On links between quantitative ergodicity and absorption

Laurent Miclo

Institut de Mathématiques de Toulouse

based on joint works with Daniel Bump, Persi Diaconis, Angela Hicks and Harold Widom

Plan of the talk

1. Convergence to quasistationarity
   - Qualitative versus quantitative convergence
   - Finite quasi-stationarity
   - A reduction by comparison
   - Approach by functional inequalities
   - Some examples
   - Estimates on the amplitude a_φ
   - Birth and death processes with ∞ as entrance boundary
   - Some references
2. Heisenberg random walks
   - Motivations and main result
   - Representation theory
   - Eigenvalues bounds
   - Further results

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A very simple example

For fixed $N \in \mathbb{N}$, consider the usual discrete time random walk $(X_t)_{t\in\mathbb{Z}_+}$ on $[\![0, N]\!]$, with holding at 0 and N. The invariant (and reversible) probability measure is the uniform distribution η.

Qualitative result: whatever the initial condition $\mathcal{L}(X_0)$, $\mathcal{L}(X_t)$ converges to η as $t \in \mathbb{Z}_+$ goes to infinity.

Quantitative result:

$$\|\mathcal{L}(X_t) - \eta\|_{\mathrm{tv}} \quad \leqslant \quad \sqrt{2\exp(-s)}$$

for

$$t \quad \geqslant \quad \frac{1}{4}(N+1)^2(1+s)$$

Total variation

Let us recall the definition of the **total variation** norm: for any signed measure m on a discrete space,

$$\begin{aligned} \|m\|_{\mathrm{tv}} &:= 2 \sup_{A \subset S} |m(A)| \\ &= \sup_{f \in \mathcal{F},\, \|f\|_\infty \leqslant 1} m(f) \\ &= \sum_{x \in S} |m(x)| \end{aligned}$$

Mixing time associated to $(\mathcal{L}(X_t))_{t \in \mathbb{Z}_+}$:

$$T_{\mathrm{mix}} := \sup_{\mathcal{L}(X_0)} \inf\{t \in \mathbb{Z}_+ \,:\, \|\mathcal{L}(X_t) - \eta\|_{\mathrm{tv}} \leqslant 1\}$$

Absorption

In the previous example: $T_{\mathrm{mix}} \leqslant (N+1)^2(1+\ln(2))/4$. This is a refined bound using the whole spectral decomposition (in particular the boundedness of the eigenvectors). If just the spectral gap is used: $T_{\mathrm{mix}} \leqslant (N+1)^2(1+\ln(N+1))/4$.

What happens if 0 is absorbing?
Then $\mathcal{L}(X_t) \to \delta_0$. This can be quantified also, using the first Dirichlet eigenvalue. Our interest here:

$$\begin{aligned} \tau &:= \inf\{t \in \mathbb{Z}_+ \,:\, X_t = 0\} \\ \mu_t &:= \mathcal{L}(X_t | \tau > t) \end{aligned}$$

Then (if $\mathcal{L}(X_0) \neq \delta_0$!), qualitative result:

$$\lim_{t \to +\infty} \mu_t = \nu$$

Quasi-stationary measure

where the probability ν on $[\![1, N]\!]$ is called the **quasi-stationary distribution** and is given by

$$\nu(x) \quad := \quad Z^{-1} \cos\left(\frac{(2N+1-2x)\pi}{2(2N+1)}\right)$$

with $Z^{-1} := 2\tan\left(\frac{\pi}{2(2N+1)}\right)$, the normalizing constant.
If $\mathcal{L}(X_0) = \nu$, then $\mu_t = \nu$ for all $t \geqslant 0$ and

$$\mathbb{P}[\tau > t] \quad = \quad \left(\cos\left(\frac{\pi}{2N+1}\right)\right)^t$$

(geometric distribution)

Our purpose: to obtain corresponding quantitative results, for instance $T_{\text{quasi-mix}} \leqslant \text{cst } N^2 \ln(N)$ in the above case.
We will rather work in the continuous time setting, where results are simpler to state.

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The framework

The whole finite state space is $\bar{S} := S \sqcup \{0\}$, where 0 is the absorbing point. It means that $\bar{S}$ is endowed with a Markov generator matrix $\bar{L} := (\bar{L}(x,y))_{x,y\in\bar{S}}$ whose restriction to $S \times S$ is irreducible and which is such that

$$\begin{array}{rrcl} \forall\, x \in \bar{S}, & \bar{L}(0,x) & = & 0 \\ \exists\, x \in S\ : & \bar{L}(x,0) & > & 0 \end{array}$$

If μ_0 is an initial distribution on S, we can associate a Markov process $X := (X_t)_{t\geqslant 0}$. The absorbing time τ and conditional distribution μ_t, for $t \geqslant 0$, are constructed as before. There exists a quasi-stationary distribution ν such that

$$\lim_{t\to+\infty} \mu_t \quad = \quad \nu$$

We want to quantify this convergence.

Notations

Let K be the $S \times S$ minor of $\bar{L}$. It can be written under the Schrödinger form $L - V$ where $V(x) = \bar{L}(x, 0)$, L an irreducible Markov generator on S. Let η be the invariant probability for L. Perron-Frobenius theory: there are $\lambda_0 > 0$ and a positive function φ such that

$$K[\varphi] \quad = \quad -\lambda_0 \varphi$$

Adjoint L^* of L in $\mathbb{L}^2(\eta)$: still a Markov generator given by

$$\forall\ x, y \in S, \qquad L^*(x, y) \quad = \quad \frac{\eta(y)}{\eta(x)} L(y, x)$$

$K^* = L^* - V$ and there is a positive function φ^* such that

$$K^*[\varphi^*] \quad = \quad -\lambda_0 \varphi^*$$

Quasi-stationary distribution

Different normalizations: $\eta[\varphi^*] = 1$ and $\eta[\varphi\varphi^*] = 1$.
We have $\nu = \varphi^* \cdot \eta$: it can be easily checked that for any function f on S, $\nu[K[f]] = -\lambda_0 \nu[f]$. As a consequence, for any $t \geqslant 0$,

$$\nu \exp(tK) \quad = \quad \exp(-\lambda_0 t)\nu$$

namely, if $\mathcal{L}(X_0) = \nu$, then

$$\mathcal{L}(X_t) = \exp(-\lambda_0 t)\nu + (1 - \exp(-\lambda_0 t))\delta_0$$

In particular

$$\mathbb{P}_\nu[\tau > t] = \mathbb{P}_\nu[X_t \in S] = \exp(-\lambda_0 t)$$

(exponential distribution).

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Doob transform

Consider the Markovian operator $\widetilde{L}$ on S defined by its off-diagonal entries via

$$\forall\ x \neq y \in S, \qquad \widetilde{L}(x,y) \ := \ L(x,y)\frac{\varphi(y)}{\varphi(x)}$$

(the diagonal entries are such the row sums vanish).
It is called the **Doob transform** of L through φ. We have for any function f,

$$\widetilde{L}[f] \ = \ \frac{1}{\varphi}(L - V + \lambda_0 I)[\varphi f]$$

$\widetilde{L}$ is irreducible and its invariant measure $\widetilde{\eta}$ is given by

$$\forall\ x \in S, \qquad \widetilde{\eta}(x) \ = \ \varphi(x)\varphi^*(x)\eta(x)$$

Indeed:

$$\begin{aligned}\widetilde{\eta}[\varphi^{-1}(L - V + \lambda_0)[\varphi f]] &= \eta[\varphi^*(L - V + \lambda_0)[\varphi f]] \\ &= \eta[\varphi f(L^* - V + \lambda_0)[\varphi^*]] \\ &= 0\end{aligned}$$

Reduction to ergodicity

Let $(\widetilde{P}_t)_{t\geqslant 0}$ the ergodic semi-group generated by $\widetilde{L}$. Its interest:

Theorem

For any probability measure μ_0 on S and for any $t \geqslant 0$, we have

$$\frac{\varphi_\wedge}{2\varphi_\vee}\left\|\widetilde{\mu}_0\widetilde{P}_t - \widetilde{\eta}\right\|_{\mathrm{tv}} \;\leqslant\; \|\mu_t - \nu\|_{\mathrm{tv}} \;\leqslant\; 2\frac{\varphi_\vee}{\varphi_\wedge}\left\|\widetilde{\mu}_0\widetilde{P}_t - \widetilde{\eta}\right\|_{\mathrm{tv}}$$

where $\widetilde{\mu}_0$ is the probability on S whose density with respect to μ_0 is proportional to φ. In particular the asymptotic exponential rate of convergence of $\|\mu_t - \nu\|_{\mathrm{tv}}$ and $\left\|\widetilde{\mu}_0\widetilde{P}_t - \widetilde{\eta}\right\|_{\mathrm{tv}}$ are the same.

We used the notation

$$\varphi_\vee := \max_{x\in S}\varphi(x) \qquad \varphi_\wedge := \min_{x\in S}\varphi(x)$$

Amplitude

Thus it appears that the first Dirichlet eigenfunction φ is the main ingredient needed to reduce the quantitative study of the convergence to quasi-stationarity to that of the convergence to equilibrium. A crucial quantity seems to be the **amplitude** of φ:

$$a_\varphi \quad := \quad \frac{\varphi_\vee}{\varphi_\wedge}$$

The convergence to equilibrium has been intensively investigated, through various approaches: Lyapounov functions, coupling, strong stationary times, isoperimetry, spectral theory, functional inequalities... The above bounds enable to recycle them for convergence to quasi-stationarity.

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The simplest of these methods: the $\mathbb{L}^2$ approach.
Let $\widehat{L}$ be the **additive symmetrization** of $\widetilde{L}$ in $\mathbb{L}^2(\widetilde{\eta})$: it is $(\widetilde{L}+\widetilde{L}^*)/2$, where $\widetilde{L}^*$ is the adjoint operator of $\widetilde{L}$ in $\mathbb{L}^2(\widetilde{\eta})$. By self-adjointness, $\widehat{L}$ is diagonalizable in $\mathbb{R}$. Let $\widehat{\lambda} > 0$ stand for the smallest non-zero eigenvalue (spectral gap) of $-\widehat{L}$.

Theorem

For any $t \geqslant 0$, we have

$$\sup_{\mu_0 \in \mathcal{P}} \|\mu_t - \nu\|_{\mathrm{tv}} \leqslant \sqrt{\frac{1}{(\varphi\varphi^*\eta)_\wedge} \frac{\varphi_\vee}{\varphi_\wedge}} \exp(-\widehat{\lambda} t)$$

where $\mathcal{P}$ stands for the set of probability measures on S.

Logarithmic Sobolev inequality

It is possible to improve the pre-exponential factor in the above result, but at the expense of the rate $\widehat{\lambda}$, via the **logarithmic Sobolev inequalities** associated to $\widehat{L}$.

Let $\widehat{\alpha} > 0$ be the largest constant such that for all $f \in \mathcal{F}$,

$$\widehat{\alpha} \sum_{x \in S} f^2(x) \ln\left(\frac{f^2(x)}{\widetilde{\eta}[f^2]}\right) \varphi^*(x)\varphi(x)\eta(x)$$
$$\leqslant \quad \sum_{x,y \in S} (f(y) - f(x))^2 \, \varphi^*(x)\varphi(y)\eta(x)L(x,y)$$

Then we have

$$\sup_{\mu_0 \in \mathcal{P}} \|\mu_t - \nu\|_{\mathrm{tv}} \quad \leqslant \quad \sqrt{2 \ln\left(\frac{\eta[\varphi\varphi^*]}{(\varphi\varphi^*\eta)_\wedge}\right) \frac{\varphi_\vee}{\varphi_\wedge}} \exp(-(\widehat{\alpha}/2)t)$$

This is interesting for not too large t and when $\widehat{\alpha}$ can be computed (e.g. by tensorization).

Reversible case

Assume that η is reversible for L:

$$\forall\, x, y \in S, \quad \eta(x)L(x,y) \;=\; \eta(y)L(y,x)$$

Then $-K = V - L$ is diagonalizable in $\mathbb{R}$, we have already met its smallest eigenvalue λ_0. Let $\lambda_1 > \lambda_0$ be the second eigenvalue. The spectral gap bound can be rewritten:

Theorem

Under the reversibility assumption, for any $t \geqslant 0$, we have

$$\begin{aligned}\sup_{\mu_0\in\mathcal{P}} \|\mu_t - \nu\|_{\mathrm{tv}} &\leqslant \sqrt{\frac{1}{(\varphi^2\eta)_\wedge}\frac{\varphi_\vee}{\varphi_\wedge}}\exp(-(\lambda_1-\lambda_0)t)\\ &\leqslant \sqrt{\frac{1}{\eta_\wedge}\left(\frac{\varphi_\vee}{\varphi_\wedge}\right)^2}\exp(-(\lambda_1-\lambda_0)t)\end{aligned}$$

(In this situation $\varphi^* = \varphi$).

Plan of the talk

A finite birth and death example with $\lambda_0 \sim \lambda_1 - \lambda_0$ (1)

The 3 first examples are **birth and death processes** on $\bar{S} := [\![0, N]\!]$, with $N \in \mathbb{N}$, absorbing at 0. So L gives positive rates only to the oriented edges $(x, x+1)$ and $(x+1, x)$ where $x \in [\![1, N-1]\!]$ and admits a reversible probability η. Assume that the killing rate at 1 is 1, namely $V(1) = \bar{L}(1, 0) = 1$. The other values of V are taken to be zero.

Specifically for the first example, we choose

$$\begin{aligned} \forall\ x \in [\![1, N-2]\!], \qquad L(x, x+1) &:= L(x+1, x) := 1 \\ L(N-1, N) &= 1 \text{ and } L(N, N-1) = 2 \end{aligned}$$

The reversible probability η is almost the uniform distribution on $S = [\![1, N]\!]$ (N has a weight divided by 2). The function φ is defined by

$$\forall\ x \in S, \qquad \varphi(x) := \frac{1}{Z} \sin(\pi x/(2N))$$

where Z is the renormalization constant such that $\eta[\varphi^2] = 1$.

A finite birth and death example with $\lambda_0 \sim \lambda_1 - \lambda_0$ (2)

We compute that λ_0 and $\lambda_1 - \lambda_0$ are of the same order (as $N \to \infty$), meaning that absorption and convergence to quasi-stationarity happen at similar rates:

$$\begin{aligned}\lambda_0 &= \frac{\pi^2}{4N^2}(1+\mathcal{O}(N^{-2}))\\ \lambda_1 - \lambda_0 &= 2\frac{\pi^2}{N^2}(1+\mathcal{O}(N^{-2}))\end{aligned}$$

Furthermore, we have

$$a_\varphi = \frac{2N}{\pi}(1+\mathcal{O}(N^{-2}))$$

It follows from the previous bound that for any given $s > 0$, if

$$t = \frac{5}{4\pi^2}N^2\ln(N) + \frac{s}{2\pi^2}N^2$$

then

$$\sup_{\mu_0 \in \mathcal{P}} \|\mu_t - \nu\|_{\mathrm{tv}} \leqslant \frac{4}{\pi^2}(1+\mathcal{O}(N^{-1}))\exp(-s)$$

A finite birth and death example with $\lambda_0 \ll \lambda_1 - \lambda_0$ (1)

It is similar to the previous example, except that for some $r > 1$, we take

$$\forall\ x \in [\![1, N-2]\!], \qquad \begin{cases} L(x, x+1) & := & r \\ L(x+1, x) & := & 1 \end{cases}$$

$$L(N-1, N) \ = \ r \ \text{ and } \ L(N, N-1) \ = \ 1 + r$$

The reversible probability η is given by

$$\forall\ x \in S, \qquad \eta(x) \ = \ \begin{cases} \frac{r^2-1}{2r^N-r-1} r^{x-1} & , \text{if } x \in [\![N-1]\!] \\ \frac{r-1}{2r^N-r-1} r^{N-1} & , \text{if } x = N \end{cases}$$

More involved computations are needed to get information on the eigenvalues and eigenfunctions, but finally we get, for large N,

$$\begin{aligned} \lambda_0 & \sim & \frac{1}{2}(r+1)(r-1)^2 \frac{1}{r^{N+1}} \\ a_\varphi & = & \frac{r}{r-1}(1 + \mathcal{O}(r^{-N})) \end{aligned}$$

A finite birth and death example with $\lambda_0 \ll \lambda_1 - \lambda_0$ (2)

$$\lambda_1 \quad > \quad (1-\sqrt{r})^2$$

It implies that

$$\lambda_1 - \lambda_0 \;\sim\; \lambda_1 \;\gg\; \lambda_0$$

meaning that convergence to quasi-stationarity happens at a much faster rate than absorption. It follows that for any fixed $s \geqslant 0$, if for N large enough we consider the time

$$t \quad := \quad \frac{1}{2(1-\sqrt{r})^2}\left(\ln(r)N + 2s\right)$$

then

$$\sup_{\mu_0\in\mathcal{P}} \|\mu_t - \nu\|_{\mathrm{tv}} \quad \leqslant \quad \frac{r^2}{(r-1)^{5/2}}\,(1+\circ(1))\exp(-s)$$

It can be shown that the relaxation time to quasi-stationarity is at least of order N, so the order is optimal here.

A finite birth and death example with $\lambda_0 \gg \lambda_1 - \lambda_0$ (1)

The setting is as in the previous example, except that now $r < 1$. But the behavior of our quantities of interest are very different for large N:

$$\begin{aligned} a_\varphi &\leqslant \frac{2N}{(1-r)r^{(N-1)/2}}(1+\circ(1)) \\ \lambda_0 &\sim (1-\sqrt{r})^2 \end{aligned}$$

and

$$\frac{(1-r)^2\sqrt{r}}{2N^2}(1+\circ(1)) \leqslant \lambda_1 - \lambda_0 \leqslant \frac{16\pi^2-(1-r)^2}{4N^2}\sqrt{r}(1+\circ(1))$$

In particular absorption happens at a much faster rate than convergence to quasi-stationarity, since $\lambda_0 \gg \lambda_1 - \lambda_0$.

A finite birth and death example with $\lambda_0 \gg \lambda_1 - \lambda_0$ (2)

Taking furthermore into account that $\eta_\wedge \sim (1-r)r^{N-1}$, we get

$$\sup_{\mu_0 \in \mathcal{P}} \|\mu_t - \nu\|_{\mathrm{tv}} \leqslant \frac{4\sqrt{1+r}N^2}{(1-r)^{5/2}r^{3(N-1)/2}}(1+\circ(1)) \exp\left(-\frac{(1-r)^2\sqrt{r}}{2N^2}(1+\circ(1))t\right)$$

In particular, for any given $\epsilon > 0$, if we consider

$$t_N := 3(1+\epsilon)\frac{\ln(1/r)}{(1-r)^2\sqrt{r}}N^3$$

then

$$\lim_{N\to\infty} \sup_{\mu_0 \in \mathcal{P}} \|\mu_{t_N} - \nu\|_{\mathrm{tv}} = 0$$

A non-reversible example

For fixed $N \in \mathbb{N}$, consider $S = \mathbb{Z}_N$ endowed with the (turning) generator L

$$\forall\ x, y \in \mathbb{Z}_N, \qquad L(x,y) \quad := \quad \begin{cases} 1 & \text{, if } y = x+1 \\ -1 & \text{, if } y = x \\ 0 & \text{, otherwise} \end{cases}$$

whose invariant probability measure η is the uniform distribution.
The potential V takes the value 1 at 0 and 0 otherwise.
We can show that

$$\sup_{\mu_0 \in \mathcal{P}} \|\mu_t - \nu\|_{\mathrm{tv}} \quad \leqslant \quad 2\sqrt{N}(1+\circ(1)) \exp\left(\frac{2\pi^2}{N^2}(1+\circ(1))t\right)$$

In particular, for any given $\epsilon > 0$, if we consider

$$t_N \quad := \quad (1+\epsilon)\frac{N^2 \ln(N)}{4\pi^2}$$

then

$$\lim_{N\to\infty} \sup_{\mu_0 \in \mathcal{P}} \|\mu_{t_N} - \nu\|_{\mathrm{tv}} \quad = \quad 0$$

Plan of the talk

Probabilistic interpretation of a_φ

How to estimate a_φ in practice?
First resort to the following probabilistic interpretation due to **Jacka and Roberts** [1995]. For any $x, y \in \bar{S}$, denote by τ_y^x the reaching time of y by X^x, the Markov process starting from x:

$$\tau_y^x \quad := \quad \inf\{t \geqslant 0 \,:\, X_t^x = y\} \;\in\; \mathbb{R}_+ \sqcup \{+\infty\}$$

Proposition

For any $x, y \in S$, we have

$$\frac{\varphi(x)}{\varphi(y)} \quad = \quad \mathbb{E}[\exp(\lambda_0 \tau_y^x) \mathbf{1}_{\tau_y^x < \tau_0^x}]$$

In particular, with $O := \{x \in S \,:\, \bar{L}(x, 0) > 0\}$, we have

$$a_\varphi \quad = \quad \max_{x \in S,\, y \in O} \mathbb{E}[\exp(\lambda_0 \tau_y^x) \mathbf{1}_{\tau_y^x < \tau_0^x}]$$

A path method

This result leads to two methods of estimating a_φ.
The first one is through a path argument.
If $\gamma = (\gamma_0, \gamma_1, ..., \gamma_l)$ is a path in S, denote

$$P(\gamma) \quad := \quad \prod_{k\in[\![0,l-1]\!]} \frac{\overline{L}(\gamma_k, \gamma_{k+1})}{|\overline{L}(\gamma_k, \gamma_k)| - \lambda_0}$$

Proposition

Assume that for any $y \in O$ and $x \in S$, we are given a path $\gamma_{y,x}$ going from y to x. Then we have

$$a_\varphi \quad \leqslant \quad \left(\min_{y\in O,\, x\in S} P(\gamma_{y,x}) \right)^{-1}$$

An example for the path method

Let G be the oriented graph induced by L on S, denote by d its maximum outgoing degree and by D its "oriented diameter". Let $0 < \rho_* \leqslant \rho^*$ be such that

$$\forall\ x \neq y \in \bar{S}, \qquad \bar{L}(x,y) \ \in \ \{0\} \sqcup [\rho_*, \rho^*]$$

Then we get

$$a_\varphi \ \leqslant \ \left(\frac{\rho^* d}{\rho_*}\right)^D$$

In the previous finite birth and death examples, it gives a bound on a_φ exploding exponentially in $D = N$, which is the true behavior only if $0 < r < 1$. The second method based on spectral estimates enables to recover the fact that φ_N explodes linearly in N for $r = 1$ and is bounded if $r > 1$.

Spectral estimates

Assume that η is reversible for L. The operator $-K$ is then diagonalizable, denote $\lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{N-1}$ its eigenvalues ($N = \mathrm{card}(S)$). For any $x \in S$, let $\lambda_0(S\backslash\{x\})$ be the first eigenvalue of the $(S\backslash\{x\}) \times (S\backslash\{x\})$ minor of $-K$. Finally, consider

$$\lambda_0' \quad := \quad \min_{x \in O} \lambda_0(S\backslash\{x\})$$

Proposition

Under the reversibility assumption, we have

$$a_\varphi \quad \leqslant \quad \left(\left(1 - \frac{\lambda_0}{\lambda_0'}\right) \prod_{k \in [\![N-1]\!]} \left(1 - \frac{\lambda_0}{\lambda_k}\right) \right)^{-1}$$

Under appropriate assumptions, it can be extended to denumerable state spaces.

Plan of the talk

Denumerable birth and death (1)

Consider $S := \mathbb{N}$ and $\bar{S} := \mathbb{Z}_+$, endowed with a **birth and death generator** $\bar{L}$: namely of the form

$$\forall\ x \neq y \in \bar{S}, \qquad \bar{L}(x,y) \quad = \quad \begin{cases} b_x & \text{, if } y = x+1 \\ d_x & \text{, if } y = x-1 \\ -d_x - b_x & \text{, if } y = x \\ 0 & \text{, otherwise} \end{cases}$$

where $(b_x)_{x\in\mathbb{Z}_+}$ and $(d_x)_{x\in\mathbb{N}}$ are the positive birth and death rates, except that $b_0 = 0$: 0 is the absorbing state and the restriction of $\bar{L}$ to $\mathbb{N}$ is irreducible.

The boundary point ∞ is said to be an **entrance boundary** for $\bar{L}$ if the following conditions are met:

$$\sum_{x=1}^{\infty} \frac{1}{\pi_x b_x} \sum_{y=1}^{x} \pi_y \quad = \quad +\infty \qquad (1)$$

$$\sum_{x=1}^{\infty} \frac{1}{\pi_x b_x} \sum_{y=x+1}^{\infty} \pi_y \quad < \quad +\infty \qquad (2)$$

Denumerable birth and death (2)

where

$$\forall\ x \in \mathbb{N}, \qquad \pi_x \quad := \quad \begin{cases} 1 & , \text{ if } x = 1 \\ \frac{b_1 b_2 \cdots b_{x-1}}{d_2 d_3 \cdots d_x} & , \text{ if } x \geqslant 2 \end{cases}$$

The probabilistic meanings: (1): for $x \in \mathbb{Z}_+$, X^x, does not explode to ∞ in finite time, (2): these processes "go down from infinity". One consequence of (2): $\sum_{x \in \mathbb{N}} \pi_x < +\infty$ and η is the normalization of π into a probability measure.
(1) and (2) imply that the operator $-K$ has only eigenvalues of multiplicity 1, say the $(\lambda_n)_{n \in \mathbb{Z}_+}$ in increasing order, and **Gong, Mao and Zhang** [2012] have shown that they are well approximated by the eigenvalues of the Neumann restriction of $\bar{L}$ to $[\![0, N]\!]$ for large $N \in \mathbb{N}$. Finally, define

$$\lambda_0' \quad := \quad \lambda_0(\mathbb{N} \backslash \{1\})$$

Extension of the spectral estimate

Theorem

Under the assumptions (1) and (2), we have

$$\left(1-\frac{\lambda_0}{\lambda_0'}\right)\prod_{n\in\mathbb{N}}\left(1-\frac{\lambda_0}{\lambda_n}\right) > 0$$

The eigenvector φ is bounded and its amplitude satisfies:

$$\frac{\sup_{x\in\mathbb{N}}\varphi(x)}{\inf_{y\in\mathbb{N}}\varphi(y)} = \frac{\lim_{x\to\infty}\varphi(x)}{\varphi(1)} \leqslant \left(\left(1-\frac{\lambda_0}{\lambda_0'}\right)\prod_{n\in\mathbb{N}}\left(1-\frac{\lambda_0}{\lambda_n}\right)\right)^{-1}$$

There is a somewhat converse statement, via the Lyapounov function approach.

Toward Hardy's bounds

The drawback: not really quantitative! Here is a more relevant bound:

Corollary

We have

$$\begin{aligned} a_\varphi &\leqslant \left(\left(1-\frac{\lambda_0}{\lambda_0'}\right)^2 \prod_{n\in\mathbb{N}\setminus\{1\}} \left(1-\frac{\lambda_0}{\lambda_0^{(n)}}\right)\right)^{-1} \\ &\leqslant \left(\left(1-\frac{\lambda_0}{\lambda_0'}\right)^{1+m} \prod_{n\in\mathbb{N}\setminus[\![1,m]\!]} \left(1-\frac{\lambda_0}{\lambda_0^{(n)}}\right)\right)^{-1} \end{aligned}$$

for any given $m \in \mathbb{N}$*, where for any* $n \in \mathbb{Z}_+$*,*

$$\lambda^{(n)} := \lambda_0(\mathbb{Z}_+ \setminus [\![0,n]\!])$$

Hardy's bounds

The advantage of the latter inequality, are the Hardy bounds on the $\lambda^{(n)}$. Define

$$\forall\ n \in \mathbb{Z}_+, \qquad A_n \;=\; \sup_{m>n} \sum_{k\in[\![n+1,m]\!]} \frac{1}{\pi(k)d_k} \sum_{l\geqslant m} \pi(l)$$

It is known that

$$\forall\ n \in \mathbb{Z}_+, \qquad A_n^{-1}/4 \;\leqslant\; \lambda_0^{(n)} \;\leqslant\; A_n^{-1} \tag{3}$$

so $m \in \mathbb{N}$ is chosen so that $A_n \leqslant 2A_0$ for all $n > m$, then

$$a_\varphi \;\leqslant\; \left(\left(1-\frac{\lambda_0}{\lambda_0'}\right)^{1+m} \prod_{n\in\mathbb{N}\setminus[\![1,m]\!]} \left(1-\frac{A_n}{4A_0}\right)\right)^{-1}$$

Example

Assume that there exists $\rho > 1$ such that the rates satisfy

$$\forall\ k \in \mathbb{N}, \qquad \begin{cases} b_{k+1} & \leqslant & \rho b_k \\ d_{k+1} & \geqslant & \rho d_k \end{cases} \tag{4}$$

Under the additional assumptions (1) and (2), we get

$$a_\varphi \quad \leqslant \quad \left(\left(1 - \rho^{-2}\right)^2 \prod_{n \geqslant 2} \left(1 - \rho^{-n}\right) \right)^{-1}$$

As a true example, with $\rho > 1$:

$$b_n \quad := \quad \begin{cases} 1 & , \text{ if } n \geqslant 1 \\ 0 & , \text{ if } n = 0 \end{cases}$$

$$d_n \quad := \quad \begin{cases} \rho^{n-1} & , \text{ if } n \geqslant 1 \\ 0 & , \text{ if } n = 0. \end{cases}$$

Plan of the talk

There is a huge literature on quasi-stationarity, with recent surveys provided by **Méléard and Villemonais** [2012], **Van Doorn and Pollett** [2013] or by the book of **Collet, Martínez and San Martín** [2013]. All of these review the history (**Yaglom**, **Bartlett**, **Darroch-Seneta**, ...). An annotated online bibliography is kept up to date by **Pollett** at

`http://www.maths.uf.edu.au/~pkp/papers/qsds.html`.

But the quantitative aspect was not fully investigated, usually only the asymptotical rate $\lambda_1 - \lambda_0$ was identified, but without the pre-exponential factor. See nevertheless **Van Doorn and Pollett** [2013], **Barbour and Pollett** [2010, 2012] or recent preprints of **Cloez and Thai** and of **Champagnat and Villemonais**.
The amplitude a_φ was used by **Jacka and Roberts** [1995] to investigate the process conditioned to have never been absorbed.

Plan of the talk

Discrete Heisenbeg group

For $n \in \mathbb{N} \backslash \{1, 2\}$, let

$$H(n) \quad := \quad \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : \ x, y, z \in \mathbb{Z}_n \right\}$$

Denoting (x, y, z) the above matrix, the matrix multiplication can be written

$$(x, y, z)(x', y', z') \quad = \quad (x + x', y + y', z + z' + xy')$$

Consider the symmetric generating set

$$S \quad := \quad \{(1, 0, 0), (-1, 0, 0), (0, 1, 0), (0, -1, 0)\}$$

and Q the uniform distribution on S.

Random walk

The associated random walk consists in sampling independent elements according to Q and in multiplying them. At any time $k \in \mathbb{Z}_+$, the resulting distribution is Q^{*k} and if n is odd, it converges for k large to the uniform distribution U on $H(n)$. Quantitatively order n^2 steps are necessary for convergence:

Theorem

There exists constants $A < B$ such that

$$A \exp(-2\pi^2 k/n^2) \;\leqslant\; \left\| Q^{*k} - U \right\|_{\mathrm{tv}} \;\leqslant\; B \exp(-2\pi^2 k/n^2)$$

These estimates were first proved by Diaconis and Saloff-Coste [94-96], using geometric analysis on nilpotent groups (Nash inequalities, moderate growth conditions, ...). Related results can be found in Stong [94-95], Alexopoulos [02], Breuillard [04-05], Diaconis [10] or Peres and Sly [13].

An interesting matrix

The goal of this part is to present an alternative approach based on Fourier analysis (representation theory) and absorption estimates. In particular to quantify the fact that the convergence of the z-coordinate should be faster (order $n \ln(n)$, but it should be n) than that of the x- and y-coordinates (order n^2).
Everything boils down to bounding the spectrum of the matrix $M(c) := (P + D(c))/2$, with $c \in \mathbb{Z}_n$, where P is the transition kernel of the random walk on $\mathbb{Z}_n$ and $D(c)$ is the diagonal matrix with entries

$$\forall\ j \in \mathbb{Z}_n, \qquad D_{j,j}(c) \quad := \quad \cos(2\pi cj/n)$$

Surprisingly, variants of these simple matrices come up in a variety of solid state physics and ergodic theory problems, such as Harper's Operators, Hofstader's Butterfly or the Ten Martinis Problem of Simon. They are also connected to the fast Fourier transform and to the Lévy's area of Brownian motions.

Plan of the talk

Basic formulas (1)

Let G be a finite group with $\widehat{G}$ the irreducible unitary representations. For Q a probability on G and $\rho \in \widehat{G}$, define the Fourier transform

$$\widehat{Q}(\rho) := \sum_{g \in G} Q(g)\rho(g)$$

The Fourier Inversion Theorem enables to reconstruct Q from these matrices:

$$\forall\, g \in G, \qquad Q(g) = \frac{1}{|G|} \sum_{\rho \in \widehat{G}} d_\rho \mathrm{tr}(\widehat{Q}(\rho)\rho(g)^*)$$

where d_ρ is the dimension of the representation ρ.

Basic formulas (2)

The Fourier transform takes convolutions into products, thus $\widehat{Q * Q}(\rho) = (\widehat{Q}(\rho))^2$. It can also be shown (Schur's lemma) that if U is the uniform distribution on G, then $\widehat{U}(\rho) = 0$, except for the trivial representation $\mathbf{1}$ of $\widehat{G}$.
Diaconis [88] has deduced the following upper bound: for any $k \in \mathbb{N}$,

$$\left\| Q^{*k} - U \right\|_{\text{tv}}^2 \leqslant \frac{1}{4} \sum_{\rho \in \widehat{G} \setminus \{\mathbf{1}\}} d_\rho \left\| \widehat{Q}^k(\rho) \right\|^2$$

where the norm of a matrix M is given by $\|M\|^2 := \mathrm{tr}(MM^*)$.

Representation theory for $H(n)$

Fix an integer m dividing n and let $V := \{f : \mathbb{Z}_m \to \mathbb{C}\}$. Consider $a, b \in \mathbb{Z}_{n/m}$ and $c \in \mathbb{Z}_m$ which is prime with m. Denote $q_m = e^{2\pi i/m}$, $q_n = e^{2\pi i/n}$ and define $\rho_{a,b,c} \in \mathrm{GL}(V)$ via

$$\forall\ f \in V, \forall\ j \in \mathbb{Z}_m,$$
$$\rho_{a,b,c}(x,y,z)f(j) \quad := \quad q_n^{ax+by} q_m^{c(yj+z)} f(j+x)$$

It can be checked that such $\rho_{a,b,c}$ are distinct irreducible representations of $H(n)$. For fixed m their number is $\left(\frac{n}{m}\right)^2 \phi(m)$ where ϕ is the Euler phi function. Since

$$\sum_{m|n} \left(\frac{n}{m}\right)^2 \phi(m) m^2 \;=\; n^3 \;=\; |H(n)|$$

they form a complete set of irreducible representations of $H(n)$.

Exemples (1)

• When $m = 1$, this gives n^2 1-dimensional representations (or characters):

$$\rho_{a,b,1}(x, y, z) \quad := \quad q_n^{ax+by}$$

• Consider the uniform probability Q on S and let us compute $\widehat{Q}(a, b, c)$. When $m = 1$, we get

$$\widehat{Q}(a, b, 1) \quad = \quad \frac{1}{2} \cos\left(\frac{2\pi a}{n}\right) + \frac{1}{2} \cos\left(\frac{2\pi b}{n}\right)$$

When $m = 2$ (so n is even), we get

$$\widehat{Q}(a, b, 1) \quad = \quad \frac{1}{2} \begin{pmatrix} \cos\left(\frac{2\pi b}{n}\right) & \cos\left(\frac{2\pi a}{n}\right) \\ \cos\left(\frac{2\pi a}{n}\right) & -\cos\left(\frac{2\pi b}{n}\right) \end{pmatrix}$$

Exemples (2)

More generally, for $m \geqslant 3$,

$$\forall\ l, k \in \mathbb{Z}_m, \quad \widehat{Q}(a,b,1)_{l,k} \quad = \quad \frac{1}{4}\begin{cases} 2\Re(q_n^b q_m^{lc}) & \text{, if } l = k \\ q_n^a & \text{, if } k = l+1 \\ q_n^{-a} & \text{, if } k = l-1 \\ 0 & \text{, otherwise} \end{cases}$$

Conjugating with the diagonal matrix $(1, q_n^a, q_n^{2a}, \ldots, q_n^{(m-1)a})$ produces a matrix with 1 on the super and sub diagonals (and q_n^{-ma} and q_n^{ma} in the "special corners").

• When n is prime, there are n^2 one-dimensional 1-dimensional representations and $n-1$ n-dimensional representations, with $\widehat{Q}(0,0,c)$ conjugate to $M(c)$.

Proof of the theorem when n is prime (1)

When n is prime, one thus gets, for any $k \in \mathbb{N}$,

$$4 \left\| Q^{*k} - U \right\|_{\mathrm{tv}}^2 \leqslant \sum_{(a,b)\neq(0,0)} \left(\frac{1}{2} \cos\left(\frac{2\pi a}{n}\right) + \frac{1}{2}\cos\left(\frac{2\pi b}{n}\right)\right)^{2k} + \sum_{c=1}^{n-1} n \left\| M(c)^k \right\|^2$$

For the first sum, consider for instance the case $a = 0$ and $b = 1$. If k is of order n^2, say $k = \eta n^2$, it appears the

$$\left(\frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\pi}{n}\right)\right)^{2k} = \left(1 - \frac{\pi^2}{n^2} + O\left(\frac{1}{n^4}\right)\right)^{2\eta n^2} \sim \exp(-2\pi^2\eta)$$

The other terms can be similarly bounded so that in the end, the whole first sum is of order $\exp(-2\pi^2\eta)$ up to an universal factor.

Proof of the theorem when n is prime (2)

For the second sum, consider the eigenvalues of $M(c)$:

$$1 > \beta_1(c) \geqslant \beta_2(c) \geqslant \cdots \geqslant \beta_n(c) > -1$$

and denote $\beta^*(c) := \max(\beta_1(c), -\beta_n(c))$. The following bounds enable to end the proof of the theorem, taking into account that $\beta^*(c) = \beta^*(n-c)$.

Proposition

There exists an universal constant $\theta > 0$ such that: for $0 < c < n/\ln(n)$,

$$\beta^*(c) \leqslant 1 - \frac{\theta}{n^{4/3}}$$

for $n/\ln(n) \leqslant c < n/2$,

$$\beta^*(c) \leqslant 1 - \frac{3}{4}\left(\frac{c}{n}\right)^2$$

Plan of the talk

Symmetries and inclusions

More generally, consider for $n, c \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, the diagonal matrix $D(n, c, \alpha)$ with entries

$$\cos\left(\frac{2\pi c}{n}(j+\alpha)\right), \quad 0 \leqslant j \leqslant n-1$$

and the matrix $M(n, c, \alpha) := (P + D(n, c, \alpha))/2$. Let $S(n, c, \alpha)$ be the spectrum of $M(n, c, \alpha)$.
We have the following properties:

(a) $\forall\ k \in \mathbb{N}, \quad S(n, c, \alpha) \subset S(kn, kc, \alpha)$
(b) If n is even, $S(n, c, \alpha) = -S\left(n, c, \alpha + \frac{n}{2c}\right)$
(c) If n is odd, $S(n, c, \alpha) \subset -S\left(2n, 2c, \alpha + \frac{n}{2c}\right)$

(a) comes from a juxtaposition of the eigenvectors, (b) from the multiplication of an eigenvector by $((-1)^j)_{0 \leqslant i \leqslant n-1}$ and (c) from (b) and (a) with $k = 2$.

Path arguments: principle

Consider on K a transition kernel on $\bar{S} = S \sqcup \{\infty\}$ where
• K is irreducible on the finite set S and reversible with respect to a positive measure μ.
• ∞ is an absorbing point which is reachable from S.

Let be given for any $x \in S$ a path

$$\gamma_x := (x_0 = x, x_1, x_2, ..., x_{|\gamma_x|} = \infty)$$

going from x to ∞ with $K(x_l, x_{l+1}) > 0$ for $0 \leqslant l \leqslant |\gamma_x| - 1$. Then the largest eigenvalue of K is bounded above by $1 - 1/A$ where

$$A \quad := \quad \max_{x \in S,\, y \in \bar{S}} \frac{2}{\mu(x) K(x,y)} \sum_{z \in S \,:\, (x,y) \in \gamma_z} |\gamma_z| \, \mu(z)$$

Path arguments (1)

The matrix $M(c)$ can be transformed into a subMarkov kernel by considering

$$K_c \quad := \quad (I + 2M(c))/3$$

We come back to the previous situation by taking $S := \mathbb{Z}_n$ and completing K_c into an absorbed kernel (with $K_c(x, \infty) := (1 - \cos(2\pi cx/n))/3)$. Thus it remains to find convenient paths in order to prove:

Lemma

There exists an universal constant $\theta > 0$ such that for all positive integer n and $1 \leqslant c \leqslant n/\ln(n)$, we have

$$\beta_1(c) \quad \leqslant \quad 1 - \theta \left(\frac{c}{n}\right)^{4/3}$$

Path arguments (2)

The idea is to use paths exiting $\mathbb{Z}_n$ where the probability to go in one step to ∞ is no longer negligible (it is the weakest at 0 mod (n/c)). So to get out of the "bad positions" (where the chain is not sufficiently killed!), we use paths of length $(n/c)^{2/3}$ to join the nearest "good positions". From the latter ones, just go directly to ∞. For a good x, we have

$$\begin{aligned} K_c(x,\infty) &\geqslant \frac{1}{3}\left(\frac{1}{2}\left(\frac{2\pi c}{n}\left(\frac{n}{c}\right)^{2/3}\right)^2 + O\left(\frac{2\pi c}{n}\left(\frac{n}{c}\right)^{2/3}\right)^4\right) \\ &\sim \frac{2\pi^2}{3}\left(\frac{c}{n}\right)^{2/3} \end{aligned}$$

The wanted bound follows easily (after an appropriate optimization which led to the exponent $2/3$).

Path arguments (3)

For $\frac{n}{\log(n)} \leqslant c \leqslant \frac{n}{2}$, we get a similar bound:

$$\beta_1(c) \quad \leqslant \quad 1 - \frac{3}{4}\left(\frac{c}{n}\right)^2$$

The proof is the same, except that to join good positions from bad ones, use paths of length $\lceil \lfloor n/c \rfloor / 4 \rceil$.

Note that these path arguments are quite robust, e.g. by real shift of the diagonal, namely if $(\cos(2\pi cj/n))_{j\in\mathbb{Z}_n}$ is replaced by $(\cos(2\pi c(\alpha + j)/n))_{j\in\mathbb{Z}_n}$ where $\alpha \in \mathbb{R}$. With the inclusion $S(n, c, 0) \subset -S\left(2n, 2c, \frac{n}{2c}\right)$, this the key for the minoration of the lowest eigenvalue:

Lowest eigenvalue

Lemma

There exists an universal constant $\theta > 0$ such that for all positive integer n and $1 \leqslant c \leqslant n/\ln(n)$, we have

$$\beta_n(c) \quad \geqslant \quad -1 + \theta \left(\frac{c}{n}\right)^{4/3}$$

and for $\frac{n}{\log(n)} \leqslant c < \frac{n}{2}$,

$$\beta_n(c) \quad \geqslant \quad -1 + \frac{3}{4}\left(\frac{c}{n}\right)^2 (1 + o(1))$$

It ends the proof of the announced proposition.

Plan of the talk

1. Convergence to quasistationarity
   - Qualitative versus quantitative convergence
   - Finite quasi-stationarity
   - A reduction by comparison
   - Approach by functional inequalities
   - Some examples
   - Estimates on the amplitude a_φ
   - Birth and death processes with ∞ as entrance boundary
   - Some references
2. Heisenberg random walks
   - Motivations and main result
   - Representation theory
   - Eigenvalues bounds
   - Further results

More precise asymptotics (1)

So we got an order $n^{-4/3}$ for the spectral gap of the matrix $M := M(1)$. In fact the right order is n^{-1} and one can go further:

Theorem

For n large, the k-th largest eigenvalue β_k of M, $k \in \mathbb{N}$, behaves like

$$\beta_k \quad = \quad 1 - \frac{\mu_k}{n} + o\left(\frac{1}{n}\right)$$

where $\mu_k = (2k-1)\pi$ is the k-th smallest eigenvalue of the (harmonic oscillator) Schrödinger operator on $\mathbb{R}$:

$$-\frac{1}{4}\frac{d^2}{dx^2} + \pi^2 x^2$$

More precise asymptotics (2)

This is proven by showing the convergence of the corresponding eigenfunctions, appropriately renormalized (via classical Fourier transform analysis).

There is a similar result for the lowest eigenvalues: for n large, the k-th smallest eigenvalue β_{n-k+1} of M, $k \in \mathbb{N}$, behaves like

$$\beta_{n-k+1} = -1 + \frac{\mu_k}{n} + o\left(\frac{1}{n}\right)$$

Scrambled diagonals

In the previous examples the diagonal was quite regular, what does happen if one shuffles it by replacing $(\cos(2\pi cj/n))_{j\in\mathbb{Z}_n}$ by $(\cos(2\pi c\sigma(j)/n))_{j\in\mathbb{Z}_n}$, where σ belongs to the permutation group $\mathcal{S}_n$?
Consider the case $c=1$. Heuristically the non-scrambled diagonal should be the worst case, because it is constructing the safer place for the particle not to be killed. We have not been able to prove that, but here is a result in this direction: there exists a universal constant $c>0$ such that for all $n\in\mathbb{N}\setminus\{1,2\}$, whatever $\sigma\in\mathcal{S}_n$, the corresponding largest eigenvalue is bounded above by $1-c/n$.

Quality of a niche (1)

The proof is based on a probabilistic estimate of the quality of a niche. Consider $(X_t)_{t\geqslant 0}$ the continuous-time random walk on $\mathbb{Z}$, with jump rates 1 between neighbors. Make it starts from 0. Fix $n \in \mathbb{Z}_+$ and denote

$$\tau_n \quad := \quad \inf\{t \geqslant 0 \,:\, |X_t| = n+1\}$$

Let $(u_x)_{x\in[\![-n,n]\!]}$ be some killing rates and $\mathcal{E}$ be an independent exponential variable of parameter 1. An associated absorption time $\bar{\tau} \in [0,+\infty]$ is defined by

$$\bar{\tau} \quad := \quad \inf\left\{t \geqslant 0 \,:\, \int_0^t u_{X_s}\, ds \geqslant \mathcal{E}\right\}$$

Using Ray-Knight type ideas in a discrete context, we get:

Quality of a niche (2)

Proposition

We have for $n \geqslant 1$,

$$\mathbb{P}[\bar{\tau} > \tau_n] \leqslant G_n(v) := \left(\prod_{y \in [\![0, n-1]\!]} \frac{1}{1 + (n+1)(n+1-y)v_y} \right)^{1/(n+1)}$$

where $v := (v_y)_{y \in [\![0,n]\!]} := \min(u_{-y}, u_y)$

The functional G_n has nice monotonicity properties:

$$v \leqslant v' \quad \Rightarrow \quad G_n(v) \geqslant G_n(v')$$
$$G_n(v) \quad \leqslant \quad G_n(\bar{v})$$

where $\bar{v}$ is the non-decreasing ordering of the entries of v. It enables to compare niches and to prove the announced result on the scrambled diagonals.

The previous improved bounds are the key to good estimates on the speed of convergence of the center of $H(n)$, which is $\{(0,0,z)\ :\ z \in \mathbb{Z}_n\}$. Let us consider the case where n is prime. the Fourier inversion theorem implies that for any $k \in \mathbb{Z}_+$ and $z \in \mathbb{Z}_n$,

$$\mathbb{P}[Z_k = z] \quad = \quad \frac{1}{n} + \frac{1}{n} \sum_{c=1}^{n-1} e^{\frac{-2\pi icz}{n}} \sum_{l=1}^{n} (\widehat{Q}(0,0,c)^k)_{1,l}$$

where $(X_k, Y_k, Z_k)_{k \in \mathbb{Z}_+}$ is the underlying random walk.
Indeed, a priori we have for any $k \in \mathbb{Z}_+$ and $(x,y,z) \in H(n)$,

$$\mathbb{P}[X_k = x, Y_k = y, Z_k = z] \quad = \quad \frac{1}{n^3} \sum_{\rho \in \widehat{H}(n)} d_\rho \mathrm{tr}\left(\widehat{Q}(\rho))^k \rho((x,y,z)^{-1}) \right)$$

Using the explicit description of $\widehat{H}(n)$, we get that

$$\sum_{x,y \in \mathbb{Z}_n} \rho((x,y,z)^{-1}) \quad = \quad 0$$

if ρ is one of the non-trivial representations of dimension 1 and

Convergence of the center (2)

$$\sum_{x,y\in\mathbb{Z}_n} \rho((x,y,z)^{-1}) \quad = \quad ne^{\frac{-2\pi icz}{n}} \begin{pmatrix} 1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix}$$

for the n-dimensional representations $(0,0,c)$, with $c \in \mathbb{Z}_n \backslash \{0\}$.

Corollary

There exists a universal constant $\theta > 0$ such that for any prime n and any time $k \in \mathbb{Z}_+$,

$$\|\mathcal{L}(Z_k) - U\|_{\mathrm{tv}} \quad \leqslant \quad ne^{-\theta k/n}$$

It is an immediate consequence of the previous results, using for any $k \in \mathbb{Z}_+$, $c \in \mathbb{Z}_n \backslash \{0\}$ and $l \in [\![1,n]\!]$, the rough bound

$$\left|(\widehat{Q}(0,0,c)^k)_{1,l}\right| \leqslant \beta^*(c)^k \leqslant (1-\theta/n)^k$$