that eigenvalues of $\tilde{M}_d$ are non-negative. Therefore, eigenvalues of M_d are larger than or equal to $d\alpha_{N+1}$. Combining this with (1) we obtain $-L_\alpha^{(N)}|_{\mathscr{Q}_d} \geq (d\alpha_{N+1}) I_{\mathscr{Q}_d}$. □

Proof of Theorem 1.1. By Lemma 2.1, it suffices to prove that the smallest eigenvalue of $-L_\alpha^{(N)}|_{\mathscr{Q}_1}$ is α_{N+1}. To this end, we take $\theta_i = (\theta_{ij})_{1 \leq j \leq N} \in \mathbb{R}^N (1 \leq i \leq N-1)$ such that

$$\sum_{k=1}^N \theta_{ik} \alpha_k = 0, \quad \sum_{k=1}^N \theta_{ik} \theta_{jk} \alpha_k = \delta_{ij}, \quad 1 \leq i, j \leq N-1.$$

So, $\{\theta_i\}_{i=1}^N$ is a basis of $\mathbb{R}^{N-1}$. Let

$$u_i(x) = \sum_{j=1}^N \theta_{ij} x_j, \quad 1 \leq i \leq N-1;$$
$$u_N(x) = \sum_{k=1}^N x_k - \frac{\tilde{\alpha}}{|\alpha|_1}, \quad \tilde{\alpha} := |\alpha|_1 - \alpha_{N+1} = \sum_{k=1}^N \alpha_k.$$

We intend to prove that $\{u_i\}_{1 \leq i \leq N}$ is an orthogonal basis of $\mathscr{Q}_1$ with respect to the inner product $\langle f, g \rangle_\alpha^{(N)} := \mu_\alpha^{(N)}(fg) = \int_{\Delta^{(N)}} fg \mathrm{d}\mu_\alpha^{(N)}$, and $L_\alpha^{(N)} u_N = -|\alpha|_1 u_N$ while $L_\alpha^{(N)} u_i = -\alpha_{N+1} u_i$ for $1 \leq i \leq N-1$. Thus, the smallest eigenvalue of $-L_\alpha^{(N)}|_{\mathscr{Q}_1}$ is α_{N+1}.

It is easy to see that

$$\mu_\alpha^{(N)}(x_i) := \int_{\Delta^{(N)}} x_i \mu_\alpha^{(N)}(\mathrm{d}x) = \frac{\Gamma(\bar{\alpha})\Gamma(\alpha_i + 1)}{\Gamma(|\alpha|_1 + 1)\Gamma(\alpha_i)} = \frac{\alpha_i}{|\alpha|_1},$$
$$\mu_\alpha^{(N)}(x_i^2) = \frac{\Gamma(\bar{\alpha})\Gamma(\alpha_i + 2)}{\Gamma(|\alpha|_1 + 2)\Gamma(\alpha_i)} = \frac{\alpha_i(\alpha_i + 1)}{|\alpha|_1(|\alpha|_1 + 1)}, \qquad 1 \leq i \leq N-1;$$
$$\mu_\alpha^{(N)}(x_i x_j) = \frac{\Gamma(\bar{\alpha})\Gamma(\alpha_i + 1)\Gamma(\alpha_j + 1)}{\Gamma(|\alpha|_1 + 2)\Gamma(\alpha_i)\Gamma(\alpha_j)} = \frac{\alpha_i \alpha_j}{|\alpha|_1(|\alpha|_1 + 1)}, \quad 1 \leq i \neq j \leq N-1.$$

Then

$$\mu_\alpha^{(N)}(u_i) = \frac{1}{|\alpha|_1}\sum_{k=1}^N \theta_{ik}\alpha_k = 0, \quad 1 \le i \le N-1;$$

$$\mu_{\alpha,\lambda}^{(N)}(u_N) = \sum_{i=1}^N \frac{\alpha_i}{|\alpha|_1} - \frac{\tilde{\alpha}}{|\alpha|_1} = 0.$$

So, $\{u_i\}_{1\le i\le N} \subset \mathscr{Q}_1$. Moreover, for $1 \le i \ne j \le N-1$,

$$\begin{aligned}\mu_\alpha^{(N)}(u_iu_j) &= \frac{1}{|\alpha|_1(|\alpha|_1+1)}\Bigg(\sum_{1\le k\le N}\theta_{ik}\theta_{jk}\alpha_k(\alpha_k+1) + \sum_{1\le k\ne l\le N}\theta_{ik}\theta_{jl}\alpha_k\alpha_l\Bigg)\\ &= \frac{1}{|\alpha|_1(|\alpha|_1+1)}\Bigg\{\Big(\sum_{1\le k\le N}\theta_{ik}\alpha_k\Big)\sum_{1\le l\le N}\theta_{jl}\alpha_l + \sum_{1\le k\le N}\theta_{ik}\theta_{jk}\alpha_k\Bigg\} = 0,\end{aligned}$$

and for any $1 \le i \le N-1$,

$$\begin{aligned}\mu_\alpha^{(N)}(u_iu_N) &= \sum_{1\le k,j\le N}\theta_{ij}\mu(x_jx_k)\\ &= \frac{1}{|\alpha|_1(|\alpha|_1+1)}\sum_{1\le j\le N}\theta_{ij}\alpha_j(\alpha_j+1) + \frac{1}{|\alpha|_1(|\alpha|_1+1)}\sum_{1\le k\ne j\le N}\theta_{ij}\alpha_j\alpha_k\\ &= \frac{1}{|\alpha|_1(|\alpha|_1+1)}\sum_{1\le k,j\le N}\theta_{ij}\alpha_j\alpha_k + \frac{1}{|\alpha|_1(|\alpha|_1+1)}\sum_{1\le j\le N}\theta_{ij}\alpha_j = 0.\end{aligned}$$

Since $\{\theta_i\}_{i=1}^{N-1}$ is a basis of $\mathbb{R}^{N-1}$, we have

$$\dim \operatorname{span}\{u_i : 1 \le i \le n-1\} = N-1 = \dim \mathscr{Q}_1.$$

In conclusion, $\{u_i\}_{1\le i\le N}$ is an orthogonal basis of $\mathscr{Q}_1$.

Finally, we have

$$L_\alpha^{(N)}u_i(x) = \sum_{j=1}^N(\alpha_jx_{N+1} - \alpha_{N+1}x_j)\theta_{ij} = -\alpha_{N+1}u_i, \quad 1 \le i \le N-1,$$

and

$$L_\alpha^{(N)}u_N(x) = \sum_{j=1}^N(\alpha_jx_{N+1} - \alpha_{N+1}x_j) = -|\alpha|_1\sum_{j=1}^N x_j + \sum_{j=1}^N \alpha_j = -|\alpha|_1u_N(x).$$

Therefore, the proof is finished. □

3 The whole spectrum of $L_\alpha^{(N)}$

For $d \in \mathbb{Z}_+$, let $\mathscr{H}_d$ be the space of homogeneous polynomials of total degree d in the variables x_1, ..., x_N. Denote by $\tilde{\pi}_d$ the natural projection from $\mathscr{P}_\infty$ to $\mathscr{H}_d$ which only keeps the d-homogeneous part of a polynomial. Let $L_{\alpha,d}^{(N)} = (\tilde{\pi}_d L_\alpha^{(N)})|_{\mathscr{H}_d}$ be the restriction of the operator $\tilde{\pi}_d L_\alpha^{(N)}$ to $\mathscr{H}_d$ and denote $-\Lambda_d$ its spectrum, seen as a multi-set (namely with multiplicities). From the above considerations, the spectrum Λ of $-L_\alpha^{(N)}$ is equal to $\cup_{d\in\mathbb{Z}_+}\Lambda_d$, as a multi-set. We can write

$$\bar{L}_{\alpha,d}^{(N)} = -|\cdot|_1 \tilde{L}_{\alpha,d}^{(N)} - \alpha_{N+1}\hat{L}_{\alpha,d}^{(N)},$$

where $\tilde{L}_{\alpha,d}^{(N)} : \mathscr{H}_d \to \mathscr{H}_{d-1}$ and $\hat{L}_{\alpha,d}^{(N)} : \mathscr{H}_d \to \mathscr{H}_d$ are respectively the restriction to $\mathscr{H}_d$ of the operators

$$\tilde{L}_\alpha^{(N)} := \sum_{1\le n\le N} \left(x_n\partial_n^2 + \alpha_n\partial_n\right), \quad \hat{L}_\alpha^{(N)} := \sum_{1\le n\le N} x_n\partial_n.$$

The crucial point of the previous decomposition is that $\hat{L}_{\alpha,d}^{(N)} = dI_{\mathscr{H}_d}$. Denote by $\tilde{\Lambda}_d$ the spectrum of $|\cdot|_1\tilde{L}_{\alpha,d}^{(N)}$, we thus have

$$\Lambda_d = \tilde{\Lambda}_d + d\alpha_{N+1}.$$

Note that $\Lambda_0 = \tilde{\Lambda}_0 = \{0\}$. The next result enables to compute by iteration $\tilde{\Lambda}_d$ for all $d \in \mathbb{Z}_+$.

pro2 **Proposition 3.1.** *For any $d \in \mathbb{Z}_+$, we have*

$$\tilde{\Lambda}_{d+1} = (2d + \tilde{\alpha} + \tilde{\Lambda}_d) \cup \{0[C(N, d+1) - C(N, d)]\},$$

where $\{0[l]\}$ is the multi-set with 0 repeated l times, for $l \in \mathbb{Z}_+$ (more generally $[l]$ will stand for the multiplicity l), and where $C(N, d)$ is the dimension of $\mathscr{H}_d$, namely

$$C(N, d) = \binom{d + N - 1}{d}.$$

Proof. Consider $\lambda \in \tilde{\Lambda}_{d+1}$ and let $\varphi \in \mathscr{H}_d$ be an associated eigenvector (non-zero). We have

$$|\cdot|_1\tilde{L}_{\alpha,d+1}^{(N)}\varphi = \lambda\varphi.$$

Since $L_{\alpha,d+1}^{(N)}[\varphi]$ belongs to $\mathscr{H}_d$, there are two possibilities: either $\lambda = 0$, or $\varphi = |\cdot|_1\psi$ for some $\psi \in \mathscr{H}_d$ such that

*W (3.1) $$\tilde{L}_{\alpha,d+1}^{(N)}(|\cdot|_1\psi) = \lambda\psi.$$

We consider the latter situation, since the former case leads to the multi-set $\{0[C(N, d+1) - C(N, d)]\}$.

We compute at point x that

*W2

$$\begin{aligned}\tilde{L}^{(N)}_{\alpha,d+1}(|\cdot|_1\psi) &= |x|_1\tilde{L}^{(N)}_{\alpha}\psi + \psi\tilde{L}^{(N)}_{\alpha}|\cdot|_1 + 2\sum_{1\leq n\leq N} x_n\partial_n\psi \\ &= |x|_1\tilde{L}^{(N)}_{\alpha,d}\psi + \psi\sum_{1\leq n\leq N}\alpha_n + 2\sum_{1\leq n\leq N} x_n\partial_n\psi \\ &= |x|_1\tilde{L}^{(N)}_{\alpha,d}\psi + (\tilde{\alpha}+2d)\psi.\end{aligned} \tag{3.2}$$

So, it follows from (3.1) that $\lambda - \tilde{\alpha} - 2d$ is an eigenvalue of the operator $|\cdot|_1\tilde{L}^{(N)}_{\alpha,d}$, namely belongs to $\tilde{\Lambda}_d$. Thus,

$$\tilde{\Lambda}_{d+1} \subset (2d + \tilde{\alpha} + \tilde{\Lambda}_d) \cup \{0[C(N,d+1) - C(N,d)]\}.$$

On the other hand, if $\lambda' \in \tilde{\Lambda}_d$ then $|\cdot|_1\tilde{L}^{(N)}_{\alpha,d}\psi = \lambda'\psi$ for some $0 \neq \psi \in \mathscr{H}_d$. Then (3.2) implies

$$\tilde{L}^{(N)}_{\alpha,d+1}(|\cdot|_1\psi) = |\cdot|_1\tilde{L}^{(N)}_{\alpha,d}\psi + (\tilde{\alpha}+2d)\psi = (\lambda' + \tilde{\alpha} + 2d)\psi.$$

Therefore, $\lambda' + \tilde{\alpha} + 2d \in \tilde{\Lambda}_{d+1}$; that is, $\tilde{\Lambda}_{d+1} \supset (2d + \tilde{\alpha} + \tilde{\Lambda}_d)$. Then the proof is finished. □

The previous arguments amount to an iterative construction of the eigenvectors: for any $d \in \mathbb{Z}_+$, let $\tilde{\mathscr{F}}_d$ be the set of eigenvectors of $\tilde{L}^{(N)}_{\alpha,d}$ and $\mathscr{G}_d$ be the kernel of $\tilde{L}^{(N)}_{\alpha,d}$. Then we have

$$\forall d \in \mathbb{Z}_+, \qquad \tilde{\mathscr{F}}_{d+1} = \mathscr{G}_{d+1} \cup y_N\tilde{\mathscr{F}}_d.$$

Indeed, in the above proof, functions $\varphi \in \tilde{\mathscr{F}}_{d+1}$ of the form $y_N\psi$ with $\psi \in \tilde{\mathscr{F}}_d$ are associated to eigenvalues of the form $\tilde{\alpha} + 2d + \lambda$, where $\lambda \in \tilde{\Lambda}_d$. From Lemma 2.1, we know that $\lambda \geq 0$, so that $\tilde{\alpha} + 2d + \lambda > 0$ and φ does not belong to the kernel of $\tilde{L}^{(N)}_{d+1}$. Conversely, we have seen that all the other eigenvectors belong to the kernel of $\tilde{L}^{(N)}_{d+1}$. Thus we get the following characterization of the kernel of $\tilde{L}^{(N)}_{\alpha,d}$: it consists exactly into the eigenvectors of $\tilde{L}^{(N)}_{\alpha,d}$ which don't admit y_N as a factor.

Note that $\tilde{\mathscr{F}}_d$ is also the set of eigenvectors of $L^{(N)}_{\alpha,d}$. To get the eigenvectors of our initial operator L, we construct by iteration on $d \in \mathbb{Z}_+$ the following subsets $\mathscr{F}_d$ of $\mathscr{P}_d$. First we take $\mathscr{F}_0 := \tilde{\mathscr{F}}_0 = \mathscr{P}_0$. Next, if $\mathscr{F}_d$ has been constructed, then for any $f \in \tilde{\mathscr{F}}_{d+1}$, there exists a unique $g_f \in \mathscr{P}_d$ such that $f + g_f$ is orthogonal to $\mathscr{P}_d$ in $\Lambda^2(\mu)$. Then we define

$$\mathscr{F}_{d+1} := \{f + g_f : f \in \tilde{\mathscr{F}}_{d+1}\}.$$

The set of eigenvectors of L is $\cup_{d\in\mathbb{Z}_+}\mathscr{F}_d$.

From Proposition 3.1, it is possible to parametrize the spectrum Λ of $-L$ in the following way. Let $\mathscr{K}$ be the set of elements of the form $(k_1, k_2, ..., k_r, k_{r+1})$, where $r \in \mathbb{Z}_+$ and $0 \leq k_1 < k_2 < \cdots < k_r < k_{r+1}$. Define a mapping $K : \mathscr{K} \to \Lambda$ via

$$\forall k := (k_1, k_2, ..., k_r, k_{r+1}) \in \mathscr{K}, \qquad K(k) := 2(k_1 + \cdots k_r) + r\tilde{\alpha} + k_{r+1}\alpha_{N+1}.$$

Then K is surjective. It is truly one-to-one, if and only if 1, $\tilde{\alpha}$ and α_{N+1} are independent when $\mathbb{R}$ is seen as a vector space over $\mathbb{Q}$. Let us call this situation generical over the choice of the parameters $\alpha := (\alpha_n)_{1\le n\le N+1}$.

The multiplicities can also be recovered. Consider the mapping $D : \mathscr{K} \to \mathbb{N}$ defined by

$$D(k) := \sum_{1\le l\le r} C(N, k_l) + \sum_{1\le l\le k_{r+1}-1, l\notin\{k_1,k_2,\dots,k_r\}} \big\{C(N, l+1) - C(N, l)\big\}$$

for $k := (k_1, k_2, \dots, k_r, k_{r+1}) \in \mathscr{K}$. Then the multiplicity of an eigenvalue $\lambda \in \Lambda$ is given by

$$\sum_{k\in K^{-1}(\lambda)} D(k).$$

In particularly, generically, we have $\Lambda = \{K(k)[D(k)] : k \in \mathscr{K}\}$.

4 Proofs of Theorems 1.2 and 1.3

To prove the first assertion, let W be the L^1-Wasserstein distance induced by $\rho(x, y) := |x - y|_1$ on $\mathscr{P}(\Delta^{(\infty)})$, the set of all probability measures on $\Delta^{(\infty)}$. That is, for any $\mu, \nu \in \mathscr{P}(\Delta^{(\infty)})$,

$$W(\mu, \nu) := \int_{\pi\in\mathscr{C}(\mu,\nu)} \int_{\Delta^{(\infty)}\times\Delta^{(\infty)}} |x - y|\pi(\mathrm{d}x, \mathrm{d}y),$$

where $\mathscr{C}(\mu, \nu)$ is the set of all couplings for μ and ν; i.e. $\pi \in \mathscr{C}(\mu, \nu)$ if and only if it is a probability measure on $\Delta^{(\infty)} \times \Delta^{(\infty)}$ such that

$$\pi(\mathrm{d}x \times \Delta^{(\infty)}) = \mu(\mathrm{d}x), \quad \pi(\Delta^{(\infty)} \times \mathrm{d}y) = \nu(\mathrm{d}y).$$

It is well known that the metric W is complete and induces the weak topology on $\mathscr{P}(\Delta^{(\infty)})$, see e.g. [2, Theorems 5.4 and 5.6]. So, for the proof of Theorem 1.2 we only need to show that $\{\mu^{(n)}_{\alpha,\alpha_\infty}\}_{n\ge1}$ is W-Cauchy sequence.

Proof of Theorem 1.2. Let $\mathscr{L}\{\xi\}$ denote the law of a random variable ξ.

(1) To prove that $\{\mu^{(n)}_{\alpha,\alpha_\infty}\}_{n\ge1}$ is a W-Cauchy sequence, we use the partition property of the Dirichlet distribution mentioned in Section 1. For any $n > m \ge 1$, let $(X^{(n)}_1, \cdots, X^{(n)}_{n+1})$ have law $\tilde{\mu}^{(n+1)}_{\alpha^{(n)}}$. By the partition property, $\tilde{\mu}^{(m+1)}_{\alpha^{(m)}} = \mathscr{L}\big\{\big(X^{(n)}_1, \cdots, X^{(n)}_{m-1}, \sum_{i=m}^n X^{(n)}_i, X^{(n)}_{n+1}\big)\big\}$. So,

$$\mu^{(m)}_{\alpha^{(m)}} = \mathscr{L}\Big\{\Big(X^{(n)}_1, \cdots, X^{(n)}_{m-1}, \sum_{i=m}^n X^{(n)}_i\Big)\Big\}, \quad \mu^{(n)}_{\alpha^{(n)}} = \mathscr{L}\big\{(X^{(n)}_1, \cdots, X^{(n)}_n)\big\}.$$

Thus,

$$\mu^{(m)}_{\alpha,\alpha_\infty} = \mathscr{L}\Big\{\Big(X^{(n)}_1, \cdots, X^{(n)}_{m-1}, \sum_{i=m}^n X^{(n)}_i, 0, 0, \cdots, 0\Big)\Big\},$$
$$\mu^{(n)}_{\alpha,\alpha_\infty} = \mathscr{L}\big\{(X^{(n)}_1, \cdots, X^{(n)}_n, 0, 0, \cdots, 0)\big\}.$$

Then, by the definition of W and noting that $|\alpha|_1 < \infty$, we have

$$\begin{aligned}
&\limsup_{m\to\infty} \sup_{n\geq m+1} W(\mu_{\alpha,\alpha_\infty}^{(m)}, \mu_{\alpha,\alpha_\infty}^{(n)}) \leq 2 \limsup_{m\to\infty} \sup_{n\geq m+1} \sum_{i=m+1}^{n} \mathbb{E}|X_i^{(n)}| \\
&\leq \limsup_{m\to\infty} \frac{2\sum_{i=m+1}^{\infty} \alpha_i}{\alpha_\infty + \|\alpha\|_1} = 0.
\end{aligned}$$

Therefore, $\{\mu_{\alpha,\alpha_\infty}^{(n)}\}_{n\geq 1}$ is a W-Cauchy sequence and the proof of the first assertion is finished.

(2) It suffices to prove

CL (4.1)
$$\mathscr{E}_{\alpha,\alpha_\infty}^{(\infty)}(f,g) = -\int_{\Delta^{(\infty)}} (f L_{\alpha,\alpha_\infty}^{(\infty)} g)\, \mathrm{d}\mu_{\alpha,\alpha_\infty}^{(\infty)}, \quad f,g \in \mathscr{F}C^2. \tag{4.1}$$

For any $f,g \in \mathscr{F}C^2$, there exist $m \in \mathbb{N}$ and $f_m, g_m \in C^2(\mathbb{R}^m)$ such that

$$f(x) = f_m(x_1, \cdots, x_m), \quad g(x) = g_m(x_1, \cdots, x_m), \quad x \in \Delta^{(\infty)}.$$

So, by the definition of $\mu_{\alpha,\alpha_\infty}^{(n)}$ and using (1.3), we have

GG (4.2)
$$-\int_{\Delta^{(\infty)}} (f L_{\alpha^{(n)}}^{(n)} g) \mathrm{d}\mu_{\alpha,\alpha_\infty}^{(n)} = \int_{\Delta^{(\infty)}} \Big\{\Big(1 - \sum_{1\leq i\leq n} x_i\Big) \sum_{i=1}^{m} x_i (\partial_i f)(\partial_i g)\Big\} \mathrm{d}\mu_{\alpha,\alpha_\infty}^{(n)}. \tag{4.2}$$

Since $\mu_{\alpha,\alpha_\infty}^{(n)} \to \mu_{\alpha,\alpha_\infty}^{(\infty)}$ weakly, and it is easy to see that

$$\begin{aligned}
&\lim_{n\to\infty} \sup_{x\in\Delta^{(\infty)}} |f L_{\alpha^{(n)}}^{(n)} g - f L_{\alpha,\alpha_\infty} g|(x) = 0, \\
&\lim_{n\to\infty} \sup_{x\in\Delta^{(\infty)}} \Big|\Big(1 - \sum_{1\leq i\leq n} x_i\Big) \sum_{i=1}^{m} x_i (\partial_i f)(\partial_i g) - \Big(1 - \sum_{i=1}^{\infty} x_i\Big) \sum_{i=1}^{m} x_i (\partial_i f)(\partial_i g)\Big| = 0,
\end{aligned}$$

by letting $n \to \infty$ in (4.2) we prove (4.1).

(3) Finally, as was shown in (2) that the desired Poincaré inequality follows by applying Theorem 1.1 to $\mu_{\alpha^{(n)}}^{(n)}$ on $\Delta^{(n)}$ then letting $n \to \infty$. So, $\mathrm{gap}(L_{\alpha,\alpha_\infty}^{(\infty)}) \geq \alpha_\infty$. On the other hand, let

$$u(x) = \alpha_2 x_1 - \alpha_1 x_2, \quad x \in \Delta^{(\infty)}.$$

We have

$$L_{\alpha,\alpha_\infty}^{(\infty)} u(x) = \big\{\alpha_1(1 - |x|_1) - \alpha_\infty x_1\big\}\alpha_2 - \big\{\alpha_1(1 - |x|_1) - \alpha_\infty x_2\big\}\alpha_1 = -\alpha_\infty u(x), \quad x \in \Delta^{(\infty)}.$$

This implies $\mathrm{gap}(L_{\alpha,\alpha_\infty}^{(\infty)}) \leq \alpha_\infty$. In conclusion, we have $\mathrm{gap}(L_{\alpha,\alpha_\infty}^{(\infty)}) = \alpha_\infty$. □

Proof of Theorem 1.3. (a) For the first assertion, we only need to prove that $\{P_{x,T}^{(n)}\}_{n\geq 1}$ is a Cauchy sequence with respect to the L^1-Wassertein distance

$$W_T(P, P') := \inf_{\Pi\in\mathscr{C}(P,P')} \int_{\Omega_T\times\Omega_T} \|\xi - \eta\|_{1,\infty} \Pi(\mathrm{d}\xi, \mathrm{d}\eta).$$

To this end, for any $n > m \geq 2$, we construct a coupling of $P_{x,T}^{(n)}$ and $P_{x,T}^{(m)}$ as follows.

Firstly, let $(X_i^{(n)}(t))_{1\leq i\leq n}$ solve the following SDE with $X_0^{(n)} = x^{(n)}$:

FY (4.3)
$$\begin{aligned} \mathrm{d}X_i^{(n)}(t) =& \Big[\alpha_i\big(1-|X^{(n)}(t)|_1\big) - \alpha_\infty X_i^{(n)}(t)\Big]\mathrm{d}t \\ &+ \sqrt{2(1-|X^{(n)}(t)|_1)X_i^{(n)}(t)}\,\mathrm{d}B_i(t),\ 1\leq i\leq n-1; \\ \mathrm{d}X_n^{(n)}(t) =& \Big[\sum_{j=n}^\infty \alpha_j\big(1-|X^{(n)}(t)|_1\big) - \alpha_\infty X_n^{(n)}(t)\Big]\mathrm{d}t \\ &+ \sqrt{2(1-|x^{(n)}(t)|_1)X_n^{(n)}(t)}\,\mathrm{d}B_n(t),\quad t\in[0,T], \end{aligned}$$

where $(B_i(t)_{1\leq i\leq n}$ are independent one-dimensional Brownian motions. Then $P_{x,T}^{(n)}$ is the distribution of $(X^{(n)}(t))_{t\in[0,T]}$.

Next, let

FY2 (4.4)
$$X_i^{(m)}(t) = X_i^{(n)}(t) \text{ for } 1\leq i\leq m-1, \text{ and } X_m^{(m)}(t) = \sum_{j=m^n} X_j^{(n)}(t),\quad t\in[0,T].$$

Then $X^{(m)}(0) = x^{(m)}$ and by (4.3),

$$\begin{aligned} \mathrm{d}X_i^{(m)}(t) =& \Big[\alpha_i\big(1-|X^{(m)}(t)|_1\big) - \alpha_\infty X_i^{(m)}(t)\Big]\mathrm{d}t \\ &+ \sqrt{2(1-|x^{(m)}(t)|_1)X_i^{(m)}(t)}\,\mathrm{d}B_i(t),\ 1\leq i\leq m-1; \\ \mathrm{d}X_m^{(m)}(t) =& \Big[\sum_{j=m}^\infty \alpha_j\big(1-|X^{(m)}(t)|_1\big) - \alpha_\infty X_m^{(m)}(t)\Big]\mathrm{d}t \\ &+ \sqrt{2(1-|x^{(m)}(t)|_1)X_m^{(m)}(t)}\,\mathrm{d}\tilde{B}_m(t),\quad t\in[0,T], \end{aligned}$$

where $\mathrm{d}\tilde{B}_m(t) := \frac{1}{\sqrt{X_m^{(m)}(t)}}\sum_{i=m}^n \sqrt{X_i^{(n)}(t)}\,\mathrm{d}B_i(t)$ is a one-dimensional Brownian motion independent of $(B_i(t))_{1\leq i\leq m-1}$. Therefore, $(X^{(m)}(t))_{t\in[0,T]}$ has law $P_{x,T}^{(m)}$.

Now, by (4.4) and the definition of W_T, we have

FY3 (4.5)
$$W_T(P_{x,T}^{(n)}, P_{x,T}^{(m)}) \leq \mathbb{E}\sup_{t\in[0,T]}|X^{(m)}(t) - X^{(n)}(t)|_1 = \mathbb{E}\sup_{t\in[0,T]}\sum_{j=m+1}^n X_j^{(n)}(t).$$

Let $Z(t) = \sum_{j=m+1}^n X_j^{(n)}(t)$. By (4.3) we have

$$\mathrm{d}Z(t) \leq \Big(\sum_{j=m+1}^\infty \alpha_j\Big)\mathrm{d}t + \sum_{j=m+1}^n \sqrt{s(1-|X^{(n)}(t)|_1)X_i^{(n)}(t)}\,\mathrm{d}B_i(t).$$

So,

$$Z(t) \le \sum_{j=1+m}^{\infty} (x_j + t\alpha_j) + \sum_{j=m+1}^{n} \int_0^t \sqrt{s(1-|X^{(n)}(s)|_1)X_i^{(n)}(s)}\, \mathrm{d}B_i(s) =: \bar{Z}(t), \quad t \in [0,T].$$

Since $Z(t) \ge 0$, $\bar{Z}(t)$ is a nonnegative submartingale. Then by Kolmogorov's inequality,

$$\mathbb{P}\Big(\sup_{t\in[0,T]} Z(t) \ge \lambda\Big) \le \mathbb{P}\Big(\sup_{t\in[0,T]} \bar{Z}(t) \ge \lambda\Big) \le \frac{1}{\lambda}\mathbb{E}\bar{Z}(T) = \frac{1}{\lambda}\sum_{j=m+1}^{\infty}(x_j + \alpha_j T), \quad \lambda > 0.$$

Since $Z(t) \le 1$, this implies

$$\mathbb{E}\sup_{t\in[0,T]} Z(t) \le \lambda + \mathbb{P}\Big(\sup_{t\in[0,T]} Z(t) \ge \lambda\Big) \le \lambda + \frac{1}{\lambda}\sum_{j=m+1}^{\infty}(x_j + \alpha_j T), \quad \lambda > 0.$$

Taking $\lambda = \sqrt{\sum_{j=m+1}^{\infty}(x_j + \alpha_j T)}$, and combining with (4.5), we obtain

$$\lim_{m\to\infty}\sup_{n\ge m+1} W_T(P_{x,T}^{(n)}, P_{x,T}^{(m)}) \le 2\lim_{m\to\infty}\sqrt{\sum_{j=m+1}^{\infty}(x_j + \alpha_j T)} = 0.$$

Therefore, the first assertion is proved.

(b) Let $f \in \mathscr{F}C^2$. We have $f(x) = f(x_1, \cdots, x_m)$ for some $m \ge 1$ and $f \in C^2(\Delta^{(m)})$. For the coordinate process $X(t)$, define

$$M^{(n)}(t) = f(X(t)) - \int_0^t L_{\alpha,\alpha_\infty}^{(n)} f(X(s))\mathrm{d}s, \quad n \ge m, t \in [0,T].$$

Then $(M_t^{(n)})_{t\in[0,T]}$ is a $P_{x,T}^{(n)}$-martingale; that is, for any $0 < s < t \le T$, and any bounded Lipschitz continuous function g on Ω_T measurable with respect to $\mathscr{F}_s$,

WF4 (4.6)
$$\int_{\Omega_T} M^{(n)}(t)(\omega)g(\omega)\mathrm{d}P_{x,T}^{(n)} = \int_{\Omega_T} M^{(n)}(s)(\omega)g(\omega)\mathrm{d}P_{x,T}^{(n)}.$$

We intend to prove the same equality for $P_{x,T}^{(\infty)}$ and

$$M^{(\infty)}(t) := f(X(t)) - \int_0^t L_{\alpha,\alpha_\infty}^{(\infty)} f(X(s))\mathrm{d}s, \ t \in [0,T].$$

By an approximation argument, we may and do assume that $f \in C_b^3(\Delta^{(m)})$. In this case, $M^{(n)}(t)$ is bounded and Lipschitz on Ω_T uniformly in $n \ge m$ and $t \in [0,T]$. Since g is bounded and Lipschitz on Ω_T as well, there exists a constant $C > 0$ such that

$$|(M^{(n)}(t)g)(\xi) - (M^{(n)}g)(t)(\eta)| \le C\|\xi - \eta\|_{1,\infty}, \quad n \ge m, \xi, \eta \in \Omega_T, t \in [0,T].$$

Therefore,

$$\left| \int_{\Omega_T} M^{(n)}(t) g \mathrm{d} P^{(n)}_{x,T} - \int_{\Omega_T} M^{(n)}(t) g \mathrm{d} P^{(\infty)}_{x,T} \right| \leq C W_T(P^{(n)}_{x,T}, P^{(\infty)}_{x,T}), \quad n \geq m, t \in [0,T].$$

Combining this with (4.6), $\lim_{n\to\infty} W_T(P^{(n)}_{x,T}, P^{(\infty)}_{x,T}) = 0$, $\lim_{n\to\infty} M^{(n)} = M^{(\infty)}$ and noting that $\{M^{(n)}g\}_{n\geq m}$ are uniformly bounded, we conclude that

$$\begin{aligned}
&\left| \int_{\Omega_T} \big[M^{(\infty)}(t) - M^{(\infty)}(s)\big] g \, \mathrm{d} P^{(\infty)}_{x,T} \right| \\
&= \lim_{n\to\infty} \left| \int_{\Omega_T} \big[M^{(n)}(t) - M^{(n)}(s)\big] g \, \mathrm{d} P^{(\infty)}_{x,T} \right| \\
&\leq 2C \limsup_{n\to\infty} W_T(P^{(n)}_{x,T}, P^{(\infty)}_{x,T}) = 0.
\end{aligned}$$

Then the proof is finished. □

5 A Discrete Model

For any $N \geq 1$, $M \geq N+1$, consider a population of M individuals of $N+1$ different types. Divide the population into two groups: group I of types $1, \ldots, N$ and group II of type $N+1$. Focusing on group I and treat group II as outsiders or external sources. Initially the number of type i individuals is $m_i, i = 1, \ldots, N+1$. The group I evolves as follows: a type i individual independent of all others will wait for an exponential time at rate α_{N+1} and at the end of the waiting emigrates to the outside becoming type $N+1$; an outsider will independently wait an exponential time with rate α_i and immigrate to group I becoming type i; in addition to emigration and immigration, each couple between a type I and a type II waits for an exponential time with rate 2 and when the clock rings, either the group I individual moves out becoming an outsider or the group II individual moves in becoming the type of the selected individual in group I.

Let $X(t) = M^{-1}(M_1(t), \ldots, M_N(t))$ denote the relative frequencies of individuals of different types in group I among the whole population at time t. For $\alpha \in (0,\infty)^{N+1}$, we construct $X(t)$ as a multivariate Markov chain with generator

$$\begin{aligned}
\mathscr{A}^{(N)}_{M,\alpha} f(x) &= M \sum_{i=1}^{N} \Big\{ \alpha_{N+1} x_i \Big[f\Big(x - \frac{e_i}{M}\Big) - f(x) \Big] + \alpha_i (1 - |x|_1) \Big[f\Big(x + \frac{e_i}{M}\Big) - f(x) \Big] \Big\} \\
&\quad + M^2 \sum_{i=1}^{N} (1 - |x|_1) x_i \Big\{ f\Big(x - \frac{e_i}{M}\Big) + f\Big(x + \frac{e_i}{M}\Big) - 2f(x) \Big\}, \quad f \in C^2(\Delta^{(N)})
\end{aligned}$$

for $x \in \Delta^{(N)}_M := \big\{ x \in \frac{1}{M}\mathbb{Z}^N_+ : \ |x|_1 = \sum_{i=1}^N x_i \leq 1 \big\}$, where e_i is the unit vector in the ith direction. Letting $M \to \infty$ and $x \to y \in \Delta^{(N)}$, one gets $\mathscr{A}^{(N)}_{M,\alpha} f(x) \to L^{(N)}_\alpha f(y)$.

We will see that the finite Markov chain generated by $\mathscr{A}_{M,\alpha}^{(N)}$ on $\Delta_M^{(N)}$ is reversible with respect to the probability measure $\mu_{M,\alpha}^{(N)}$:

$$\mu_{M,\alpha}^{(N)}(x) := \frac{[\alpha_{N+1}]_{M(1-|x|_1)}}{Z\{M(1-|x|_1)\}!}\prod_{i=1}^N \frac{[\alpha_i]_{Mx_i}}{(Mx_i)!}, \quad x \in \Delta_M^{(N)},$$

where $[\alpha]_m := \prod_{i=0}^{m-1}(\alpha+i)$ for $\alpha \geq 0$ and $m \geq 1$, $[\alpha]_0 := 1$, and

$$Z := \sum_{x\in\Delta_M^{(N)}} \frac{[\alpha_{N+1}]_{M(1-|x|_1)}}{\{M(1-|x|_1)\}!}\prod_{i=1}^N \frac{[\alpha_i]_{Mx_i}}{(Mx_i)!}$$

is the normalization. Moreover, for $N \geq 2$, $\mathscr{A}_{M,\alpha}^{(N)}$ has the same spectral gap α_{N+1} as $L_\alpha^{(N)}$.

Theorem 5.1. *Let $N \geq 2$. The Markov chain generated by $\mathscr{A}_{M,\alpha}^{(N)}$ is irreducible and reversible with respect to $\mu_{M,\alpha}^{(N)}$. Moreover, $\mathscr{A}_{M,\alpha}^{(N)}$ has spectral gap α_{N+1} in $L^2(\mu_{M,\alpha}^{(N)})$.*

Proof. (a) Denote $\gamma_i = \frac{e_i}{M}$ for $1 \leq i \leq N$. For any $x, y \in \Delta_M^{(N)}$, let

$$q_{x,y} = \begin{cases} Mx_i\alpha_{N+1} + M^2x_i(1-|x|_1), & \text{if } y = x-\gamma_i, 1 \leq i \leq N; \\ \alpha_i M(1-|x|_1) + M^2x_i(1-|x|_1), & \text{if } y = x+\gamma_i, 1 \leq i \leq N; \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$\mathscr{A}_{M,\alpha}^{(N)}f(x) = \sum_{y\in\Delta_M^{(N)}} q_{xy}\{f(y)-f(x)\}, \quad x \in \Delta_M^{(N)}.$$

Since $q_{x,y} > 0$ when $x, y \in \Delta_M^{(N)}$ with $y = x \pm \gamma_i$ for $1 \leq i \leq N$, and $\Delta_M^{(N)}$ is connected by the edges $x \to x \pm \gamma_i$, we see that the Markov chain is irreducible.

Next, it is well known that $\mathscr{A}_{M,\alpha}^{(N)}$ is symmetric in $L^2(\mu_{M,\alpha}^{(N)})$ if and only if

Z2 (5.1)
$$\mu_{M,\alpha}^{(N)}(x)q_{x,y} = \mu_{M,\alpha}^{(N)}(y)q_{y,x}, \quad x, y \in \Delta_M^{(N)}.$$

To verify this condition, we only need to consider the following two situations.

(a1) $y = x + \gamma_i$ for some $1 \leq i \leq N$. In this case we have $M|x|_1 \leq M-1$, and by the definition of $\mu_{M,\alpha}^{(N)}$,

$$\frac{\mu_{M,\alpha}^{(N)}(y)}{\mu_{M,\alpha}^{(N)}(x)} = \frac{M(1-|x|_1)(\alpha_i+Mx_i)}{(\alpha_{N+1}+M(1-|x|_1)-1)(Mx_i+1)} = \frac{q_{xy}}{q_{yx}}.$$

(a2) $y = x - \gamma_i$ for some $1 \leq i \leq N$. In this case we have $Mx_i \geq 1$, and by the definition of $\mu_{M,\alpha}^{(N)}$,

$$\frac{\mu_{M,\alpha}^{(N)}(y)}{\mu_{M,\alpha}^{(N)}(x)} = \frac{(\alpha_{N+1}+M(1-|x|_1))Mx_i}{(M(1-|x|_1)+1)(Mx_i-1+\alpha_i)} = \frac{q_{xy}}{q_{yx}}.$$

In conclusion, (5.1) holds and thus, $\mathscr{A}^{(N)}_{M,\alpha}$ is symmetric in $L^2(\mu^{(N)}_{M,\alpha})$.

(b) For any $d \in \mathbb{Z}_+$, consider again $\mathscr{P}_d$ the space of all polynomials (in N variables) whose total degree is less than or equal to d. For any $f \in \mathscr{P}_d$ and $1 \le i \le N$, $x \mapsto f(x-\gamma_i) - f(x)$ and $x \mapsto f(x+\gamma_i) - f(x)$ are polynomials belonging to $\mathscr{P}_{d-1}$, while $x \mapsto f(x-\gamma_i) + f(x+\gamma_i) - 2f(x)$ is a polynomial belonging to $\mathscr{P}_{d-2}$. From the definition of $\mathscr{A}^{(N)}_{M,\alpha}$, it follows that $\mathscr{P}_d$ is preserved by $\mathscr{A}^{(N)}_{M,\alpha}$. As in Section 2, we consider for $d \in \mathbb{Z}_+$,

$$\mathscr{Q}_d \;:=\; \{f \in \mathscr{P}_d \cap L^2(\mu^{(N)}_{M,\alpha}) :\; \mu^{(N)}_{M,\alpha}[fg] = 0, \forall g \in \mathscr{P}_{d-1}\}$$

(with the convention $\mathscr{Q}_0 = \mathscr{P}_0$). Note that for d large enough, $\mathscr{Q}_d = \{0\}$, nevertheless, we still have

$$L^2(\mu^{(N)}_{M,\alpha}) \;=\; \bigoplus_{d \in \mathbb{Z}_+} \mathscr{Q}_d$$

and the $\mathscr{Q}_d$ are orthogonal. Furthermore by symmetry of $\mathscr{A}^{(N)}_{M,\alpha}$ in $L^2(\mu^{(N)}_{M,\alpha})$, each of the $\mathscr{Q}_d$ is preserved by $\mathscr{A}^{(N)}_{M,\alpha}$. Thus it is sufficient to study the spectral decompositions of the restrictions of $\mathscr{A}^{(N)}_{M,\alpha}$ to the $\mathscr{Q}_d$. But this is exactly the same analysis as in Section 2, because there we only used the highest monomials. Indeed, note that for all $f \in \mathscr{Q}_d$ and $1 \le i \le N$,

$$x \mapsto f(x-\gamma_i) - f(x) + \frac{\partial_i f(x)}{M}, \quad x \mapsto f(x+\gamma_i) - f(x) - \frac{\partial_i f(x)}{M}$$

are polynomials belonging to $\mathscr{P}_{d-2}$, and

$$x \mapsto f(x-\gamma_i) + f(x+\gamma_i) - 2f(x) - \frac{\partial_i^2 f(x)}{M^2}$$

belong to $\mathscr{P}_{d-3}$, where we set $\mathscr{P}_k = \{0\}$ if $k < 0$. Thus, for any polynomial $f \in \mathscr{Q}_d$, the polynomials $\mathscr{A}^{(N)}_{M,\alpha} f$ and $L^{(N)}_\alpha f$ have the same highest order term (i.e. the term of degree d), so that these two operators have the same spectral gap.

□

Finally, we show that $\mu^{(N)}_{M,\alpha}$ converges weakly to $\mu^{(N)}_\alpha$ as $M \to \infty$.

Proposition 5.2. *Under the topology on $\Delta^{(N)}$ induced by $|\cdot|_1$, $\mu^{(N)}_{M,\alpha}$ converges weakly to $\mu^{(N)}_\alpha$ as $M \to \infty$.*

Proof. It suffices to prove that $\mu^{(N)}_{M,\alpha}(f) \to \mu^{(N)}_\alpha(f)$ for any polynomial f. We first consider $\mu(f) = 0$, i.e. $f \in \mathscr{P}_{0,d}$ for some $d \ge 1$. Since $L^{(N)}_\alpha|_{\mathscr{P}_{0,d}}$ is bounded with eigenvalues not larger than $-\alpha_{N+1}$, $L^{(N)}_\alpha$ is invertible on $\mathscr{P}_{0,d}$. So, there exists $g \in \mathscr{P}_{0,d}$ such that $f = L^{(N)}_\alpha g$. Noting that

$$\lim_{M\to\infty} \|\mathscr{A}^{(N)}_{M,\alpha} g - L^{(N)}_\alpha g\|_\infty = 0, \quad \mu^{(N)}_{\alpha,M}(\mathscr{A}^{(N)}_{M,\alpha} g) = 0,$$

we obtain

$$\lim_{M\to\infty}|\mu_{\alpha,M}^{(N)}(f)| = \lim_{M\to\infty}|\mu_{\alpha,M}^{(N)}(L_\alpha^{(N)}g)| = \lim_{M\to\infty}|\mu_{\alpha,M}^{(N)}(\mathscr{A}_{\alpha,M}^{(N)}g)| = 0.$$

That is, $\lim_{M\to\infty}\mu_{\alpha,M}^{(N)}(f) = \mu_\alpha^{(N)}(f)$ holds for any polynomial f with $\mu(f) = 0$. In general, if $\mu(f) \neq 0$, by letting $\hat{f} = f - \mu(f)$ we obtain

$$0 = \lim_{M\to\infty}\mu_{\alpha,M}^{(N)}(\hat{f}) = \lim_{M\to\infty}\mu_{\alpha,M}^{(N)}(f) - \mu(f).$$

Then the proof is finished. □

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