Estimates on the amplitude of the first Dirichlet eigenvector in discrete frameworks

Persi Diaconis[†] and Laurent Miclo[‡]

[†]Department of Mathematics
Stanford University, California, USA

[‡]Institut de Mathématiques de Toulouse, UMR 5219
Université de Toulouse and CNRS, France

Abstract

Consider a finite absorbing Markov generator, irreducible on the non-absorbing states. Perron-Frobenius theory ensures the existence of a corresponding positive eigenvector φ. The goal of the paper is to give bounds on the amplitude $\max\varphi/\min\varphi$. Two approaches are proposed: one using a path method and the other one, restricted to the reversible situation, based on spectral estimates. The latter approach is extended to denumerable birth and death processes absorbing at 0 for which infinity is an entrance boundary. The interest of estimating the ratio is the reduction of the quantitative study of convergence to quasi-stationarity to the convergence to equilibrium of related ergodic processes, as seen in [8].

Keywords: finite absorbing Markov process, first Dirichlet eigenvector, path method, spectral estimates, denumerable absorbing birth and death process, entrance boundary.

MSC2010: primary: 60J27, secondary: 15B48, 15A42, 05C50, 60J80, 47B36.

[†]Partially supported by ANR-11-LABX-0040-CIMI within the program ANR-11-IDEX-0002-02
[‡]Partially supported by ANR-12-BS01-0019

1 Introduction

This paper, a companion to [8], develops tools to get useful quantitative bounds on rates of convergence to quasi-stationarity for absorbing Markov processes. With notation explained below, the bounds in [8] are of the form

$$\frac{\varphi_\wedge}{2\varphi_\vee}\left\|\widetilde{\mu}_0\widetilde{P}_t-\widetilde{\eta}\right\|_{\mathrm{tv}} \;\leqslant\; \|\mu_t-\nu\|_{\mathrm{tv}} \;\leqslant\; 2\frac{\varphi_\vee}{\varphi_\wedge}\left\|\widetilde{\mu}_0\widetilde{P}_t-\widetilde{\eta}\right\|_{\mathrm{tv}}.$$

In the middle is the term of interest: μ_t is the transition probability conditioned on non-absorbtion at time $t \geqslant 0$ and ν is the quasi-stationary distribution. On both sides, $\widetilde{P}_t$ is the Doob transform (forced to be non-absorbing), $\widetilde{\mu}_0$ is an associated starting distribution and $\widetilde{\eta}$ is the stationary distribution of the transformed process. The point is that quantitative rates of convergence to quasi-stationarity are hard to come by, requiring new tools which are not readily available. The pair $(\widetilde{P}_t, \widetilde{\eta})$ is a usual ergodic Markov chain with many techniques available.
The two sides differ by a factor $\varphi_\wedge/2\varphi_\vee$. Here φ is the usual Peron-Forbenius eigenfunction for the matrix restricted to the non-absorbing sites and $\varphi_\wedge \coloneqq \min\varphi$, $\varphi_\vee \coloneqq \max\varphi$. For the bounds to be useful, we must get control of this ratio. In [8], this control was achieved in special examples where analytic expressions are available with explicit diagonalization. The purpose of the present paper is to give a probabilistic interpretation of this ratio as well as several bounding techniques. For background on quasi-stationarity see Méléard and Villemonais [17], Collet, Martínez and San Martín [6], van Doorn and Pollett [27], Champagnat and Villemonais [3] or the discussion in [8]. We proceed to a more careful description.

Let us begin by introducing the finite setting. The whole finite state space is $\bar{S} \coloneqq S \sqcup \{\infty\}$, where ∞ is the absorbing point. This means that $\bar{S}$ is endowed with a Markov generator matrix $\bar{L} \coloneqq (\bar{L}(x,y))_{x,y\in\bar{S}}$ whose restriction to $S\times S$ is irreducible and such that

$$\begin{array}{rrcl} \forall\ x\in\bar{S}, & \bar{L}(\infty,x) & = & 0 \\ \exists\ x\in S\ : & \bar{L}(x,\infty) & > & 0. \end{array}$$

Recall that a Markov (respectively subMarkovian) generator is a matrix whose off-diagonal entries are non-negative and such that the sums of the entries of a row all vanish (resp. are non-positive).

An eigenvalue λ of $\bar{L}$ is said to be of Dirichlet type if an associated eigenvector vanishes at ∞. Equivalently, λ is an eigenvalue of the $S\times S$ minor K of $\bar{L}$. Since the matrix K is an irreducible subMarkovian generator, the Perron-Frobenius theorem implies that K admits a unique eigenvalue λ_0 whose associated eigenvector is positive. The eigenvalue λ_0 is simple and we denote by φ an associated positive eigenvector. Its renormalization is not very important for us, because we will be mainly concerned by its *amplitude* defined by

$$a_\varphi \;\coloneqq\; \frac{\varphi_\vee}{\varphi_\wedge},$$

with

$$\varphi_\vee \;\coloneqq\; \max_{x\in S}\varphi(x), \qquad \varphi_\wedge \;\coloneqq\; \min_{x\in S}\varphi(x).$$

We refer to [8] for the importance of a_φ in the investigation of the convergence to quasi-stationarity of the absorbing Markov processes generated by $\bar{L}$. Our purpose here is to estimate this quantity.

Our approach is based on a probabilistic interpretation of φ and, more precisely, of the ratios of its values. For any $x\in S$, let $X^x \coloneqq (X^x_t)_{t\geqslant 0}$ be a càdlàg Markov process generated by $\bar{L}$ and starting from x. For any $y\in\bar{S}$, denote by τ^x_y the first hitting time of y by X^x:

$$\tau^x_y \;\coloneqq\; \inf\{t\geqslant 0\ :\ X^x_t = y\}, \tag{1}$$

with the convention that $\tau_y^x = +\infty$ if X^x never reaches y. The first identity below comes from Jacka and Roberts [14].

Proposition 1 *For any $x, y \in S$, we have*

$$\frac{\varphi(x)}{\varphi(y)} \quad = \quad \mathbb{E}[\exp(\lambda_0 \tau_y^x) \mathbb{1}_{\tau_y^x < \tau_\infty^x}].$$

In particular, with $O \coloneqq \{x \in S \,:\, \bar{L}(x, \infty) > 0\}$, we have

$$a_\varphi \quad = \quad \max_{x\in S,\, y\in O} \mathbb{E}[\exp(\lambda_0 \tau_y^x) \mathbb{1}_{\tau_y^x < \tau_\infty^x}].$$

This probabilistic interpretation leads to two methods of estimating a_φ. The first one is through a path argument.

If $\gamma = (\gamma_0, \gamma_1, ..., \gamma_l)$ is a path in S, with $\bar{L}(\gamma_k, \gamma_{k+1}) > 0$ for all $k \in [\![0, l-1]\!]$, denote

$$P(\gamma) \quad \coloneqq \quad \prod_{k\in[\![0,l-1]\!]} \frac{\bar{L}(\gamma_k, \gamma_{k+1})}{|\bar{L}(\gamma_k, \gamma_k)| - \lambda_0} \tag{2}$$

(for any $l' \leqslant l'' \in \mathbb{Z}$, $[\![l', l'']\!] \coloneqq \{l', l'+1, ..., l''-1, l''\}$ and for $l' \in \mathbb{N}$, $[\![l']\!] \coloneqq [\![1, l']\!]$).

Proposition 2 *Assume that for any $y \in O$ and $x \in S$, we are given a path $\gamma_{y,x}$ going from y to x. Then we have*

$$a_\varphi \quad \leqslant \quad \left(\min_{y\in O,\, x\in S} P(\gamma_{y,x})\right)^{-1}.$$

The second method requires that K (the generator restricted to S) admit a reversible probability η on S, namely satisfying

$$\forall\ x, y \in S, \qquad \eta(x)\bar{L}(x, y) \quad = \quad \eta(y)\bar{L}(y, x).$$

The operator $-K$ is then diagonalizable. Let $\lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{N-1}$ be its eigenvalues, where N is the cardinality of S (the first inequality is strict, due to the Perron Frobenius theorem and to the irreducibility of K). For any $x \in S$, let $\lambda_0(S\backslash\{x\})$ be the first eigenvalue of the $(S\backslash\{x\}) \times (S\backslash\{x\})$ minor of $-K$ (or of $-\bar{L}$). Finally, consider

$$\lambda_0' \quad \coloneqq \quad \min_{x\in O} \lambda_0(S\backslash\{x\}). \tag{3}$$

Proposition 3 *Under the reversibility assumption, we have*

$$a_\varphi \quad \leqslant \quad \left(\left(1 - \frac{\lambda_0}{\lambda_0'}\right) \prod_{k\in[\![N-2]\!]} \left(1 - \frac{\lambda_0}{\lambda_k}\right)\right)^{-1}.$$

One advantage of the last result is that it can be extended to absorbing processes on denumerable state spaces, at least under appropriate assumptions. We won't develop a whole theory here, so let us just give the example of birth and death processes on $\mathbb{Z}_+$ absorbing at 0 and for which ∞ is an entrance boundary. To follow the usual terminology in this domain, we change the notation, 0 being the absorbing point and ∞ being the boundary point at infinity of $\mathbb{Z}_+$. We consider $S \coloneqq \mathbb{N} \coloneqq \{1, 2, 3, ...\}$ and $\bar{S} \coloneqq \mathbb{Z}_+ \coloneqq \{0, 1, 2, 3, ...\}$, endowed with a birth and death generator $\bar{L}$, namely of the form

$$\forall\ x \neq y \in \bar{S}, \qquad \bar{L}(x, y) \quad = \quad \begin{cases} b_x & \text{, if } y = x + 1 \\ d_x & \text{, if } y = x - 1 \\ -d_x - b_x & \text{, if } y = x \\ 0 & \text{, otherwise,} \end{cases}$$

where $(b_x)_{x\in\mathbb{Z}_+}$ and $(d_x)_{x\in\mathbb{N}}$ are the positive birth and death rates, except that $b_0 = 0$: 0 is the absorbing state and the restriction of $\bar{L}$ to $\mathbb{N}$ is irreducible.

The boundary point ∞ is said to be an entrance boundary for $\bar{L}$ (cf. for instance Section 8.1 of the book [2] of Anderson) if the following conditions are met:

$$\sum_{x=1}^{\infty} \frac{1}{\pi_x b_x} \sum_{y=1}^{x} \pi_y \quad = \quad +\infty \tag{4}$$

$$\sum_{x=1}^{\infty} \frac{1}{\pi_x b_x} \sum_{y=x+1}^{\infty} \pi_y \quad < \quad +\infty, \tag{5}$$

where

$$\forall\ x \in \mathbb{N}, \qquad \pi_x \quad \coloneqq \quad \begin{cases} 1 & \text{, if } x = 1 \\ \frac{b_1 b_2 \cdots b_{x-1}}{d_2 d_3 \cdots d_x} & \text{, if } x \geqslant 2. \end{cases} \tag{6}$$

The meaning of (4) is that it is not possible (a.s.) for the underlying process X^x, for $x \in \mathbb{Z}_+$ to explode to ∞ in finite time, while (5) says it can come back in finite time from as close as wanted to ∞.

One consequence of (5) is that $Z \coloneqq \sum_{x\in\mathbb{N}} \pi_x < +\infty$, so we can consider the probability

$$\forall\ x \in \mathbb{N}, \qquad \eta(x) \quad \coloneqq \quad Z^{-1}\pi_x.$$

Denote by $\mathcal{F}$ the space of functions which vanish outside a finite subset of points from $\mathbb{N}$ and by K the restriction of the operator $\bar{L}$ to $\mathcal{F}$. It is immediate to check that K is symmetric on $\mathbb{L}^2(\eta)$. Thus we can consider its Freidrich's extension (see e.g. the book of Akhiezer and Glazman [1]), still denoted K, which is a self-adjoint operator in $\mathbb{L}^2(\eta)$. The fact that ∞ is an entrance boundary ensures indeed that such a self-adjoint extension is unique. It is furthermore known that the spectrum of $-K$ only consists of eigenvalues of multiplicity one, say the $(\lambda_n)_{n\in\mathbb{Z}_+}$ in increasing order, see for instance Gong, Mao and Zhang [12]. Let φ be an eigenvector associated to the eigenvalue $\lambda_0 > 0$ of $-K$. As in (3), since the absorbing point is only reachable from 1, we also introduce

$$\lambda_0' \quad \coloneqq \quad \lambda_0(\mathbb{N}\backslash\{1\}),$$

which is the first eigenvalue of the restriction of $-K$ to functions which vanish at 1.

We can now state the extension of Proposition 3:

Theorem 4 *Under the above assumptions, we have $\lambda_0' > \lambda_0$ and*

$$\sum_{n\in\mathbb{Z}_+} \frac{1}{\lambda_n} \quad < \quad +\infty. \tag{7}$$

In particular, we deduce that

$$\left(1 - \frac{\lambda_0}{\lambda_0'}\right) \prod_{n\in\mathbb{N}} \left(1 - \frac{\lambda_0}{\lambda_n}\right) \quad > \quad 0.$$

Up to a change of sign, the eigenvector φ is increasing on $\mathbb{Z}_+$ (with the convention $\varphi(0) = 0$). It is furthermore bounded and its amplitude satisfies:

$$\begin{aligned} a_\varphi \quad \coloneqq \quad \frac{\sup_{x\in\mathbb{N}} \varphi(x)}{\inf_{y\in\mathbb{N}} \varphi(y)} \quad &= \quad \frac{\lim_{x\to\infty} \varphi(x)}{\varphi(1)} \\ &\leqslant \quad \left(\left(1 - \frac{\lambda_0}{\lambda_0'}\right) \prod_{n\in\mathbb{N}} \left(1 - \frac{\lambda_0}{\lambda_n}\right)\right)^{-1}. \end{aligned}$$

A priori, this result is interesting only if one has at his disposal some estimates on the gap $\lambda_0' - \lambda_0$ and on the spectrum $(\lambda_n)_{n\in\mathbb{Z}_+}$. For the latter, it can be useful to take into account the other principal Dirichlet eigenvalues: for $n \in \mathbb{Z}_+$, define

$$\lambda^{(n)} \quad \coloneqq \quad \lambda_0(\mathbb{Z}_+ \backslash [\![0, n]\!])$$

(namely the first eigenvalue of the restriction of $-K$ to functions which vanish on $[\![1, n-1]\!]$, so that $\lambda_0^{(0)} = \lambda_0$ and $\lambda_0^{(1)} = \lambda_0'$).

Corollary 5 *With the notation and assumptions of Theorem 4, we have*

$$\begin{aligned} a_\varphi &\leqslant \left(\left(1-\frac{\lambda_0}{\lambda_0'}\right)^2 \prod_{n\in\mathbb{N}\backslash\{1\}} \left(1-\frac{\lambda_0}{\lambda_0^{(n)}}\right)\right)^{-1} \\ &\leqslant \left(\left(1-\frac{\lambda_0}{\lambda_0'}\right)^{1+m} \prod_{n\in\mathbb{N}\backslash[\![1,m]\!]} \left(1-\frac{\lambda_0}{\lambda_0^{(n)}}\right)\right)^{-1} \end{aligned}$$

for any given $m \in \mathbb{N}$.

The advantage of the latter inequality, is that there exist Hardy bounds on the $\lambda^{(n)}$ which are semi-explicit and relatively sharp: define

$$\forall\, n \in \mathbb{Z}_+, \qquad A_n \quad = \quad \sup_{m>n} \sum_{k\in[\![n+1,m]\!]} \frac{1}{\pi(k)d_k} \sum_{l\geqslant m} \pi(l).$$

It was proven in [19] that

$$\forall\, n \in \mathbb{Z}_+, \qquad A_n^{-1}/4 \;\leqslant\; \lambda_0^{(n)} \;\leqslant\; A_n^{-1}. \tag{8}$$

This suggests the following procedure to apply Corollary 5 on concrete examples. Compute or evaluate the A_n, for $n \in \mathbb{Z}_+$, and find $m \in \mathbb{N}$ such that $A_n \leqslant 2A_0$ for all $n > m$. It follows from (8) that for $n \geqslant m$, $1 - \lambda_0/\lambda_n \geqslant 1 - A_n/(4A_0) \geqslant 1/2$, so we get

$$a_\varphi \quad \leqslant \quad \left(\left(1-\frac{\lambda_0}{\lambda_0'}\right)^{1+m} \prod_{n\in\mathbb{N}\backslash[\![1,m]\!]} \left(1-\frac{A_n}{4A_0}\right)\right)^{-1}$$

It would then remain to estimate the gap $\lambda_0' - \lambda_0$ (e.g. Lemma 15 below can serve as a first step in this direction). Nevertheless, the above approach can quickly lead to involved computations. In the final Example 19, we present a class of birth and death processes which can be treated directly through Corollary 5.

There is a classical result converse to Theorem 4, showing that the criterion of entrance boundary is in some sense optimal for effective absorption at 0 and boundedness of φ. It is typically based on the Lyapounov function approach of convergence of Markov processes (cf. the book of Meyn and Tweedie [18]), Proposition 6 below gives a more precise statement for an example.

Let $\bar{L}$ be a birth and death generator on $\mathbb{Z}_+$, absorbing at 0 and irreducible on $\mathbb{N}$. It is always possible to associate to it the minimal Markov processes $X^x \coloneqq (X_t)_{0\leqslant t<\sigma_\infty}$, starting from $x \in \mathbb{Z}_+$ and defined up to the explosion time σ_∞. These are constructed in the following probabilistic way, where all the used random variables are independent (conditionally to the parameters entering in the definition of their laws). We take $X_t^x = x$ for $0 \leqslant t < \sigma_1$, where σ_1 is distributed according to an exponential variable of parameter $|\bar{L}(x,x)|$ (if $x = 0$, $\bar{L}(0,0) = 0$, so $\sigma_1 = +\infty = \sigma_\infty$, namely the trajectory stays at the absorbing point 0). Next, if $x \neq 0$, the position $X_{\sigma_1}^x = y$ is chosen according to the distribution $(L(x,y)/|L(x,x)|)_{y\in\bar{S}\setminus\{x\}}$. The process stays at this position for $t \in [\sigma_1, \sigma_2)$, where $\sigma_2 \coloneqq \sigma_1 + \mathcal{E}_2$, with $\mathcal{E}_2$ an exponential variable of parameter $\left|\bar{L}(X^x_{\sigma_1}, X^x_{\sigma_1})\right|$. If $X^x_{\sigma_1} \neq 0$, the next position $X^x_{\sigma_2} = y$ is chosen according to the distribution $(L(X^x_{\sigma_1}, y)/|L(X^x_{\sigma_1}, X^x_{\sigma_1})|)_{y\in\bar{S}\setminus\{X^x_{\sigma_1}\}}$. This procedure goes on up to the time $\sigma_\infty \coloneqq \lim_{n\to+\infty}\sigma_n$ (by convention $\sigma_\infty = +\infty$ if one of the σ_n, $n \in \mathbb{N}$, is infinite, which a.s. means that 0 has been reached).

We consider again the first hitting times τ^x_y defined in (1), now for $x, y \in \mathbb{Z}_+$.

Proposition 6 *Assume on one hand, that there exist $x \in \mathbb{N}$ such that τ^x_0 is a.s. finite, namely the process X^x a.s. ends up being absorbed at 0. Then this is true for all $x \in \mathbb{Z}_+$. On the other hand, that there exist a positive number $\lambda > 0$ and a positive function φ on $\mathbb{N}$, with finite amplitude $a_\varphi < +\infty$, which satisfy $K[\varphi] \leqslant -\lambda\varphi$. Then ∞ is an entrance boundary for $\bar{L}$.*

Condition (5) (coming back from infinity in finite time) for birth and death processes satisfying (4) (non explosion) and admitting a positive generalized eigenvector (i.e. not necessarily belonging to $\mathbb{L}^2(\eta)$) associated to a positive eigenvalue of $-K$ is also known to be equivalent to the uniqueness of the quasi-invariant probability distribution, see Theorem 3.2 of Van Doorn [26] (or Theorem 5.4 of the book [6] of Collet, Martínez and San Martín). Thus the quantitative reduction (through the amplitude a_φ) of convergence to quasi-stationarity to the convergence to equilibrium presented in [8] can be applied to such birth and death processes, if and only if they admit a unique quasi-invariant distribution.

The uniqueness of the quasi-stationary probability was characterized in a general setting by Champagnat and Villemonais [3]. It appears that a_φ may be infinite in this situation, in particular if diffusion processes are considered (then $\inf\varphi = 0$).

Recently Gao and Mao [11] also gave probabilist and spectral representations of the first Dirichlet eigenvector in the context of Theorem 4, see their Lemmas 3.2 and 3.4. Their formula (3.15) for its amplitude corresponds to the (extension to the infinite setting of the) r.h.s. of (25) below, due to the fact that in the absorbed (at the boundary) birth and death framework, (23) and (24) are in fact equalities.

The paper is constructed according to the following plan. In the next section, Proposition 1 is recovered along with a probabilistic interpretation of the first Dirichlet eigenvector φ. As a consequence, Propositions 2 and 3 are obtained in Section 3. The situation of denumerable absorbing at 0 birth and death processes is treated in Section 4, where an example is given.

2 Probabilistic interpretation of φ

Our main purpose here is to recover the stochastic representation of the ratio of the first Dirichlet eigenvector φ given in Proposition 1. This is due to Jacka and Roberts [14], who deduce it from the corresponding discrete time result proven by Seneta [25]. Since these authors work with denumerable state spaces, for the sake of simplicity and completeness, we present here a direct proof for finite state spaces.

We start by recalling three simple and classical results. Consider $\mathcal{P}(S)$ the set of probability measures on S. Generalizing (1), let us define, for any initial distribution $\mu \in \mathcal{P}(S)$ and for any $y \in \bar{S}$,

$$\tau^\mu_y \quad\coloneqq\quad \inf\{t \geqslant 0 \,:\, X^\mu_t = y\},$$

where $(X^\mu_t)_{t\geqslant 0}$ is a càdlàg Markov process generated by $\bar{L}$ and starting from μ.

Lemma 7 *For any $\lambda \geqslant 0$, we have*

$$\exists\ \mu \in \mathcal{P}(S)\,:\, \mathbb{E}[\exp(\lambda\tau^\mu_\infty)] < +\infty \quad\Leftrightarrow\quad \forall\ \mu \in \mathcal{P}(S),\ \mathbb{E}[\exp(\lambda\tau^\mu_\infty)] < +\infty.$$

Proof

It is sufficient to consider the direct implication, the reverse one being obvious. Since for any $\mu \in \mathcal{P}(S)$, we have

$$\mathbb{E}[\exp(\lambda\tau_\infty^\mu)] = \sum_{x\in S} \mu(x)\mathbb{E}[\exp(\lambda\tau_\infty^x)],$$

we just need to check that

$$\forall\ x, y \in S, \qquad \mathbb{E}[\exp(\lambda\tau_\infty^x)] < +\infty \ \Leftrightarrow\ \mathbb{E}[\exp(\lambda\tau_\infty^y)] < +\infty,$$

namely

$$\forall\ x, y \in S, \qquad \mathbb{E}[\exp(\lambda\tau_\infty^x)] < +\infty \ \Rightarrow\ \mathbb{E}[\exp(\lambda\tau_\infty^y)] < +\infty. \tag{9}$$

For given $x, y \in S$, let $\gamma = (\gamma_0, \gamma_1, ..., \gamma_l)$ be a path in S going from x to y and satisfying $\bar{L}(\gamma_k, \gamma_{k+1}) > 0$. Such a path exists, by irreducibility of K. Let A^γ be the event that the first jump of the trajectory X^x is from $x = \gamma_0$ to γ_1, that the second jump of X^x is from γ_1 to γ_2, ..., that the l-th jump of X^x is from γ_{l-1} to γ_l. By the probabilistic construction of X^x, we have that

$$\begin{aligned}\mathbb{P}[A^\gamma] &= \frac{\bar{L}(\gamma_0, \gamma_1)}{|L(\gamma_0, \gamma_0)|}\frac{\bar{L}(\gamma_1, \gamma_2)}{|L(\gamma_1, \gamma_1)|}\cdots\frac{\bar{L}(\gamma_{l-1}, \gamma_l)}{|L(\gamma_{l-1}, \gamma_{l-1})|} \\ &> 0.\end{aligned} \tag{10}$$

Using the strong Markov property of X^x at the minimum time between the time of the l-th jump time and the absorbing time, we get

$$\begin{aligned}\mathbb{E}[\exp(\lambda\tau_\infty^x)] &\geqslant \mathbb{E}[\mathbb{1}_{A^\gamma}\exp(\lambda\tau_\infty^x)] \\ &\geqslant \mathbb{E}[\mathbb{1}_{A^\gamma}\mathbb{E}[\exp(\lambda\tau_\infty^y)]] \\ &= \mathbb{P}[A^\gamma]\mathbb{E}[\exp(\lambda\tau_\infty^y)],\end{aligned}$$

which implies (9).

■

Define

$$\Lambda := \{\lambda \geqslant 0 : \forall\ \mu \in \mathcal{P}(S),\ \mathbb{E}[\exp(\lambda\tau_\infty^\mu)] < +\infty\}.$$

Lemma 8 *We have*

$$\Lambda = [0, \lambda_0).$$

Proof

Consider $\nu \in \mathcal{P}(S)$ the quasi-stationary distribution associated to $\bar{L}$, namely the left eigenvector of K (extended to vanish at ∞) associated to the eigenvalue $-\lambda_0$. For any $t \geqslant 0$, the distribution of X_t^ν is $\exp(-\lambda_0 t)\nu + (1 - \exp(-\lambda_0 t))\delta_\infty$. It follows that

$$\begin{aligned}\forall\ t \geqslant 0, \qquad \mathbb{P}[\tau_\infty^\nu > t] &= \mathbb{P}[X_t^\nu \in S] \\ &= \exp(-\lambda_0 t),\end{aligned}$$

namely, τ_∞^ν is distributed according to the exponential law of parameter λ_0. In particular, we have

$$\forall\ \lambda \geqslant 0, \qquad \mathbb{E}[\exp(\lambda\tau_\infty^\nu)] = \begin{cases} \frac{\lambda_0}{\lambda_0 - \lambda} & , \text{if } \lambda < \lambda_0 \\ +\infty & , \text{if } \lambda \geqslant \lambda_0. \end{cases}$$

The announced result follows from the previous lemma, showing that

$$\Lambda \quad \coloneqq \quad \{\lambda \geqslant 0 \,:\, \mathbb{E}[\exp(\lambda\tau_\infty^\nu)] < +\infty\}.$$

■

For any $\lambda \in \Lambda$, we can consider the mapping φ_λ defined on $\bar{S}$ by

$$\forall\, x \in \bar{S}, \qquad \varphi_\lambda(x) \quad \coloneqq \quad \frac{\mathbb{E}[\exp(\lambda\tau_\infty^x)]}{\mathbb{E}[\exp(\lambda\tau_\infty^\nu)]},$$

where $\nu \in \mathcal{P}(S)$ is the quasi-stationary distribution of $\bar{L}$, whose definition was recalled in the above proof (but for our purpose, ν could be replaced by any other fixed distribution of $\mathcal{P}(S)$).

Proposition 9 *As $\lambda \in \Lambda$ converges to λ_0, the mapping φ_λ converges on S to a function φ which is a positive eigenvector associated to the eigenvalue λ_0 of $-K$.*

Proof

We begin by checking that for fixed $\lambda \in \Lambda$, φ_λ satisfies

$$\forall\, x \in S, \qquad \bar{L}[\varphi_\lambda](x) \quad = \quad -\lambda\varphi_\lambda(x). \tag{11}$$

To simplify the notation, define

$$\forall\, x \in \bar{S}, \qquad \psi_\lambda(x) \quad \coloneqq \quad \mathbb{E}[\exp(\lambda\tau_\infty^x)], \tag{12}$$

it is sufficient to show that $\bar{L}[\psi_\lambda] = -\lambda\psi_\lambda$ on S. This comes from the fact that for $x \in \bar{S}$, the quantity $\psi_\lambda(x)$ can be seen as the Feynman-Kac integral with respect to the Markov process X^x and the potential $\lambda\mathbb{1}_S$ (for the diffusion analogue, see e.g. Section 4.7 of the book of Chung and Walsh [5]). But maybe the shortest way to deduce it is to use the martingale problem associated to X^x (for a general reference, see the book of Ethier and Kurtz [9]). More precisely, consider the mapping f on $\mathbb{R}_+ \times \bar{S}$ defined by

$$\forall\, (t, y) \in \mathbb{R}_+ \times \bar{S}, \qquad f(t, y) \quad \coloneqq \quad \exp(\lambda t)\psi_\lambda(y).$$

There exists a local martingale $M = (M_t)_{t\geqslant 0}$ such that a.s.

$$\forall\, t \geqslant 0, \qquad f(t, X_t^x) \quad = \quad f(0, x) + \int_0^t \partial_s f(s, X_s^x) + \bar{L}[f(s,\cdot)](X_s^x)\, ds + M_t$$

This is sometimes known as the time-inhomogeneous Dynkin formula (or the Dynkin formula for the time-space Markov process $(t, X_t)_{t\geqslant 0}$), cf. for instance the book of Rogers and Williams [24]. The fact that $\lambda \in \Lambda$ implies that M is an actual martingale (namely that for all $t \geqslant 0$, M_t is integrable). In particular, by the stopping theorem, we get that for any $t \geqslant 0$, $\mathbb{E}[M_{t\wedge\tau_\infty^x}] = 0$, so that

$$\mathbb{E}[f(t\wedge\tau_\infty^x, X_{t\wedge\tau_\infty^x}^x)] \quad = \quad \psi_\lambda(x) + \mathbb{E}\left[\int_0^{t\wedge\tau_\infty^x} \partial_s f(s, X_s^x) + \bar{L}[f(s,\cdot)](X_s^x)\, ds\right].$$

But the strong Markov property applied to the stopping time $t\wedge\tau_\infty^x$ implies that

$$\begin{aligned}\mathbb{E}[f(t\wedge\tau_\infty^x, X_{t\wedge\tau_\infty^x}^x)] \quad &= \quad \mathbb{E}[\exp(\lambda\tau_\infty^x)] \\ &= \quad \psi_\lambda(x),\end{aligned}$$

and we get that

$$\mathbb{E}\left[\int_0^{t\wedge\tau_\infty^x} \partial_s f(s, X_s^x) + \bar{L}[f(s,\cdot)](X_s^x)\, ds\right] \quad = \quad 0.$$

Taking into account that for any $s \geqslant 0$ and $y \in \bar{S}$, $\partial_s f(s,y) = \lambda f(s,y)$, we deduce that

$$\begin{aligned}\lambda\psi_\lambda(x) + \bar{L}[\psi_\lambda](x) &= \lim_{t\to 0_+} t^{-1}\mathbb{E}\left[\int_0^{t\wedge\tau_\infty^x} \lambda f(s, X_s^x) + \bar{L}[f(s,\cdot)](X_s^x)\, ds\right] \\ &= 0,\end{aligned}$$

which amounts to (11).

Of course, the $\lambda \in \Lambda$ are not Dirichlet eigenvalues of $\bar{L}$, because $\varphi_\lambda(\infty) \neq 0$:

$$\varphi_\lambda(\infty) = \frac{1}{\mathbb{E}[\exp(\lambda\tau_\infty^\nu)]},$$

but as $\lambda \in \Lambda$ goes to λ_0, this expression converges to zero. Furthermore, if for $x, y \in S$, we call $r_{x,y}$ the r.h.s. of (10) and

$$r = \min_{x,y\in S} r_{x,y},$$

then

$$\forall\ \lambda \in \Lambda,\ \forall\ x, y \in S, \qquad r \leqslant \frac{\varphi_\lambda(y)}{\varphi_\lambda(x)} \leqslant r^{-1}. \tag{13}$$

Thus we can find a sequence $(l_n)_{n\in\mathbb{N}}$ of elements of Λ converging to λ_0 such that φ_{l_n} converges toward a function φ on $\bar{S}$, positive on S. According to the previous observation $\varphi(\infty) = 0$ and taking the limit in (11), we get

$$\bar{L}[\varphi] = -\lambda_0\varphi,$$

it follows that the restriction to S of φ is a positive eigenvector associated to the eigenvalue λ_0 of $-K$. Furthermore,

$$\nu[\varphi] = \lim_{\lambda\to\lambda_0 -} \nu[\varphi_\lambda] = 1,$$

and this normalization entirely determines φ. It follows that the mapping φ does not depend on the chosen sequence $(l_n)_{n\in\mathbb{N}}$. A usual compactness argument based on (13) shows that in fact

$$\lim_{\lambda\to\lambda_0 -} \varphi_\lambda = \varphi.$$

■

We need a last preliminary result.

Lemma 10 *For any $x \in S$, we have*

$$\lambda_0(S\backslash\{x\}) > \lambda_0,$$

where we recall that the l.h.s. is the first eigenvalue of the $(S\backslash\{x\}) \times (S\backslash\{x\})$ minor of $-\bar{L}$.

Heuristically, this result says that for any fixed $x \in S$, it is asymptotically strictly easier for the underlying processes to exit $S\backslash\{x\}$ than S. It is well-known in the reversible context, via the variational characterization of the eigenvalues, but we cannot use that argument here. Note also that in the trivial case where S is reduced to a singleton, by convention $\lambda_0(\emptyset) = +\infty$ and the above inequality is also true.

Proof

Fix $x \in S$ and let φ^x be a non-negative eigenvector associated to the eigenvalue $-\lambda_0(S\backslash\{x\})$ of the $(S\backslash\{x\}) \times (S\backslash\{x\})$ minor of $-\bar{L}$. Extending φ^x on $\bar{S}$ by making it vanish on $\{\infty, x\}$, we have that

$$\forall\, y \in S\backslash\{x\}, \qquad \bar{L}[\varphi^x](y) \;=\; -\lambda_0(S\backslash\{x\})\varphi^x(y).$$

Consider the set

$$S' \;\coloneqq\; \{y \in S \,:\, \varphi^x(y) = 0\} \;\supset\; \{x\}$$

(S' may be larger than $\{x\}$, since it can happen that the restriction of $\bar{L}$ to $S\backslash\{x\}$ is no longer irreducible). By irreducibility of K, there exists $x_0 \in S'$ and $y_0 \in S\backslash S'$ with $\bar{L}(x_0, y_0) > 0$. It follows that

$$\begin{aligned}
\bar{L}[\varphi^x](x_0) &= \sum_{y\in\bar{S}} \bar{L}(x_0, y)(\varphi^x(y) - \varphi^x(x_0)) \\
&= \sum_{y\in S} \bar{L}(x_0, y)\varphi^x(y) \\
&\geqslant \bar{L}(x_0, y_0)\varphi^x(y_0) \\
&> 0.
\end{aligned}$$

Similarly, we prove that

$$\forall\, y \in S', \qquad \bar{L}[\varphi^x](y) \;\geqslant\; 0$$

(this is the maximum principle for the Markovian generator $\bar{L}$).

Let ν be the quasi-stationary measure associated to $\bar{L}$, already encountered in the proof of Lemma 8. Since $\nu\bar{L} = -\lambda_0\nu$, we have in particular

$$\nu[\bar{L}[\varphi^x]] \;=\; -\lambda_0\nu[\varphi^x].$$

But according to the previous observations, we have

$$\begin{aligned}
\nu[\bar{L}[\varphi^x]] &= \nu[\mathbb{1}_{S\backslash S'}\bar{L}[\varphi^x]] + \nu[\mathbb{1}_{S'}\bar{L}[\varphi^x]] \\
&= -\lambda_0(S\backslash\{x\})\nu[\mathbb{1}_{S\backslash S'}\varphi^x] + \nu[\mathbb{1}_{S'}\bar{L}[\varphi^x]] \\
&= -\lambda_0(S\backslash\{x\})\nu[\varphi^x] + \nu[\mathbb{1}_{S'}\bar{L}[\varphi^x]] \\
&> -\lambda_0(S\backslash\{x\})\nu[\varphi^x].
\end{aligned}$$

It follows that

$$\lambda_0(S\backslash\{x\}) \;>\; \lambda_0.$$

■

We can now come to the

Proof of Proposition 1

Concerning the first equality, let us fix $x, y \in S$. We can assume that $x \neq y$, since the equality is trivial for $x = y$. According to Proposition 9, it is sufficient to see that

$$\lim_{\lambda\to\lambda_0-} \frac{\varphi_\lambda(x)}{\varphi_\lambda(y)} \;=\; \mathbb{E}[\exp(\lambda_0\tau_y^x)\mathbb{1}_{\tau_y^x<\tau_\infty^x}]. \tag{14}$$

Define $\tau = \tau_y^x \wedge \tau_\infty^x$. It is the exit time from $S\backslash\{y\}$ for X^x. In particular, we have

$$\forall\, l \in \mathbb{R}_+, \qquad \mathbb{E}[\exp(l\tau)] < +\infty \;\Leftrightarrow\; l < \lambda_0(S\backslash\{y\}),$$

and Lemma 10 implies that

$$\mathbb{E}[\exp(\lambda_0 \tau)] \quad < \quad +\infty \tag{15}$$

For $\lambda \in \Lambda$, consider again the function ψ_λ defined in (12). Using the strong Markov property at time τ, we have

$$\begin{aligned}
\psi_\lambda(x) &= \mathbb{E}[\exp(\lambda\tau)\mathbb{1}_{X_\tau^x = y}\psi_\lambda(y)] + \mathbb{E}[\exp(\lambda\tau)\mathbb{1}_{X_\tau^x=\infty}\psi_\lambda(\infty)] \\
&= \psi_\lambda(y)\mathbb{E}[\exp(\lambda\tau)\mathbb{1}_{X_\tau^x=y}] + \mathbb{E}[\exp(\lambda\tau)\mathbb{1}_{X_\tau^x=\infty}]
\end{aligned}$$

Dividing by $\psi_\lambda(y)$, taking into account (15) and letting λ go to λ_0, we get (14). In particular, we deduce that

$$a_\varphi \quad = \quad \max_{x\in S,\, y\in S} \mathbb{E}[\exp(\lambda_0 \tau_y^x)\mathbb{1}_{\tau_y^x < \tau_\infty^x}]$$

To show the representation of a_φ in Proposition 1, it is enough to check that $\varphi_\wedge = \min_{y\in O}\varphi(y)$. This is a consequence of the fact that for any $y \in S$, either $y \in O$ or there exists a neighbor $z \in S$ of y (namely a point satisfying $\bar{L}(y,z) > 0$) with $\varphi(z) < \varphi(y)$. Indeed this comes from

$$\begin{aligned}
\sum_{z\in\bar{S}} \bar{L}(y,z)(\varphi(z) - \varphi(y)) &= -\lambda_0\varphi(y) \\
&< 0.
\end{aligned}$$

■

3 Path and spectral arguments

It will be seen here how the probabilistic representation of the amplitude a_φ can be used to deduce more practical estimates.

We begin with a path argument, similar in spirit to the one already encountered in the proof of Lemma 7. Let $\gamma = (\gamma_0, \gamma_1, ..., \gamma_l)$ be a path in S, to which we associate the event A^γ requiring that the first jump of the trajectory X^{γ_0} is from γ_0 to γ_1, that the second jump of X^{γ_0} is from γ_1 to γ_2, ..., that the l-th jump of X^{γ_0} is from γ_{l-1} to γ_l.

Lemma 11 *For any $\lambda \in [0, \min(|\bar{L}(\gamma_k,\gamma_k)| \,:\, k \in [\![0, l-1]\!]))$, we have*

$$\mathbb{E}[\mathbb{1}_{A^\gamma}\exp(\lambda\tau_{\gamma_l}^{\gamma_0})] \quad = \quad \prod_{k\in[\![0,l-1]\!]} \frac{\bar{L}(\gamma_k,\gamma_{k+1})}{|\bar{L}(\gamma_k,\gamma_k)| - \lambda}.$$

If $\lambda \geqslant \min(|\bar{L}(\gamma_k,\gamma_k)| \,:\, k \in [\![0, l-1]\!])$, the expectation in the l.h.s. is infinite.

Proof

This result is directly based on the probabilistic construction of the trajectory X^{γ_0}. Let us recall it: X^{γ_0} stays at γ_0 for an exponential time of parameter $|\bar{L}(\gamma_0,\gamma_0)|$, then it chooses a new position x_1 according to the probability $\bar{L}(\gamma_0,x_1)/|\bar{L}(\gamma_0,\gamma_0)|$. Next it stays at x_1 for an exponential time of parameter $|\bar{L}(x_1,x_1)|$, until it chooses a new position x_2 with respect to the probability $\bar{L}(x_1,x_2)/|\bar{L}(x_1,x_1)|$, etc. To simplify the notation, denote

$$\forall\ k \in [\![0, l-1]\!], \qquad L_k \quad := \quad |\bar{L}(\gamma_k,\gamma_k)|\,.$$

It follows that if $\lambda < \min(L_k \,:\, k \in [\![0, l-1]\!])$,

$$\begin{aligned}
&\mathbb{E}[\mathbb{1}_{A^\gamma} \exp(\lambda \tau_{\gamma_l}^{\gamma_0})] \\
&= \left(\prod_{k\in[\![0,l-1]\!]} \frac{\bar{L}(\gamma_k,\gamma_{k+1})}{L_k}\right) \int\int\cdots\int e^{\lambda(t_0+t_1+\cdots+t_k)}\, L_0 e^{-L_0 t_0} dt_0\, L_1 e^{-L_1 t_1} dt_1 \cdots L_{l-1} e^{-L_{l-1} t_{l-1}} dt_{l-1} \\
&= \prod_{k\in[\![0,l-1]\!]} \left(\frac{\bar{L}(\gamma_k,\gamma_{k+1})}{L_k} L_k \int e^{(\lambda - L_k)t_k}\, dt_k\right) \\
&= \prod_{k\in[\![0,l-1]\!]} \frac{\bar{L}(\gamma_k,\gamma_{k+1})}{L_k - \lambda}.
\end{aligned}$$

The same computation shows that if for some $k \in [\![0, l-1]\!]$, $\lambda \geqslant \left|\bar{L}(\gamma_k,\gamma_k)\right|$, then $\mathbb{E}[\mathbb{1}_{A^\gamma} \exp(\lambda \tau_{\gamma_l}^{\gamma_0})] = +\infty$.

■

Proof of Proposition 2

It is now a consequence of the following observation: if $\gamma_{y,x}$ is a path going from y to x in S, then from Proposition 1, we get

$$\begin{aligned}
\frac{\varphi(y)}{\varphi(x)} &= \mathbb{E}[\exp(\lambda_0 \tau_x^y) \mathbb{1}_{\tau_x^y < \tau_\infty^y}] \\
&\geqslant \mathbb{E}[\mathbb{1}_{A^\gamma} \exp(\lambda_0 \tau_x^y)] \\
&= P(\gamma_{y,x}),
\end{aligned}$$

according to the previous lemma (where the functional P was defined in (2)). Indeed, one would have noticed that

$$\lambda_0 \quad \leqslant \quad \min_{x\in S} \left|\bar{L}(x,x)\right|.$$

Arguments similar to those given in the proof of Lemma 10 (reinterpret $\bar{L}(x,x)$ as the first Dirichlet eigenvalue associated to the $\{x\}\times\{x\}$ minor of $\bar{L}$) show that the above inequality is strict, except if S is a singleton. In the latter case, say $S = \{x_0\}$, necessarily $y = x = x_0$ and $\gamma_{x_0,x_0} = (x_0)$, so that the product defining $P(\gamma_{x_0,x_0})$ is void, meaning that $P(\gamma_{x_0,x_0}) = 1$, as it should be.

Coming back to the general case and taking the minimum over $x \in S$ and $y \in O$, we get

$$\begin{aligned}
a_\varphi &= \left(\min_{y\in O,\, x\in S} \frac{\varphi(y)}{\varphi(x)}\right)^{-1} \\
&\leqslant \left(\min_{y\in O,\, x\in S} P(\gamma_{y,x})\right)^{-1},
\end{aligned}$$

as announced.

■

One can deduce a rougher estimate, where λ_0 does not enter: with the notation of (2), define

$$Q(\gamma) \quad \coloneqq \quad \prod_{k\in[\![0,l-1]\!]} \frac{\left|\bar{L}(\gamma_k,\gamma_k)\right|}{\bar{L}(\gamma_k,\gamma_{k+1})},$$

then we have

$$a_\varphi \quad \leqslant \quad \max_{y\in O,\, x\in S} Q(\gamma_{y,x}). \tag{16}$$

Let us illustrate these computations with

Example 12 (a) Consider an oriented finite strongly connected graph G on the vertex set S and denote by E the set of its oriented edges. Let O be a non-empty subset of S. Let $\bar{G}$ be the oriented graph on $\bar{S} = S \sqcup \{\infty\}$ obtained by adding to E the edges (x, ∞), with $x \in O$. Let d be the maximum outgoing degree of $\bar{G}$ and D be the "oriented diameter" of G. Let $\bar{L}$ be the random walk generator associated to $\bar{G}$:

$$\forall\, x \neq y \in \bar{S}, \qquad \bar{L}(x,y) \;\coloneqq\; \begin{cases} 1 & \text{, if } (x,y) \in \bar{E} \\ 0 & \text{, otherwise.} \end{cases}$$

Choosing geodesics (with respect to the "oriented graph distance") for the underlying paths, the bound (16) implies that

$$a_\varphi \;\leqslant\; d^D.$$

There is an easy comparison allowing weighted edges in the case above. If the generator $\bar{L}$ is perturbed to another generator $\bar{L}$ only satisfying, for some constants $0 < r \leqslant R < +\infty$,

$$\forall\, x \neq y \in \bar{S}, \qquad \bar{L}(x,y) \;\in\; \begin{cases} [r, R] & \text{, if } (x,y) \in \bar{E} \\ \{0\} & \text{, otherwise,} \end{cases}$$

we end up with

$$a_\varphi \;\leqslant\; \left(\frac{Rd}{r}\right)^D. \tag{17}$$

(b) To see if this bound is of the right order, let us consider a specific birth and death examples on $\bar{S} = [\![0, N]\!]$, with $N \in \mathbb{N}$, absorbed in 0 (namely ∞ in the previous notation). The only non-zero jump rates of $\bar{L}$ are given by

$$\forall\, x \in [\![1, N-2]\!], \qquad \begin{cases} \bar{L}(x, x+1) & \coloneqq & \rho \\ \bar{L}(x+1, x) & \coloneqq & 1 \end{cases} \tag{18}$$

$$\bar{L}(1,0) = 1, \quad \bar{L}(N-1, N) \;=\; \rho \;\text{ and }\; \bar{L}(N, N-1) \;=\; 1+\rho, \tag{19}$$

for some fixed $\rho > 0$. The bound (17) leads to

$$a_\varphi \;\leqslant\; \left(\frac{2(1 \vee \rho)}{1 \wedge \rho}\right)^N. \tag{20}$$

It was seen in [8] that

$$a_\varphi \;=\; \begin{cases} \frac{2N}{\pi}(1 + \mathcal{O}(N^{-2})) & \text{, if } \rho = 1 \\ \frac{\rho}{\rho-1}(1 + \mathcal{O}(\rho^{-N})) & \text{, if } \rho > 1, \end{cases} \tag{21}$$

but it can be deduced from the computations presented in Section 3.3 of [8] that if $\rho < 1$ is fixed, then a_φ explodes exponentially with respect to N. So (20) corresponds to the right behavior of a_φ (i.e. it does explode exponentially with respect to N) if and only if $\rho < 1$.

□

In the previous example for $\rho \geqslant 1$, the path estimate does not catch the fact that either a_φ is bounded (for $\rho > 1$) or explodes linearly with respect to N (for $\rho = 1$). The spectral estimates we are about to present are more precise and we will see how to recover these behaviors of a_φ for $\rho \geqslant 1$ and large N.

We begin with a general result which can be deduced from Miclo [22].

Lemma 13 *In the finite setting and under the reversibility assumption, whatever $\mu \in \mathcal{P}(S)$, the time τ_∞^μ is stochastically dominated by the sum of independent exponential variables of respective parameters λ_0, λ_1, ..., λ_{N-1}, where N is the cardinality of S.*

Proof

Indeed, for any $k \in [\![0, N-1]\!]$, denote by $\mathcal{L}_k$ the convolution of $k+1$ exponential laws of parameters λ_N, λ_{N-1}, ..., λ_{N-k}. It was seen in [22] that the law of τ_∞^μ is a mixture of the $\mathcal{L}_k$, for $k \in [\![0, N-1]\!]$, the coefficients of the mixture depending on μ (and the coefficient of $\mathcal{L}_{N-1}$ being positive). The announced result follows from the fact that each of the laws $\mathcal{L}_k$, for $k \in [\![0, N-2]\!]$, is clearly stochastically dominated by $\mathcal{L}_{N-1}$.

■

We can now come to the

Proof of Proposition 3

Note that

$$a_\varphi \quad = \quad 1 \vee \max_{x\in S,\, y\in O,\, y\neq x} \mathbb{E}[\exp(\lambda_0 \tau_y^x) \mathbb{1}_{\tau_y^x < \tau_\infty^x}],$$

thus it is sufficient to show that

$$\max_{x\in S,\, y\in O,\, y\neq x} \mathbb{E}[\exp(\lambda_0 \tau_y^x) \mathbb{1}_{\tau_y^x < \tau_\infty^x}] \quad \leqslant \quad \left(\left(1 - \frac{\lambda_0}{\lambda_0'}\right) \prod_{k\in[\![N-1]\!]} \left(1 - \frac{\lambda_0}{\lambda_k}\right)\right)^{-1}.$$

Fix $y \in O$, let $\widetilde{K}$ be the $(S\backslash\{y\}) \times (S\backslash\{y\})$ minor of $\bar{L}$ and denote $\widetilde{\lambda}_0 < \widetilde{\lambda}_1 \leqslant \cdots \leqslant \widetilde{\lambda}_{N-2}$ the eigenvalues of $-\widetilde{K}$. By the usual interlacing property of the eigenvalues of minors (see for instance Theorem 4.3.8 of the book [13] of Horn and Johnson), we have

$$\lambda_0 < \widetilde{\lambda}_0 \leqslant \lambda_1 \leqslant \widetilde{\lambda}_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{N-2} \leqslant \widetilde{\lambda}_{N-2} \leqslant \lambda_{N-1}. \tag{22}$$

The first inequality is strict, due to Lemma 10. According to the previous lemma, we have that for any $x \in S\backslash\{y\}$, $\tau_y^x \wedge \tau_\infty^x$ is stochastically dominated by the sum of independent exponential variables of respective parameters $\widetilde{\lambda}_0$, $\widetilde{\lambda}_1$, ..., $\widetilde{\lambda}_{N-2}$. It follows that

$$\begin{aligned} \mathbb{E}[\exp(\lambda_0 \tau_y^x) \mathbb{1}_{\tau_y^x < \tau_\infty^x}] \quad &\leqslant \quad \mathbb{E}[\exp(\lambda_0 (\tau_y^x \wedge \tau_\infty^x))] &(23)\\ &\leqslant \quad \mathbb{E}[\exp(\lambda_0 (\mathcal{E}_0 + \cdots + \mathcal{E}_{N-2}))], &(24)\end{aligned}$$

where $\mathcal{E}_0$, ..., $\mathcal{E}_{N-2}$ are independent exponential variables of respective parameters $\widetilde{\lambda}_0$, $\widetilde{\lambda}_1$, ..., $\widetilde{\lambda}_{N-2}$. Thus the last expectation is also equal to

$$\begin{aligned} \prod_{l\in[\![0,N-2]\!]} \mathbb{E}[\exp(\lambda_0 \mathcal{E}_l)] \quad &= \quad \prod_{l\in[\![0,N-2]\!]} \frac{\widetilde{\lambda}_l}{\widetilde{\lambda}_l - \lambda_0} &(25)\\ &\leqslant \quad \left(\left(1 - \frac{\lambda_0}{\widetilde{\lambda}_0}\right) \prod_{k\in[\![N-2]\!]} \left(1 - \frac{\lambda_0}{\lambda_k}\right)\right)^{-1}\\ &\leqslant \quad \left(\left(1 - \frac{\lambda_0}{\lambda_0'}\right) \prod_{k\in[\![N-2]\!]} \left(1 - \frac{\lambda_0}{\lambda_k}\right)\right)^{-1}, \end{aligned}$$

where the interlacing (22) was used, as well as the definition of λ_0' given in (3). Proposition 3 follows, since the above upper bound no longer depends on the choice of $y \in O$.

■

Let us now show how the spectral estimate can provide a better bound than the path estimate, at least when some knowledge on the relevant eigenvalues is available.

Example 14 We return to the birth and death processes presented at the end of Example 12 with $\rho \geqslant 1$.

We first treat the case $\rho = 1$, for which we have seen in [8] that the eigenvalues of $-K$ are given by

$$\forall\, k \in [\![0, N-1]\!], \qquad \lambda_k \;=\; 2(1-\cos((2k+1)\pi/(2N))).$$

With the notation of the proof of Proposition 3, we have $O = \{1\}$ and the matrix $\widetilde{K}$ is the same as K, except that N has been replaced by $N-1$. Thus we get that

$$\forall\, k \in [\![0, N-2]\!], \qquad \widetilde{\lambda}_k \;=\; 2(1-\cos((2k+1)\pi/(2(N-1)))).$$

By using (25) directly, we get the bound

$$\begin{aligned} a_\varphi &\leqslant \prod_{l\in[\![0,N-2]\!]} \frac{\widetilde{\lambda}_l}{\widetilde{\lambda}_l - \lambda_0} \\ &= \prod_{l\in[\![0,N-2]\!]} \left(1-\frac{\lambda_0}{\widetilde{\lambda}_l}\right)^{-1}, \end{aligned}$$

and the first bound is in fact an equality, because it is known that the time needed by a finite birth and death process to go from one boundary point to the other one is exactly a sum of independent exponential variables whose parameters are the corresponding Dirichlet eigenvalues (see e.g. Fill [10] or Diaconis and Miclo [7] for a probabilistic proof as well as a review of the history of this property). In the above product, we begin by considering the first factor

$$\begin{aligned} 1-\frac{\lambda_0}{\widetilde{\lambda}_0} &= 1-\frac{\sin^2(\pi/(4N))}{\sin^2(\pi/(4(N-1)))} \\ &= \frac{\sin^2(\pi/(4(N-1)))-\sin^2(\pi/(4N))}{\sin^2(\pi/(4(N-1)))} \\ &= \frac{\sin(\pi/(4N(N-1)))\sin(\pi(2N-1)/(4N(N-1)))}{\sin^2(\pi/(4(N-1)))}, \end{aligned}$$

where we used the trigonometric formula

$$\forall\, a,b \in \mathbb{R}, \qquad \sin^2(a)-\sin^2(b) \;=\; \sin(a+b)\sin(a-b).$$

Letting N go to infinity, it appears that

$$\left(1-\frac{\lambda_0}{\widetilde{\lambda}_0}\right)^{-1} \;\sim\; \frac{N}{2}, \tag{26}$$

which provides us with the announced linear explosion in N. It remains to treat the other factors

$$1-\frac{\lambda_0}{\widetilde{\lambda}_k} \;=\; 1-\frac{\sin^2(\pi/(4N))}{\sin^2((2k+1)\pi/(4(N-1)))},$$

for $k \in [\![1, N-2]\!]$. Taking into account that

$$\lim_{N\to\infty} \frac{\sin^2(\pi/(4N))}{\sin^2(3\pi/(4(N-1)))} \;=\; \frac{1}{9},$$

and that for all $\theta \in [0, \pi/2]$, $2\theta/\pi \leqslant \sin(\theta) \leqslant \theta$, we can find a constant $c > 0$ such that for N large enough,

$$\forall\, k \in [\![1, N-2]\!], \qquad \frac{\sin^2(\pi/(4N))}{\sin^2((2k+1)\pi/(4(N-1)))} \;\leqslant\; \frac{1}{8} \wedge \frac{c}{(2k+1)^2}.$$

This bound and the dominated convergence theorem show

$$\lim_{N\to\infty} \sum_{k\in[\![1,N-2]\!]} \ln\left(1 - \frac{\sin^2(\pi/(4N))}{\sin^2((2k+1)\pi/(4(N-1)))}\right) \;=\; \sum_{k\in\mathbb{N}} \ln\left(1 - \frac{1}{(2k+1)^2}\right) \;>\; -\infty.$$

The above observations and (21) lead in fact to Wallis' formula:

$$\prod_{k\in\mathbb{N}\setminus\{1\}\,:\,k\text{ even}} 1 - \frac{1}{k^2} \;=\; \frac{\pi}{4}.$$

We now come to the case $\rho > 1$. As remarked above, for all finite birth and death processes absorbed at 0, we have an exact formula

$$a_\varphi \;=\; \prod_{l\in[\![0,N-2]\!]} \left(1 - \frac{\lambda_0}{\widetilde{\lambda}_l}\right)^{-1},$$

but to exploit it, one needs a knowledge of the eigenvalues $\lambda_0, \widetilde{\lambda}_0, \widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_{N-2}$. The only behavior provided in [8], is that for large N

$$\lambda_0 \;\sim\; \frac{1}{2}(\rho+1)(\rho-1)^2\frac{1}{\rho^{N+1}}. \tag{27}$$

Let us show how this is sufficient to deduce that a_φ remains bounded as N go to infinity. Indeed, we will just need an additional qualitative result about the number of nodal domains of the corresponding eigenvectors, which is a discrete analogue of Sturm's theorem for one dimensional diffusions.

We begin by treating the first factor. As above, by spatial homogeneity, $\widetilde{\lambda}_0$ is just λ_0 when N has been replaced by $N-1$. It follows that $\widetilde{\lambda}_0 \sim \frac{1}{2}(\rho+1)(\rho-1)^2\frac{1}{\rho^N}$, so that

$$\lim_{N\to\infty} \left(1 - \frac{\lambda_0}{\widetilde{\lambda}_0}\right)^{-1} \;=\; \left(1 - \frac{1}{\rho}\right)^{-1},$$

which in comparison with (26), is a first indication why a_φ should stay bounded as N goes to infinity.

It remains to prove that

$$\limsup_{N\to\infty} -\sum_{l\in[\![1,N-2]\!]} \ln\left(1 - \frac{\lambda_0}{\widetilde{\lambda}_l}\right) \;<\; +\infty.$$

Since we know that

$$\begin{aligned}\forall\, l \in [\![1, N-2]\!], \qquad \frac{\lambda_0}{\widetilde{\lambda}_l} \;&\leqslant\; \frac{\lambda_0}{\widetilde{\lambda}_0}\\ &\leqslant\; \frac{1+\rho^{-1}}{2} \;<\; 1,\end{aligned}$$

for N large enough, it is sufficient to find a constant $c > 0$ such that

$$\forall\, l \in [\![1, N-2]\!], \qquad \frac{\lambda_0}{\widetilde{\lambda}_l} \;\leqslant\; c\rho^{-l},$$

or similarly, such that

$$\forall\ l \in [\![1, N-1]\!], \qquad \frac{\lambda_0}{\lambda_l} \ \leqslant\ c\rho^{-l}. \tag{28}$$

For given $l \in [\![1, N-1]\!]$, let φ_l be an eigenvector of $-\bar{L}$ associated to the eigenvalue λ_l and vanishing at 0. Since $\bar{L}$ is a tri-diagonal matrix, φ_l has $l+1$ nodal domains. More precisely, extend φ_l into a continuous function on $[0, N]$ by making it affine on each of the segments $[n, n+1]$ with $n \in [\![0, N-1]\!]$. Then φ_l has exactly $l+1$ zeros: $x_0 = 0 < x_1 < \cdots < x_l$ and it was seen in Miclo [20] that if $x_k \notin [\![0, N]\!]$, there is a natural way to define the jump rates $\bar{L}(\lceil x_k \rceil, x_k)$ and $\bar{L}(\lfloor x_k \rfloor, x_k)$ such that we have

$$\forall\ k \in [\![0, l]\!], \qquad \lambda_0([x_k, x_{k+1}]) \ =\ \lambda_l$$

with the convention $x_{l+1} = N$. Since each of the segments $[x_k, x_{k+1}]$, for $k \in [\![0, l]\!]$, contains at least one integer, we have $x_l \geqslant l$ and by consequence

$$\begin{aligned} \lambda_l &= \lambda_0([x_l, N]) \\ &\geqslant \lambda_0([l, N]). \end{aligned}$$

Due to the spacial homogeneity of the initial generator $\bar{L}$, $\lambda_0([l, N])$ is the same as λ_0 where N is replaced by $N-l$. The bound (28) is now an easy consequence of (27), through the existence of a constant $C \geqslant 1$ (depending on $\rho > 1$) such that

$$\forall\ N \in \mathbb{N}, \qquad C^{-1}\rho^{-N} \ \leqslant\ \lambda_0 \ \leqslant\ C\rho^{-N}.$$

$\square$

4 Some denumerable birth and death processes

This section treats denumerable birth and death processes absorbing at 0 and with ∞ as entrance boundary via approximation by finite absorbing birth and death processes.

We begin by recalling the theory of approximation of birth and death processes with ∞ as entrance boundary, as developed by Gong, Mao and Zhang [12]. For $N \in \mathbb{N}$, consider the finite state spaces $S_N \coloneqq [\![N]\!]$ and $\bar{S}_N \coloneqq [\![0, N]\!]$ endowed with the Markovian generator $\bar{L}_N$ which is the restriction of $\bar{L}$ to $\bar{S}_N$, except that $\bar{L}_N(N, N) = -b_{N-1}$, so that a Neumann (reflection) condition is put at N. The point 0 is still absorbing for $\bar{L}_N$. Denote by

$$\lambda_{N,0} < \lambda_{N,1} < \lambda_{N,2} < \cdots < \lambda_{N,N-1},$$

the eigenvalues of the subMarkovian generator K_N, the restriction of $\bar{L}_N$ to S_N. By convention, take $\lambda_{N,n} \coloneqq +\infty$ for $n \geqslant N$.

Theorem 5.4 of Gong, Mao and Zhang [12] asserts that for any fixed $n \in \mathbb{Z}_+$, the sequence $(\lambda_{N,n})_{N \in \mathbb{N}}$ is non-increasing and that

$$\lim_{N \to \infty} \lambda_{N,n} \ =\ \lambda_n, \tag{29}$$

where $(\lambda_n)_{n \in \mathbb{Z}_+}$ are the eigenvalues of $-K$ defined in the introduction.

For $N \in \mathbb{N} \backslash \{2\}$, let $\lambda'_{N,0}$ be the smallest eigenvalue of the restriction of $\bar{L}_N$ to $[\![2, N]\!]$. Consider the absorbing Markov generator $\bar{L}'$ on $\mathbb{N}$, coinciding with the restriction of $\bar{L}$ there, except that 1 is absorbing: $\bar{L}'(1,1) = \bar{L}'(1,2) = 0$. Applying (29) with $n = 0$ and with respect to $\bar{L}'$, shows that the sequence $(\lambda'_{N,0})_{N\in\mathbb{N}\setminus\{1\}}$ is non-increasing and

$$\lim_{N\to\infty} \lambda'_{N,0} \quad = \quad \lambda'_0. \tag{30}$$

These convergence properties are the main ingredient to deduce the estimate of Theorem 4 from Proposition 3. We will also need the fact that the eigenvector φ can be chosen to be positive on $\mathbb{N}$ and increasing. This is well-known, see for instance Chen [4] or Miclo [21], whose arguments can be extended to the present denumerable absorbing birth and death setting. We present a succinct proof for the sake of completeness.

For any function f defined on $\mathbb{N}$, consider the value

$$\mathcal{E}(f) \quad = \quad \eta(1)d_1 f^2(1) + \sum_{x\geqslant 1} \eta(x)b_x(f(x+1)-f(x))^2 \;\in\; \mathbb{R}_+ \sqcup \{+\infty\}.$$

Then φ is a minimizer of $\mathcal{E}(f)/\eta(f^2)$ over all functions $f \in \mathbb{L}^2(\eta)\setminus\{0\}$.

Since $\mathcal{E}(f) \geqslant \mathcal{E}(|f|)$ for any function f, we can assume that φ is non-negative, up to replacing it by $|\varphi|$. For fixed $x \in \mathbb{N}$, considering the quantity $\varphi(x)$ as a variable in the ratio $\mathcal{E}(\varphi)/\eta(\varphi^2)$, it appears by minimization that $\varphi(x) \in [\min(\varphi(x-1), \varphi(x+1)), \max(\varphi(x-1), \varphi(x+1))]$ and even $\varphi(x) \in (\min(\varphi(x-1), \varphi(x+1)), \max(\varphi(x-1), \varphi(x+1)))$ if $\varphi(x-1) \neq \varphi(x+1)$. This property implies the monotonicity of φ. Since $\varphi(0) = 0$ and φ must be positive somewhere, it follows that φ is non-decreasing. Consider $x_0 \coloneqq \max\{x : \varphi(x) = 0\}$. From $\varphi(x) \in (0, \varphi(x+1))$, we end up with a contradiction if $x \neq 0$. So $x_0 = 0$ and φ is positive on $\mathbb{N}$. The same argument shows that if there exists $x \in \mathbb{N}$ such that $\varphi(x) = \varphi(x+1)$ then $\varphi(x-1) = \varphi(x)$. By iteration it would imply that φ vanishes identically. Thus φ is increasing, not only non-decreasing.

This observation is also valid for $\bar{L}'$: there is an eigenvector φ' associated to the eigenvalue $-\lambda'_0$ (of K', the restriction to $\mathbb{N}\setminus\{1\}$ of $\bar{L}'$) which is positive and increasing on $\mathbb{N}\setminus\{1\}$. Indeed, ∞ is also an entrance boundary for $\bar{L}'$, so that its spectrum consists equally of eigenvalues of multiplicity 1, in particular φ' exists. As a consequence we get that $\lambda'_0 > \lambda_0$:

Lemma 15 *With the above notation,*

$$\lambda'_0 - \lambda_0 \quad = \quad \eta(1)b_1\frac{\varphi'(2)\varphi(1)}{\eta[\varphi'\varphi]} \;>\; 0$$

(where φ' is seen as function defined on $\mathbb{N}$ with the convention $\varphi'(1) = 0$).

Proof

The result follows from the computation of $\eta[\varphi' K[\varphi]]$: by definition of φ,

$$\eta[\varphi' K[\varphi]] \quad = \quad -\lambda_0\eta[\varphi'\varphi].$$

By self-adjointness of K the l.h.s. is equal to $\eta[\varphi K[\varphi']]$. We remark that for $x \in \mathbb{N}$,

$$\begin{aligned} K[\varphi'](x) \quad &= \quad K'[\varphi'](x) + b_1\varphi'(2)\mathbb{1}_{\{1\}}(x) \\ &= \quad -\lambda'_0\varphi'(x) + b_1\varphi'(2)\mathbb{1}_{\{1\}}(x) \end{aligned}$$

(by convention, $K'[\varphi'](1) = 0$), so that by multiplication by $\varphi(x)$ and integration with respect to η,

$$\eta[\varphi K[\varphi']] \quad = \quad -\lambda'_0\eta[\varphi'\varphi] + \eta(1)b_1\varphi'(2)\varphi(1).$$

■

Let us next check the second assertion of Theorem 4.

Lemma 16 *Under the entrance boundary condition,* (7) *is valid.*

Proof

If we were working with an ergodic birth and death on $\mathbb{Z}_+$, this result is due to Mao [16]. To come back to this situation, let us consider the Markov generator $\widehat{L}$ on $\mathbb{N}$ which coincides with $\bar{L}$, except that $\widehat{L}(0,1) = 1 = -\widehat{L}(0,0)$. For this process, ∞ is still an entrance boundary. Let $(\widehat{\lambda}_n)_{n\in\mathbb{Z}_+}$ be the eigenvalues of $-\widehat{L}$. We have $\widehat{\lambda}_0 = 0$ and Mao [16] has shown that

$$\sum_{n\in\mathbb{N}} \frac{1}{\widehat{\lambda}_n} \quad < \quad +\infty.$$

It is well-known that the eigenvalues $(\lambda_n)_{n\in\mathbb{Z}_+}$ and $(\widehat{\lambda}_n)_{n\in\mathbb{Z}_+}$ are interlaced:

$$\widehat{\lambda}_0 < \lambda_0 \leqslant \widehat{\lambda}_1 \leqslant \lambda_1 \leqslant \cdots$$

(see for instance Miclo [23] where this kind of comparison was extensively used). This implies the validity of (7).

■

We can now readily end the

Proof of the bound of Theorem 4

For $N \in \mathbb{N}$, let φ_N be an eigenvector associated with the eigenvalue $\lambda_{N,0}$ of K_N and normalized by $\varphi_N(1) = 1$. According to Proposition 3, whose reversibility assumption is satisfied, we have

$$a_{\varphi_N} \quad \leqslant \quad \left(\left(1 - \frac{\lambda_{N,0}}{\lambda'_{N,0}}\right) \prod_{k\in[\![N-2]\!]} \left(1 - \frac{\lambda_{N,0}}{\lambda_{N,n}}\right)\right)^{-1}. \tag{31}$$

Let $N_0 \in \mathbb{N}$ be large enough so that

$$\lambda_{N_0,0} \quad \leqslant \quad \frac{\lambda_0 + \lambda'_0 \wedge \lambda_1}{2} \quad (> \lambda_0).$$

It follows that for $N \geqslant N_0$,

$$\begin{aligned}
1 - \frac{\lambda_{N,0}}{\lambda'_{N,0}} \quad &\geqslant \quad 1 - \frac{\lambda_{N,0}}{\lambda'_0} \\
&\geqslant \quad 1 - \frac{\lambda_0 + \lambda'_0}{2\lambda'_0} \\
&= \quad \frac{\lambda'_0 - \lambda_0}{2\lambda'_0}.
\end{aligned}$$

In a similar manner, we get that for any $n \in \mathbb{N}$,

$$1 - \frac{\lambda_{N,0}}{\lambda_{N,n}} \quad \geqslant \quad 1 - \frac{\lambda_0 + \lambda_1}{2\lambda_n},$$

so that for all $N \geqslant N_0$,

$$a_{\varphi_N} \quad \leqslant \quad \left(\left(\frac{\lambda'_0 - \lambda_0}{2\lambda'_0}\right) \prod_{k\in\mathbb{N}} \left(1 - \frac{\lambda_0 + \lambda_1}{2\lambda_n}\right)\right)^{-1},$$

which is finite because of Lemma 15 and (7).

The functions φ_N are also increasing on $[\![N-1]\!]$. Consider them as non-decreasing mappings defined on $\mathbb{N}$ by taking $\varphi_N(n) = \varphi_N(N)$ for all $n \geqslant N$. Due to this monotonicity property and to the above uniform bound on a_{φ_N} over $N \geqslant N_0$, we can find an increasing subsequence $(N_l)_{l\in\mathbb{N}}$ and a non-decreasing and bounded function $\widetilde{\varphi}$ on $\mathbb{N}$ with $\widetilde{\varphi}(1) = 1$ such that φ_{N_l} converge uniformly on $\mathbb{N}$ toward $\widetilde{\varphi}$ as l goes to infinity. We are then allowed to pass to the limit in the equation $K_N[\varphi_N] = -\lambda_{N,0}\varphi_N$ to get on $\mathbb{N}$,

$$K[\widetilde{\varphi}] = -\lambda_0\widetilde{\varphi}.$$

Since the function $\widetilde{\varphi}$ is bounded, it also belongs to $\mathbb{L}^2(\eta)$ and so it is an eigenvector associated to the eigenvalue $-\lambda_0$ of K. It must thus be proportional to φ.

The previous considerations also enable to pass to the limit in (31) and this ends the proof of Theorem 4.

■

Corollary 5 is an easy consequence of the above considerations.

Proof of Corollary 5

For any $n \in \mathbb{Z}_+$, consider the birth and death rates given by

$$b_k^{(n)} \quad := \quad \begin{cases} b_{n+k} & , \text{ if } k \geqslant 1 \\ 0 & , \text{ if } k = 0 \end{cases} \tag{32}$$

$$d_k^{(n)} \quad := \quad \begin{cases} d_{n+k} & , \text{ if } k \geqslant 1 \\ 0 & , \text{ if } k = 0 \end{cases} \tag{33}$$

Denote by $\bar{L}^{(n)}$ and $\pi^{(n)}$ the associated generator and invariant measure. Since the rates $(b_k^{(n)})_{k\in\mathbb{Z}_+}$ and $(d_k^{(n)})_{k\in\mathbb{Z}_+}$ satisfy Conditions (4) and (5), the spectrum of $\bar{L}^{(n)}$ in $\mathbb{L}^2(\pi^{(n)})$ is discrete and simple. Let $\lambda_0^{(n)} < \lambda_1^{(n)} < \lambda_2^{(n)} < \cdots$ be the eigenvalues of $-\bar{L}^{(n)}$. By the interlacing property, we have

$$\forall\ n, k \in \mathbb{Z}_+, \qquad \lambda_k^{(n)} \ \leqslant\ \lambda_k^{(n+1)} \ \leqslant\ \lambda_{k+1}^{(n)}$$

In particular, by iteration, it follows that

$$\forall\ n \in \mathbb{Z}_+, \qquad \lambda_n \ =\ \lambda_n^{(0)} \ \geqslant\ \lambda_0^{(n)}$$

Injecting these inequalities in Theorem 4, we get the first bound of Corollary 5. The second bound comes from the fact that $(\lambda_0^{(n)})_{n\in\mathbb{Z}_+}$ is non-decreasing, as another consequence of the interlacing property.

■

Remark 17 It is theoretically possible to improve the first bound of Corollary 5, up to a certain knowledge of the eigenfunctions of $\bar{L}$. More precisely, let $(\varphi_k)_{k\in\mathbb{Z}_+}$ be a family of eigenvectors of $\bar{L}$ associated to the eigenvalues $(\lambda_k)_{k\in\mathbb{Z}_+}$. For any $k \in \mathbb{Z}_+$, by the discrete Courant theorem in dimension 1, the function φ_k changes sign exactly k times. Let n_k be the last time it changes sign (i.e. the smallest $n \in \mathbb{Z}_+$ such that for any $m \geqslant n$, the values $\varphi_k(m)$ have the same sign (with the convention that 0 has both signs). Using the considerations developed in [20] (and extending them to the infinite entrance boundary setting by approximation), it appears that $\lambda_k \geqslant \lambda_0^{(n_k)}$. It follows that

$$a_\varphi \quad \leqslant \quad \left(\left(1 - \frac{\lambda_0}{\lambda_0^{(n_1)}}\right)^2 \prod_{k\in\mathbb{N}\setminus\{1\}} \left(1 - \frac{\lambda_0}{\lambda_0^{(n_k)}}\right)\right)^{-1}$$

One recovers Corollary 5 by observing that $n_k \geqslant k$ for all $k \in \mathbb{Z}_+$, due to the discrete Courant theorem.

$\square$

Let us recall a classical

Proof of Proposition 6

According to Karlin and McGregor [15], the a.s. absorption of the processes X^x, for $x \in \mathbb{N}$, is equivalent to

$$\sum_{x=1}^{\infty} \frac{1}{\pi_x b_x} \quad = \quad +\infty,$$

and this divergence clearly implies that of (4).

Similarly to the proof of Proposition 9, consider next the mapping f on $\mathbb{R}_+ \times \mathbb{Z}_+$ defined by

$$\forall\ (t, x) \in \mathbb{R}_+ \times \mathbb{Z}_+, \qquad f(t, x) \quad \coloneqq \quad \exp(\lambda t)\varphi(x)$$

(as usual, we impose $\varphi(0) = 0$). By the martingale problem solved by the law of X^x, for $x \in \mathbb{N}$, the process $M \coloneqq (M_t)_{t \geqslant 0}$ given by

$$\forall\ t \geqslant 0, \qquad M_t \quad \coloneqq \quad f(t, X_t^x) - f(0, x) - \int_0^t \partial_s f(s, X_s^x) + \bar{L}[f(s, \cdot)](X_s^x)\, ds$$

is a local martingale and even a martingale, because for any fixed $t \geqslant 0$, M_t is bounded, due to the assumption $a_\varphi < +\infty$. The stopped stochastic process $(M_{t \wedge \tau_{x,1}})_{t \geqslant 0}$ is also a martingale, so taking expectations at time $t \geqslant 0$, we get

$$\begin{aligned}
\mathbb{E}[f(t, X_{t \wedge \tau_{x,1}}^x)] \quad &= \quad \varphi(x) + \mathbb{E}\left[\int_0^{t \wedge \tau_{x,1}} \partial_s f(s, X_s^x) + \bar{L}[f(s, \cdot)](X_s^x)\, ds\right] \\
&= \quad \varphi(x) + \mathbb{E}\left[\int_0^{t \wedge \tau_{x,1}} (\lambda\varphi + \bar{L}[\varphi])(X_s^x)\ \exp(\lambda s) ds\right].
\end{aligned}$$

By assumption, we have $\lambda\varphi + \bar{L}[\varphi] \leqslant 0$ on $\mathbb{N}$, so that

$$\mathbb{E}[\varphi(X_{t \wedge \tau_{x,1}}^x) \exp(\lambda(t \wedge \tau_{x,1}))] \quad \leqslant \quad \varphi(x),$$

and it follows that

$$\mathbb{E}[\exp(\lambda(t \wedge \tau_{x,1}))] \quad \leqslant \quad a_\varphi.$$

Letting t go to infinity, we deduce that

$$\begin{aligned}
\mathbb{E}[\tau_{x,1}] \quad &\leqslant \quad \frac{\mathbb{E}[\exp(\lambda \tau_{x,1})] - 1}{\lambda} \\
&\leqslant \quad \frac{a_\varphi}{\lambda}.
\end{aligned}$$

But it is well-known (see for instance Paragraph 8.1 of Anderson [2]) that

$$\mathbb{E}[\tau_{x,1}] \quad = \quad \sum_{y=1}^{x-1} \frac{1}{\pi_y b_y} \sum_{z=y+1}^{x} \pi_z,$$

thus letting x go to infinity we obtain

$$\sum_{y=1}^{\infty} \frac{1}{\pi_y b_y} \sum_{z=y+1}^{\infty} \pi_z \quad \leqslant \quad \frac{a_\varphi}{\lambda} \quad < \quad +\infty,$$

namely (5) is satisfied. This ends the proof that ∞ is an entrance boundary for $\bar{L}$.

■

Let us now discuss the conditions (4) and (5). The following example is quite academic, it shows that (5) is not true for the benchmark M/M/∞ (absorbed) queueing process, but it is almost at the frontier of applicability, since a mild modification of it enters in the setting of Theorem 4.

Example 18 Consider the rates given for all $n \in \mathbb{Z}_+$ by

$$b_n := \begin{cases} 1 & , \text{ if } n \geqslant 1 \\ 0 & , \text{ if } n = 0 \end{cases}$$

$$d_n := \begin{cases} n & , \text{ if } n \geqslant 1 \\ 0 & , \text{ if } n = 0. \end{cases}$$

The measure π defined in (6) is proportional to the restriction on $\mathbb{N}$ of the Poisson distribution of parameter 1:

$$\forall\, n \in \mathbb{N}, \qquad \pi_n = \frac{1}{n!}. \tag{34}$$

It is easily computed that (5) is not satisfied.

To transform ∞ into an entrance boundary, the underlying Markov process must be accelerated near ∞: consider the rates given for all $n \in \mathbb{Z}_+$ by

$$b_n := \begin{cases} \ln^2(e+n) & , \text{ if } n \geqslant 1 \\ 0 & , \text{ if } n = 0 \end{cases}$$

$$d_n := \begin{cases} n\ln^2(e-1+n) & , \text{ if } n \geqslant 1 \\ 0 & , \text{ if } n = 0. \end{cases}$$

The measure π is not modified, still given by (34). Nevertheless Conditions (4) and (5) are satisfied and Theorem 4 can be applied.

□

It is quite painful to try to apply to the previous example the procedure described after Corollary 5. So let us finish the paper by a class of examples where Corollary 5 does lead to quantitative estimates.

Example 19 Assume that there exists $\rho > 1$ such that the rates satisfy

$$\forall\, k \in \mathbb{N}, \qquad \begin{cases} b_{k+1} & \leqslant & \rho b_k \\ d_{k+1} & \geqslant & \rho d_k \end{cases} \tag{35}$$

Under the additional assumptions (4) and (5), we get

$$a_\varphi \leqslant \left(\left(1-\rho^{-2}\right)^2 \prod_{n\geqslant 2} \left(1-\rho^{-n}\right) \right)^{-1}$$

Indeed, let us show that

$$\forall\, n \in \mathbb{Z}_+, \qquad \lambda^{(n+1)} \geqslant \rho\lambda^{(n)} \tag{36}$$

so that the previous bound is a consequence of Corollary 5 and of $\lambda^{(n)} \geqslant \rho^n\lambda_0$, for all $n \in \mathbb{Z}_+$.

Fix $n \in \mathbb{Z}_+$. Consider the birth and death rates given by (32) and (33), as well as the associated generator $\bar{L}^{(n)}$ and invariant measure $\pi^{(n)}$. Since the rates $(b_k^{(n+1)})_{k\in\mathbb{Z}_+}$ and $(d_k^{(n+1)})_{k\in\mathbb{Z}_+}$ satisfy

Conditions (4) and (5), there exists an increasing function $\varphi^{(n+1)}$ on $\mathbb{Z}_+$, with $\varphi^{(n+1)}(0) = 0$, such that $\bar{L}^{(n+1)}[\varphi^{(n+1)}] = -\lambda^{(n+1)}\varphi^{(n+1)}$. Namely, we have for all $k \in \mathbb{Z}_+$,

$$b^{(n+1)}(k)(\varphi^{(n+1)}(k+1) - \varphi^{(n+1)}(k)) + d^{(n+1)}(k)(\varphi^{(n+1)}(k-1) - \varphi^{(n+1)}(k)) \quad = \quad -\lambda^{(n+1)}\varphi^{(n+1)}(k)$$

Taking into account that $\varphi^{(n+1)}(k+1) - \varphi^{(n+1)}(k) \geqslant 0$, that $\varphi^{(n+1)}(k-1) - \varphi^{(n+1)}(k) \leqslant 0$ and (35), we get by dividing by ρ,

$$b^{(n)}(k)(\varphi^{(n+1)}(k+1) - \varphi^{(n+1)}(k)) + d^{(n)}(k)(\varphi^{(n+1)}(k-1) - \varphi^{(n+1)}(k)) \quad \geqslant \quad -\frac{\lambda^{(n+1)}}{\rho}\varphi^{(n+1)}(k)$$

i.e. $\bar{L}^{(n)}[\varphi^{(n+1)}] \geqslant -\lambda^{(n+1)}\varphi^{(n+1)}/\rho$. Multiplying this relation by $\varphi^{(n+1)}$ and integrating with respect to $\pi^{(n)}$, we obtain

$$\frac{\lambda^{(n+1)}}{\rho} \quad \geqslant \quad \frac{-\pi^{(n)}[\varphi^{(n+1)}\bar{L}^{(n)}[\varphi^{(n+1)}]]}{\pi^{(n+1)}[(\varphi^{(n+1)})^2]}$$

The r.h.s. is bounded below by $\lambda^{(n)}$, so (36) follows. An example in the spirit of Example 18 satisfying (4), (5) and (35), is given by

$$\begin{aligned} b_n &\coloneqq \begin{cases} 1 & \text{, if } n \geqslant 1 \\ 0 & \text{, if } n = 0 \end{cases} \\ d_n &\coloneqq \begin{cases} \rho^{n-1} & \text{, if } n \geqslant 1 \\ 0 & \text{, if } n = 0. \end{cases} \end{aligned}$$

with $\rho > 1$.

□

Acknowledgments:

The work of Persi Diaconis was partially supported by the CIMI (Centre International de Mathématiques et d'Informatique) Excellence program while he was invited in Toulouse during the first semester of 2014. The work of Laurent Miclo was partially supported by the ANR STAB (Stabilité du comportement asymptotique d'EDP, de processus stochastiques et de leurs discrétisations).

References

[1] N. I. Akhiezer and I. M. Glazman. *Theory of linear operators in Hilbert space. Vol. II*, volume 10 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, Mass.-London, 1981. Translated from the third Russian edition by E. R. Dawson, Translation edited by W. N. Everitt.

[2] William J. Anderson. *Continuous-time Markov chains*. Springer Series in Statistics: Probability and its Applications. Springer-Verlag, New York, 1991. An applications-oriented approach.

[3] N. Champagnat and D. Villemonais. Exponential convergence to quasi-stationary distribution and Q-process. *ArXiv e-prints*, April 2014.

[4] Mufa Chen. Analytic proof of dual variational formula for the first eigenvalue in dimension one. *Sci. China Ser. A*, 42(8):805–815, 1999.

[5] Kai Lai Chung and John B. Walsh. *Markov processes, Brownian motion, and time symmetry*, volume 249 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, New York, second edition, 2005.

[6] Pierre Collet, Servet Martínez, and Jaime San Martín. *Quasi-stationary distributions*. Probability and its Applications (New York). Springer, Heidelberg, 2013. Markov chains, diffusions and dynamical systems.

[7] Persi Diaconis and Laurent Miclo. On times to quasi-stationarity for birth and death processes. *J. Theoret. Probab.*, 22(3):558–586, 2009.

[8] Persi Diaconis and Laurent Miclo. On quantitative convergence to quasi-stationarity. Available at `http://hal.archives-ouvertes.fr/hal-01002622`, June 2014.

[9] Stewart N. Ethier and Thomas G. Kurtz. *Markov processes*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, 1986. Characterization and convergence.

[10] James Allen Fill. The passage time distribution for a birth-and-death chain: strong stationary duality gives a first stochastic proof. *J. Theoret. Probab.*, 22(3):543–557, 2009.

[11] Wu-Jun Gao and Yong-Hua Mao. Quasi-stationary distribution for the birth-death process with exit boundary. *Journal of Mathematical Analysis and Applications*, 427:114–125, 2015.

[12] Yu Gong, Yong-Hua Mao, and Chi Zhang. Hitting time distributions for denumerable birth and death processes. *J. Theoret. Probab.*, 25(4):950–980, 2012.

[13] Roger A. Horn and Charles R. Johnson. *Matrix analysis*. Cambridge University Press, Cambridge, 1990. Corrected reprint of the 1985 original.

[14] S. D. Jacka and G. O. Roberts. Weak convergence of conditioned processes on a countable state space. *J. Appl. Probab.*, 32(4):902–916, 1995.

[15] Samuel Karlin and James McGregor. The classification of birth and death processes. *Trans. Amer. Math. Soc.*, 86:366–400, 1957.

[16] Yong-Hua Mao. The eigentime identity for continuous-time ergodic Markov chains. *J. Appl. Probab.*, 41(4):1071–1080, 2004.

[17] Sylvie Méléard and Denis Villemonais. Quasi-stationary distributions and population processes. *Probab. Surv.*, 9:340–410, 2012.

[18] Sean Meyn and Richard L. Tweedie. *Markov chains and stochastic stability*. Cambridge University Press, Cambridge, second edition, 2009. With a prologue by Peter W. Glynn.

[19] L. Miclo. An example of application of discrete Hardy's inequalities. *Markov Process. Related Fields*, 5(3):319–330, 1999.

[20] Laurent Miclo. On eigenfunctions of Markov processes on trees. *Probab. Theory Related Fields*, 142(3-4):561–594, 2008.

[21] Laurent Miclo. Monotonicity of the extremal functions for one-dimensional inequalities of logarithmic Sobolev type. In *Séminaire de Probabilités XLII*, volume 1979 of *Lecture Notes in Math.*, pages 103–130. Springer, Berlin, 2009.

[22] Laurent Miclo. On absorption times and Dirichlet eigenvalues. *ESAIM Probab. Stat.*, 14:117–150, 2010.

[23] Laurent Miclo. On ergodic diffusions on continuous graphs whose centered resolvent admits a trace. Available at `http://hal.archives-ouvertes.fr/hal-00957019`, March 2014.

[24] L. C. G. Rogers and David Williams. *Diffusions, Markov processes, and martingales. Vol. 2*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2000. Itô calculus, Reprint of the second (1994) edition.

[25] E. Seneta. *Nonnegative matrices and Markov chains*. Springer Series in Statistics. Springer-Verlag, New York, second edition, 1981.

[26] Erik A. van Doorn. Quasi-stationary distributions and convergence to quasi-stationarity of birth-death processes. *Adv. in Appl. Probab.*, 23(4):683–700, 1991.

[27] Erik A. van Doorn and Philip K. Pollett. Quasi-stationary distributions for discrete-state models. *European J. Oper. Res.*, 230(1):1–14, 2013.

Stanford University
Department of Mathematics
Building 380, Sloan Hall
Stanford, California 94305, USA

‡miclo@math.univ-toulouse.fr
Institut de Mathématiques de Toulouse
Université Paul Sabatier
118, route de Narbonne
31062 Toulouse Cedex 9, France