A singular large deviation phenomenon for some exit times

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Abstract

We consider small Brownian perturbations of 1-dimensional dynamical systems for which unicity of the solutions does not hold and thus in particular the classical theory of Freidlin and Wentzell cannot be applied. More precisely, by using some usual changes of scale and speed for real diffusions, we will study the asymptotic behavior of exit times from appropriate normalized neighborhoods of 0, point where a lack of Lipschitzianity is assumed. This probabilistic approach should enable to recover and extend some recent results of Gradinaru, Herrmann and Roynette concerning singular large deviations for the densities of these processes, via spectral interpretations of the appearing rates.

Keywords: Small Brownian perturbations of degenerate real dynamical systems, singular large deviations, behavior of large exit times, largest (negative) eigenvalue for diffusion and related Schrödinger operators with Dirichlet boundary condition.

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1 Motivation and expectations

In a recent paper, Gradinaru, Herrmann and Roynette [13] considered for any fixed $1<\gamma<2$, the real diffusion $(X_\epsilon(t))_{t\geq 0}$ parametrized by $\epsilon>0$ and defined by

$$\left\{\begin{array}{rcl} X_\epsilon(0) & = & 0 \\ dX_\epsilon(t) & = & \epsilon\, dB(t)+\frac{\gamma}{2}\,|X_\epsilon(t)|^{\gamma-1}\,\mathrm{sign}(X_\epsilon(t))\,dt \end{array}\right.$$

where $(B(t))_{t\geq 0}$ is a standard Brownian motion. As ϵ goes to 0_+, this stochastic process X_ϵ can be seen as a small perturbation of a dynamical system Y verifying

$$\left\{\begin{array}{rcl} Y(0) & = & 0 \\ Y'(t) & = & \frac{\gamma}{2}\,|Y(t)|^{\gamma-1}\,\mathrm{sign}(Y(t)) \end{array}\right. \tag{1}$$

Due to the non unicity of the solutions of this equation, one cannot apply directly the theory of large deviations developped by Freidlin and Wentzell [11] for similar diffusions, and indeed some new phenomena are appearing (cf [1]).

To go further in the understanding of the induced particular behaviors, Gradinaru, Herrmann and Roynette [13] have studied singular large deviations for $p_\epsilon(t,x)$, the density with respect to the Lebesgue measure of the distribution of $X_\epsilon(t)$ at time $t>0$ and position $x\in\mathbb{R}$ satisfying $(t,x)\in\triangle$, where $\triangle$ is the time-space domain under the maximal solutions of (1):

$$\triangle \;=\; \{(t,x)\in\mathbb{R}_+\times\mathbb{R}\,:\,|x|<(\gamma(1-\gamma/2))t)^{1/(2-\gamma)}\}$$

They show that there exists a constant $\lambda_\infty>0$ (admitting a spectral interpretation) such that

$$\lim_{\epsilon\to 0_+}\epsilon^{\frac{2(2-\gamma)}{\gamma}}\ln(p_\epsilon(t,x)) = -\lambda_\infty\left(\frac{|x|^{2-\gamma}}{\gamma(1-\gamma/2)}-t\right) \tag{2}$$

The proof of this result is quite technical and is based on a mix of probabilist ideas (representation of the density $p_\epsilon(t,x)$ with expectations relative to Brownian bridges, via Girsanov's formula) and analytic methods (viscosity solutions of some ordinary differential equations).

Our purpose here is to present the main step of an alternative approach, entirely probabilist (and thus subjectively simpler). In order to explain heuristically our point of view, let us recall that the set of (non identically null) solutions of (1) can be parametrized by a couple $(s,b)\in\mathbb{R}_+\times\{-1,+1\}$, which corresponds to the solution $Y_{s,b}$ given by

$$\forall\, t\geq 0,\qquad Y_{s,b}(t)=b(\gamma(1-\gamma/2))^{1/(2-\gamma)}(t-s)_+^{1/(2-\gamma)}$$

Then the idea is to obtain (singular) large deviations for approximations of this parameter relative to X_ϵ. The speed of this principle will be $\epsilon^{-\frac{2(2-\gamma)}{\gamma}}$, negligible with respect to ϵ^{-2}, which is the usual speed for Freidlin and Wentzell's type large deviations.

More precisely, we will find a nice parameter $\delta(\epsilon)>0$ such that if we consider for fixed $k>0$, the $\bar{\mathbb{R}}_+$-valued variables

$$\begin{array}{rcl} T^{(\epsilon,k)} & = & \inf\{t\geq 0\,:\,|X_\epsilon(t)|=k\delta(\epsilon)\} \\ T^{(\epsilon,k,+)} & = & \inf\{t\geq 0\,:\,X_\epsilon(t\wedge T^{(\epsilon)})=k\delta(\epsilon)\} \\ T^{(\epsilon,k,-)} & = & \inf\{t\geq 0\,:\,X_\epsilon(t\wedge T^{(\epsilon)})=-k\delta(\epsilon)\} \end{array}$$

then they satisfy a principle of large deviations: there exists a constant $\lambda_k > 0$ such that for all Borelian subset A of $[0,+\infty[$,

$$\begin{aligned} -\lambda_k \inf(\underline{A}) &\leq \liminf_{\epsilon\to 0_+} \epsilon^{\frac{2(2-\gamma)}{\gamma}} \ln(\mathbb{P}[T^{(\epsilon,k)} \in A]) \\ &\leq \limsup_{\epsilon\to 0_+} \epsilon^{\frac{2(2-\gamma)}{\gamma}} \ln(\mathbb{P}[T^{(\epsilon,k)} \in A]) \;\leq\; -\lambda_k \inf(\overline{A}) \end{aligned} \tag{3}$$

where $\underline{A}$ and $\overline{A}$ are respectively the interior and the closure of A, and idem for $T^{(\epsilon,k,+)}$ and $T^{(\epsilon,k,-)}$ (but note that this will not have been true if we had permitted to $+\infty$ to belong to A, since either $T^{(\epsilon,k,+)}$ or $T^{(\epsilon,k,-)}$ gives a weight larger than 1/2 to this point).

The link with the constant λ_∞ introduced by Gradinaru, Herrmann and Roynette [13] is that we have

$$\lim_{k\to\infty} \lambda_k = \lambda_\infty \tag{4}$$

We think (but this question won't be studied here, because it would led to different arguments to the ones we want to present) that the above principles for $T^{(\epsilon,k,+)}$ and $T^{(\epsilon,k,-)}$, for all $k>0$, are sufficient to determine all the singular large deviations of X_ϵ of the same order (in particular they imply those of $T^{(\epsilon,k)}$). The intuition is that the threshold $\delta(\epsilon)$ will have been chosen such that for large $k>0$, on $\mathbb{R}\setminus]-k\delta(\epsilon),k\delta(\epsilon)[$ one can apply (or more properly, adapt) the classical theory of Freidlin and Wentzell, and due to its stronger speed, we will only be allowed to follow there the solutions of (1). Or alternatively, the event $\{T^{(\epsilon,k,+)} = t\}$, for $t\in\mathbb{R}_+$ (and consequently that $T^{(\epsilon,k,-)} = +\infty$), should asymptotically (for small $\epsilon>0$ and next for large $k>0$) mean that X_ϵ is close to $Y_{t,1}$.

For instance, let us come back to the density $p_\epsilon(t,x)$ for $(t,x)\in\triangle$ and with $x>0$. There is a unique solution of (1) going through (t,x) (ie taking the value x at time t), say it is parametrized by $(s,1)$.

Following the previous wandering, we believe that maybe it is possible to show, in the sense of logarithmic equivalence of order $\epsilon^{-\frac{2(2-\gamma)}{\gamma}}$ for small $\epsilon>0$ and in the limit for large $k>0$, that

$$p_\epsilon(t,x) \asymp p_\epsilon(t_\epsilon, k\delta(\epsilon)) \asymp \mathbb{P}[T^{(\epsilon,k,+)} \simeq t_\epsilon] \asymp \mathbb{P}[T^{(\epsilon,k,+)} \simeq s] \tag{5}$$

where $t_\epsilon>0$ is such that $Y_{s,1}(t_\epsilon) = k\delta(\epsilon)$. The last approximation should come from the convergence of t_ϵ toward s as soon as $k\delta(\epsilon)$ goes to zero.

For the cases where $x=0$, one should rather use that for large $k>0$,

$$p_\epsilon(t,0) \asymp \mathbb{P}[T^{(\epsilon,k)} \geq t] \asymp \mathbb{P}[T^{(\epsilon,k)} \simeq t]$$

Taking into account (4) and the fact that $s = |x|^{2-\gamma}/\gamma(1-\gamma/2) - t$ in (5), it is somewhat appealing if not conforting that the above acts of faith "lead" to the exact result.

Nevertheless, as we already mentioned it, we will only prove (3) and (4), not their expected consequences. In fact, we will extend a little the setting by considering for drift of X_ϵ the vector field given by

$$\forall\, x\in\mathbb{R}, \qquad b(x) = \begin{cases} a_+\frac{\gamma}{2}x^{\gamma-1} & ,\ \text{if } x\geq 0 \\ -a_-\frac{\gamma}{2}|x|^{\gamma-1} & ,\ \text{if } x<0 \end{cases} \tag{6}$$

where $a_+, a_- > 0$ and $1<\gamma<2$.

In a quite strange manner, the lack of antisymmetry of this drift will not be strong enough to permit different large deviations principles for $T^{(\epsilon,k,+)}$ and $T^{(\epsilon,k,-)}$ (but for sure, their commun constant λ_k

will depend on a_+ and a_-). Note furthermore that for our restricted purpose, we only need that the above definition of the drift is enforced for $|x|$ small enough.

As the method we will use by changes of scale and speed for X_ϵ is quite flexible, for some times we wrongly believed that our setting could include relaxed assumptions such as

$$\left\{\begin{array}{lcl} b(x) & \underset{x\rightarrow 0_+}{\sim} & a_+\frac{\gamma_+}{2}x^{\gamma_+-1} \\ b(x) & \underset{x\rightarrow 0_-}{\sim} & -a_-\frac{\gamma_-}{2}\left|x\right|^{\gamma_--1} \end{array}\right. \tag{7}$$

where $a_+, a_- > 0$ and $1 < \gamma_+, \gamma_- < 2$. There the strong non antisymmetry of b should lead to separate behaviors for $T^{(\epsilon,k,+)}$ and $T^{(\epsilon,k,-)}$ (which would then be associated to appropriate parameters $k\delta_+(\epsilon)$ and $-k\delta_-(\epsilon)$, for $k, \epsilon > 0$, if we want to take advantage from the above considerations about Freidlin and Wentzell theory) and that should imply different speeds for the singular large deviations of $p_\epsilon(t,x)$ for $(t,x) \in \triangle$, depending on the sign of x (under some additional regularity assumptions for the drift b away from 0). But up to now, we did not manage to get round all the technical difficulties popping out in that situation (or perhaps more correctly, an important idea is still missing there), except in the case $\gamma_+ = \gamma_-$ for which we can use direct comparisons with (6).

The plan of the article is the following: in next section we recall some classical background materials about the 1-dimensional diffusion associated to a given measurable and bounded potential. Under an extra continuity assumption for the latter, the section 3 contains an alternative proof of a large deviation result for exit times from a fixed interval by the corresponding process, asymptotically for large value of this time, and makes a link with the largest eigenvalue of the relative diffusion and Schrödinger operators with Dirichlet boundary condition. These results were already known in a smooth context, but we will have to extend them a little in order to deal with particular merely continuous and unbounded situations, as they will be the main ingredient in sequel. Then in order to illustrate the computations to follow, in section 4 we treat the large deviations as ϵ goes to zero of $T^{(\epsilon,k)}$ for any fixed $k > 0$, in the nice case (6) for $0 < \gamma < 2$ and $a_-, a_+ > 0$, and we show the convergence (4), which will be in some sense meaningful only for $1 \leq \gamma < 2$. Finally the easy generalizations to the similar situations for $T^{(\epsilon,k,+)}$ and $T^{(\epsilon,k,-)}$, but where we only assume equivalences in zero, are done in the last section.

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2 Preliminaries on linear diffusions

The more general one-dimensional diffusions are the traditional setting (cf for instance [3], [15] or [18], which will be our favorite reference here) for the results we will recall. Nevertheless, we will only present this material in case of constant coefficient of "volatility", because either the diffusions considered in this article or the other ones we have in mind (with respect to (7)) will naturally satisfy this assumption, which furthermore is stable by the conditioning operation described below. Indeed the purpose of this section is twofold, in one hand we fix some usual notations and on the other one we introduce this classical theory in a convenient way for us (thus saving the reader of going through the exercises of section VII.3 of [18]).

We consider the following a priori given objects: ϵ, δ_- and δ_+ are positive reals and $U\ :]-\delta_-, \delta_+[\to \mathbb{R}$ is a measurable and locally bounded function.

For fixed $-\delta_- \leq x \leq \delta_+$, we want to give a signification to the diffusion $X_{\epsilon,x} = (X_{\epsilon,x}(t))_{t\geq 0}$ starting from x, absorbed in $-\delta_-$ and δ_+ and whose evolution between these frontiers is heuristically described by

$$dX_{\epsilon,x}(t) \quad = \quad \epsilon\, dB(t) + \frac{1}{2}U'(X_{\epsilon,x}(t))\, dt \tag{8}$$

where $B = (B(t))_{t\geq 0}$ stands for a standard Brownian motion.

But the lack of assumed regularity for U keeps us from taking a sde's point of view. By chance, in the context of dimension 1, there is another well-known and very efficient approach, via changes of speed and scale, for which we need some notations.

So let us define the next functions

$$\begin{array}{rcl} v_\epsilon\ :\ [-\delta_-, \delta_+] & \to & \bar{\mathbb{R}} \\ y & \mapsto & \displaystyle\int_0^y \exp\left(-U(z)/\epsilon^2\right)\, dz \end{array}$$

and

$$\begin{array}{rcl} m_\epsilon\ :] -\delta_-, \delta_+[& \to & \mathbb{R}_+ \\ y & \mapsto & \epsilon^{-2}\exp\left(2U(y)/\epsilon^2\right) \end{array}$$

We notice that v_ϵ is continuous and strictly increasing, thus its inverse is well-defined and we put

$$n_\epsilon \quad \stackrel{\text{def.}}{=} \quad m_\epsilon \circ v_\epsilon^{-1}\ :]v_\epsilon(-\delta_-), v_\epsilon(\delta_+)[\to \mathbb{R}_+$$

Now consider $(W^{v_\epsilon(x)}(t))_{t\geq 0}$ a Brownian motion starting from $v_\epsilon(x)$ (in the non important cases where $v_\epsilon(x) = \pm\infty$, we make the convention that for all $t \geq 0$, $W_{v_\epsilon(x)}(t) = v_\epsilon(x)$, this is possible only for the absorbing points $x = -\delta_-$ or $x = \delta_+$), we define $X_{\epsilon,x}$ as

$$\forall\ t \geq 0, \qquad X_{\epsilon,x}(t) \quad = \quad v_\epsilon^{-1}(W_{v_\epsilon(x)}[A_\epsilon^{-1}(t) \wedge T_{v_\epsilon(-\delta_-)}(W_{v_\epsilon(x)}) \wedge T_{v_\epsilon(\delta_+)}(W_{v_\epsilon(x)})]) \tag{9}$$

where A_ϵ^{-1} is the inverse of

$$A_\epsilon\ :\ \mathbb{R}_+ \ni t \quad \mapsto \quad \int_0^{t\wedge T_{v_\epsilon(-\delta_-)}(W_{v_\epsilon(x)})\wedge T_{v_\epsilon(\delta_+)}(W_{v_\epsilon(x)})} n_\epsilon(W_{v_\epsilon(x)}(s))\, ds \quad \in \bar{\mathbb{R}}_+$$

and where for any $y \in \mathbb{R}$ and any real valued continuous process Y, $T_y(Y)$ denotes the reaching time of y by Y.

The homogeneous Markov property of W^x enables to deduce it for $X_{\epsilon,x}$, ie if $\mathbb{P}_{\epsilon,x}$ denotes the law of this process on the canonical set of continous paths from $\mathbb{R}_+$ to $[-\delta_-, \delta_+]$ (endowed with the Borelian σ-field associated to the topology of locally uniform convergence), then for any $t \geq 0$, the law of $(X_{\epsilon,x}(t+s))_{s\geq 0}$ conditioned by the knowledge of $(X_{\epsilon,x}(u))_{0\leq u\leq t}$ is just ($\mathbb{P}_{\epsilon,x}$-a.s.) $\mathbb{P}_{\epsilon,X_{\epsilon,x}(t)}$.

The reason for the affirmation that $X_{\epsilon,x}$ evolves as (8) inside $]-\delta_-, \delta_+[$ is that if we assume furthermore that U is $\mathcal{C}^1$ in $]-\delta_-, \delta_+[$, then for any $\varphi \in \mathcal{C}^2_{\mathrm{c}}(]-\delta_-, \delta_+[)$ (the space of twice differentiable functions with compact support in $]-\delta_-, \delta_+[$), the process

$$\left(\varphi(X_{\epsilon,x}(t)) - \varphi(X_{\epsilon,x}(0)) - \int_0^t L_\epsilon(\varphi)(X_{\epsilon,x}(s))\, ds\right)_{t\geq 0}$$

is a martingale with respect to the natural filtration generated by $X_{\epsilon,x}$, where L_ϵ is the pregenerator defined on $\mathcal{C}^2_{\rm c}(]-\delta_-,\delta_+[)$ by

$$\forall\,\varphi\in\mathcal{C}^2_{\rm c}(]-\delta_-,\delta_+[),\,\forall\,x\in]-\delta_-,\delta_+[,\qquad L_\epsilon(\varphi)(x)\;=\;\frac{1}{2}\left[\epsilon^2\varphi''(x)+U'(x)\varphi'(x)\right]$$

This result is proved via a straightforward application of Itô's formula. We notice that if in addition, U' is for instance supposed to be locally Lipschitz on $]-\delta_-,\delta_+[$ (eg if U is $\mathcal{C}^2$), then unicity holds for the above martingale problem, so its solution is given by the previous construction. This fact is important, because it enables one to use the powerful tool of stochastic calculus. Next it is usually possible to extend the result thus deduced to less regular functions U by taking into account continuity properties with respect to these potentials. Typically, if $(U_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions on $]-\delta_-,\delta_+[$ converging uniformly to a mapping U_∞, then the respective diffusions X_{ϵ,x,U_n} (seen as functionals on the standard Wiener space, by expressing that $W_{v_{\epsilon,U_n}(x)}=v_{\epsilon,U_n}(x)+W_0$, note that each time we will need to precise the potential considered, we will add it in the subscript) converge to X_{ϵ,x,U_∞}, as n goes to infinity, for any fixed $\epsilon>0$ and $x\in[-\delta_-,\delta_+]$ (nevertheless, one should be careful there, since locally uniform convergence is not sufficient to insure this asymptotic behavior).

But one can find other heuristic interpretations of the evolution of $X_{\epsilon,x}$ inside $]-\delta_-,\delta_+[$ for some particular U. Eg, if U' is assumed to be càdlàg on $[-\delta_-,\delta_+]$, one can rather use the formula of Tanaka and that leads to the introduction of local times. We don't want to go through the whole development of this approach, nonetheless let us just present the illustrative example of the potential $U_\infty(\cdot)=\mathrm{sign}(\cdot)$ defined on $[-1,1]$ (with the inessential convention that $\mathrm{sign}(0)=1$).

The associated diffusion is then seen to be solution to

$$dX_{\epsilon,x}(t)\;=\;\epsilon\,dB(t)+\epsilon^2\frac{1+2\exp(-2/\epsilon^2)}{1+\exp(-2/\epsilon^2)}\,dl_{\epsilon,0}(t)$$

where B is a standard Brownian motion and where $(l_{\epsilon,0}(t))_{t\geq 0}$ is the process of symmetrized local time in 0 of $X_{\epsilon,x}$ (see [4]).

This remark enables to see there is no difficulty in the definition of the processes mentioned in the introduction, even those corresponding to (7) with $0\leq\gamma_-<2$ and $0\leq\gamma_+<2$, because the above considerations are valid as soon as v_ϵ and m_ϵ are well-defined (one can go even further since the local boundedness of U hypothesis can sometimes also be relaxed, but unfortunately not up to the point to allow for the treatment of the case of negative γ_+ or γ_-). For our strict case (6) and at least for what is concerning (3), we will merely assume that $0<\gamma<2$ (we remove the case $\gamma=0$, due to an embarrassing technicality appearing for non-continuous potentials). We will keep the assumption that $a_+,a_->0$, since it corresponds to the originally motivating repulsive situation, even if this study could be extended to a more general case. Nevertheless, when we will come to the proof of (4), we will have to reactivate the hypothesis $1\leq\gamma<2$, if we want the limit to be non null.

Let us come back to the general situation, but assuming now that U is bounded.

Another main interest of the mapping v_ϵ is that it admits the following probabilist interpretation (cf proposition 3.2 p. 288 of [18]) :

$$\mathbb{P}[T_{-\delta_-}(X_{\epsilon,x})<T_{\delta_+}(X_{\epsilon,x})]\;=\;\frac{v_\epsilon(\delta_+)-v_\epsilon(x)}{v_\epsilon(\delta_+)-v_\epsilon(-\delta_-)}\tag{10}$$

Before coming to a very interesting consequence of this famous fact, we simplify the notations: as we will merely be led to consider $X=(X_t)_{t\geq 0}\stackrel{\text{def.}}{=}X_{\epsilon,0}$, from now on we will not only remove the subscript x but also the ϵ. We denote by $\bar{\mathbb{P}}$ the conditioning of $\mathbb{P}$ by the event $\{T_{-\delta_-}(X)<T_{\delta_+}(X)\}$

and in order to avoid confusion, we will write $\bar{X}$ for the canonical coordinate process under this new probability (or alternatively for the image by the previous mapping of $W = (W_t)_{t\geq 0}$ under $\bar{\mathbb{P}}$, the conditioning of the standard Wiener measure by $T_{v(-\delta_-)}(W) < T_{v(\delta_+)}(W)$).

By definition, we have for any Borelian subset A of the set of trajectories,

$$\bar{\mathbb{P}}[\bar{X} \in A] \;=\; \frac{\mathbb{P}[X \in A,\, T_{-\delta_-}(X) < T_{\delta_+}(X)]}{\mathbb{P}[T_{-\delta_-}(X) < T_{\delta_+}(X)]} \tag{11}$$

The fact that the conditioning set belongs to the tail σ-algebra generated by X (due to the absorbing property of $-\delta_-$ and δ_+), shows that $\bar{X}$ is again Markovian.

Then our next objective is to explicit its generator. Following the procedure alluded to before, we start with the case where U is furthermore assumed to be $\mathcal{C}^1$ in $[-\delta_-, \delta_+]$. Let $\varphi \in \mathcal{C}^2_{\mathrm{c}}(]-\delta_-, \delta_+[)$, we have to differentiate for $t \geq 0$,

$$\partial_t \bar{\mathbb{E}}[\varphi(\bar{X}_t)] \;=\; \partial_t \left(\frac{\mathbb{E}[\varphi(X_t)\mathbf{1}_{\{T_{-\delta_-}(X) < T_{\delta_+}(X)\}}]}{\mathbb{P}[T_{-\delta_-}(X) < T_{\delta_+}(X)]} \right)$$

(due to the boundedness of U, one will have noticed that $T_{-\delta_-}(X) \wedge T_{\delta_+}(X) < +\infty$ a.s. and that $0 < \mathbb{P}[T_{-\delta_-}(X) < T_{\delta_+}(X)] < 1$).

Using (11), the fact that φ is null in $-\delta_-$ and δ_+ (implying that $\varphi(X_t) = 0$ if $t \geq T_{-\delta_-}(X) \wedge T_{\delta_+}(X)$) and conditioning $\mathbf{1}_{t \leq T_{-\delta_-}(X) < T_{\delta_+}(X)}$ by $\sigma(X_s\,;\, 0 \leq s \leq t)$, we get that this is equal to

$$\frac{\partial_t \mathbb{E}[\varphi(X_t)(v(\delta_+) - v(X_t))]}{v(\delta_+) - v(0)} \;=\; \frac{\mathbb{E}[L[\varphi(\cdot)(v(\delta_+) - v(\cdot))](X_t)]}{v(\delta_+)}$$

But one calculates at once that for all $x \in]-\delta_-, \delta_+[$,

$$\begin{aligned} L[\varphi(\cdot)(v(\delta_+) - v(\cdot))](x) &= (v(\delta_+) - v(x))L[\varphi](x) - \varphi(x)L[v](x) - \epsilon^2 \varphi'(x)v'(x) \\ &= (v(\delta_+) - v(x))(L[\varphi](x) + \epsilon^2(\ln(v(\delta_+) - v(x)))'\varphi'(x) \end{aligned}$$

because we have that

$$\begin{aligned} L[v](x) &= \frac{\epsilon^2}{2} \exp(-U(x)/\epsilon^2)\partial(\exp(U(x)/\epsilon^2)\partial v) \\ &= \frac{\epsilon^2}{2} \exp(-U(x)/\epsilon^2)\partial \mathbf{1} \;=\; 0 \end{aligned}$$

Thus it appears that $\bar{X}$ is just the diffusion starting from 0 and constructed as before, but with U replaced by the potential $\bar{U}$ defined by

$$\forall\, x \in]-\delta_-, \delta_+[, \qquad \bar{U}(x) \;=\; U(x) + 2\epsilon^2 \ln\left(\int_x^{\delta_+} \exp(-U(y)/\epsilon^2)\, dy \right) \tag{12}$$

since by putting together the above computations, we have shown that for $t \geq 0$,

$$\partial_t \bar{\mathbb{E}}[\varphi(\bar{X}_t)] \;=\; \bar{\mathbb{E}}[\bar{L}[\varphi](\bar{X}_t)]$$

with

$$\forall\, \varphi \in \mathcal{C}^2_{\mathrm{c}}(]-\delta_-, \delta_+[),\ \forall\, x \in]-\delta_-, \delta_+[, \qquad \bar{L}[\varphi](x) \;=\; \frac{1}{2}\left[\epsilon^2\varphi''(x) + \bar{U}'(x)\varphi'(x)\right]$$

(note furthermore that $\bar{X}$ is still absorbed in $-\delta_-$ and δ_+, the latter not being reached by construction).

By the above mentioned approximation procedure, one can extend this property to any continuous mapping U on $[-\delta_-, \delta_+]$. But in fact this representation of $\bar{X}$ is also true for general measurable and bounded potential U. To show it, one has to find a sequence $(U_n)_{n\geq 0}$ of bounded continuous potentials which is approximating U in a weaker sense. More precisely, taking into account the theorem of Dini dealing with uniform convergence for increasing functions and some well-known properties of Brownian motion (insuring that a.s., $T_{v_{U_n}(-\delta_-)}(W) \to T_{v_U(-\delta_-)}(W)$ if $v_{U_n} \to v_U$), one just need such a family for which v_{U_n} and A_{U_n} converge respectively to v_U and A_U (a.s. with respect to W) for large n. Indeed, it is sufficient that U_n converges to U for large n in the $\mathbb{L}^1([-\delta_-, \delta_+])$-meaning and that $(U_n)_{n\geq 0}$ is uniformly bounded (thus for instance it is enough to regularize U by convolution). To convince yourself, remark that in (9), due to the presence of the hitting times and using the family $(l_{x,t}(W))_{x\in\mathbb{R},\, t\geq 0}$ of the local times of W (in position $x \in \mathbb{R}$ and up to time $t \geq 0$), we could have rather considered

$$\begin{aligned} A_U \,:\, \mathbb{R}_+ \ni t \quad &\mapsto \quad \int_{v_U(-\delta_-)}^{v_U(\delta_+)} n_U(x)\, l_{x,t}(W)\, dx \\ &= \quad \int_{-\delta_-}^{\delta_+} m_U(x) l_{v_U(x),t}(W) v_U'(x)\, dx \\ &= \quad \epsilon^{-2} \int_{-\delta_-}^{\delta_+} \exp(U(x)/\epsilon^2) l_{v_U(x),t}(W)\, dx \end{aligned}$$

(this mapping is different from the previous one only for the non important times $t \geq T_{v(-\delta_-)}(W) \wedge T_{v(\delta_+)}(W)$), where it appears that we can take advantage in one hand of the $\mathbb{L}^1$-convergence of $\exp(U_n(x)/\epsilon^2)$ to $\exp(U(x)/\epsilon^2)$ (which is deduced immediately by Lipschitzianity from that of U_n to U) and on the other hand of the (a.s.) convergence of $l_{v_{U_n}(x),t}(W)$ to $l_{v_U(x),t}(W)$ for large n. We leave the standard details to the reader.

But it seems a fortiori that the shortest way to prove the above conditioning property is first to do it on the Brownian motion with respect to the event $\{T_{v(-\delta_-)}(W) < T_{v(\delta_+)}(W)\}$, to write the diffusion thus obtain in the form of (9) and then to compose it with the formula giving $\bar{X}$ (or X) to deduce the result.

3 Behavior of large exit times

The large deviation principle we will present here is also known (cf [6] and [14]), but it seems to us that the technicality of its proof is not corresponding to its simplicity, so we will show this result in a direct and naive way, in the specific one-dimensional case. Heuristically its signification is quite obvious and just expresses that for a fixed diffusion constructed as in the previous section, the law of the reaching time of the boundary is close to an exponential distribution for its large enough values. This fact is very natural since for such remote occurence, the process will have had plenty of time to forget its initial condition, phenomenon which is well-known to imply the exponential caracter of the exit time (this property is sometimes called unpredictability, see for instance [16]). The link with analytical considerations is that the parameter of the approximating exponential law is given by the opposite of the largest eigenvalue of the associated diffusion and Schrödinger operators with Dirichlet condition.

More precisely, let us fix some notations, which will be slightly different from those of the above section, in order to facilitate further reference. First the parameter ϵ will play no role here (the result is not of the type of those of Freidlin and Wentzell [11] nor Mathieu [16], where the asymptotics considered are when ϵ is going to zero) and our forthcoming applications will take place only for the special value $\epsilon = 1$, thus we will stick to this situation. For a reason which will also be clearer in next sections, we prefer to take $-k$ and k for boundaries, where $k > 0$ is fixed, instead of $-\delta_-$ and δ_+. And at least to begin with, assume that the potential $U : [-k, k] \to \mathbb{R}$ is bounded. As in the previous section, this leads us to define the functions

$$\begin{array}{rcl} V \,:\, [-k,k] \ni y & \mapsto & \int_0^y \exp(-U(z))\, dz \\ M \,:\, [-k,k] \ni y & \mapsto & \exp(2U(y)) \\ N \,:\, [V(-k),V(k)] \ni y & \mapsto & M(V^{-1}(y)) \end{array}$$

These ingredients enable us to construct as before a Markovian process Z, starting from 0, absorbed in $-k$ and k and whose evolution inside $]-k,k[$ is heuristically described by the sde $dZ_t = dB_t + \frac{1}{2}U'(Z_t)\, dt$, where B is a standard Brownian motion. Sometimes it will be useful to make Z start from another point $x \in [-k, k]$ and then we will have resort to the traditional notation $\mathbb{E}_x$ to indicate that we are considering expectations with respect to this process Z. It will also be convenient to believe that U is a function defined on the whole real line, feature which can be easily obtained for instance by extending it in the following manner

$$\forall\, x \in \mathbb{R}, \qquad U(x) \;=\; U(-k \vee x \wedge k)$$

or in another smoother way if some regularity is required.

Thus Z can also be constructed in the totallity of $\mathbb{R}$ (or any more convenient interval containing $[-k, k]$) and this permits to consider simultaneously for all $l > 0$, the hitting times

$$S_l \overset{\text{def.}}{=} \inf\{t \geq 0 \,:\, |Z_t| \geq l\}$$

(but our main interest will be on S_k, for Z starting from 0).

Besides, still for $l > 0$, let us introduce the constant

$$\lambda_l \overset{\text{def.}}{=} \inf_{f \in \mathcal{C}_c^1(]-l,l[)} \frac{\mu_l((f')^2)}{\mu_l(f^2)}$$

where μ_l be the probability on $[-l, l]$ whose density with respect to the Lebesgue measure is proportional to $\exp(U)$.

We will turn to spectral interpretations of this quantity latter on. For the moment it just enables us to state the result we were mentioning at the beginning:

Theorem 3.1 *Assume that U is continuous on $[-k, k]$, then for any Borelian subset A of $[0, +\infty[$, we have*

$$-\lambda_k \inf(\underline{A}) \;\leq\; \liminf_{M\to+\infty} \frac{1}{M} \ln(\mathbb{P}[\frac{S_k}{M} \in A]) \;\leq\; \limsup_{M\to+\infty} \frac{1}{M} \ln(\mathbb{P}[\frac{S_k}{M} \in A]) \;\leq\; -\lambda_k \inf(\overline{A})$$

If U was to be $\mathcal{C}^\infty$ on $[-k, k]$, this result already exists in the literature, but we will give a shortest proof in the special 1-dimensional case. Latter on we will also discuss about a multidimensional analogue and give the extension to "unilateral" exit times, which can be seen as "bilateral" ones corresponding to some unbounded potential U.

The method we present consists in two parts: the first one shows the existence of a related limit and the second one identifies it.

So here is the first step, which is true for general measurable and bounded potential U.

A priori, if S_k was really distributed as an exponential variable for some parameter $\lambda_k > 0$, then we could conclude immediately that the theorem 3.1 is true. But strictly speaking, this is false, look for example at the special case of a null potential, ie at the Brownian situation. Then the non-negative Laplace transform of the law of S_k is explicitly known and is given by

$$\forall\ 0 \leq l < 2(\pi/k)^2, \qquad \mathbb{E}[\exp(lS_k)] \;=\; \frac{1}{\cos(\sqrt{2l}k)}$$

(see for instance [18] and the references given there), which is different from that of an expected exponential distribution. This fine knowledge, and especially the behavior near the pole $2(\pi/k)^2$ of the meromorphe extension of the above mapping in $l \in \mathbb{C}$, can be used to deduce the asymptotic of $\mathbb{P}[S_k \geq M]$ for large M (cf the classical reference [9]). Nevertheless, even this approach is not so trivial, but remark that the below alternative description (15) of the constant λ_k appearing in theorem 3.1 enables to see that in this particular case its value is $2(\pi/k)^2$.

Thus a first thought is to use the manipulations of the previous section to rewrite S_k in terms of a Brownian motion W and doing so we get the formula

$$S_k \;=\; \int_0^{T_{V(-k)}(W)\wedge T_{V(k)}(W)} N(W_t)\,dt \tag{13}$$

which does not seem very helpful, except that it give a priori lower and upper bounds:

$$\min_{V(-k)\leq x\leq V(k)} N(x)(T_{V(-k)}(B)\wedge T_{V(k)}(B)) \;\leq\; S_k \;\leq\; \max_{V(-k)\leq x\leq V(k)} N(x)(T_{V(-k)}(B)\wedge T_{V(k)}(B)) \tag{14}$$

A second idea, when one look at the above large deviation result, is that it should be a consequence of the well-known theorem of Ellis and Gärtner (cf for instance [7]). Indeed, if we define

$$\bar{\lambda}_k \;\stackrel{\text{def.}}{=}\; \inf\{\lambda \geq 0\ :\ \mathbb{E}[\exp(\lambda S_k)] = +\infty\} \tag{15}$$

number which satisfies $0 < \bar{\lambda}_k < +\infty$ due to (14), then we get immediately the upper bound

$$\forall\ A \in \mathcal{B}(\mathbb{R}_+), \qquad \limsup_{M\to+\infty} \frac{1}{M}\ln(\mathbb{P}[\frac{S_k}{M} \in A]) \;\leq\; -\bar{\lambda}_k \inf(\overline{A})$$

Unfortunately this theorem is not applicable for the lower bound (this is even more frustrating as a posteriori it gives the good result!), because as in the simple case of an exponential variable, the needed hypothesis of strict convexity of the rate function is not verified. Thus we will have to work directly.

We begin by observing that to obtain the corresponding lower bounds, thanks to the order structure and particular topology of $\mathbb{R}_+$, it is sufficient to prove that

$$\liminf_{M\to+\infty} \frac{1}{M}\ln(\mathbb{P}[S_k > M]) \;\geq\; -\bar{\lambda}_k$$

Indeed, an equality is taking place here with a true limit and to go into this direction, let us show a qualitative result:

Lemma 3.2 *There exists a constant* $0 < \widetilde{\lambda}_k < +\infty$ *such that*

$$\lim_{M\to+\infty} \frac{1}{M}\ln(\mathbb{P}[S_k > M]) \;=\; -\widetilde{\lambda}_k$$

Proof:

We begin by noting that starting from 0 was not a good idea for the process Z to get out fast from $]-k,k[$: more rigorously, there exist a constant $c \geq 1$ such that

$$\forall\ x \in [-k,k],\ \forall\ M>0, \qquad \mathbb{P}_x[S_k > M] \ \leq\ c\mathbb{P}_0[S_k > M]$$

This comes from the strong Markov property, since for any fixed $x \in [-k,k]$,

$$\begin{array}{rcl}
\mathbb{E}_0[S_k > M] & \geq & \mathbb{P}_0[T_x(Z) < S_k,\, S_k > M] \\
& = & \mathbb{E}_0[\mathbb{1}_{T_x(Z)<S_k}\mathbb{P}_x[S_k(\widetilde{Z}) > M - T_x(Z)]] \\
& \geq & \mathbb{E}_0[\mathbb{1}_{T_x(Z)<S_k}\mathbb{P}_x[S_k > M]] \\
& = & \mathbb{P}_0[T_x(Z) < T_{-\mathrm{sign}(x)k}(Z)]\mathbb{P}_x[S_k > M] \\
& = & \dfrac{V(-\mathrm{sign}(x)k) - V(0)}{V(-\mathrm{sign}(x)k) - V(x)}\mathbb{P}_x[S_k > M] \\
& \geq & \dfrac{(V(k) - V(0)) \wedge (V(0) - V(-k))}{V(k) - V(-k)}\mathbb{P}_x[S_k > M]
\end{array}$$

where in the second line, the expectation $\mathbb{E}_x$ was only with respect to the process $\widetilde{Z}$, again evolving as Z but starting from x. So we can take

$$c \ = \ \frac{V(k) - V(-k)}{(V(k) - V(0)) \wedge (V(0) - V(-k))}$$

Now for any given $M_1, M_2 > 0$, we get by using the usual Markov property,

$$\begin{array}{rcl}
\mathbb{P}_0[S_k > M_1 + M_2] & = & \mathbb{E}_0[\mathbb{1}_{S_k > M_1}\mathbb{P}_{Z(M_1)}[S_k > M_2]] \\
& \leq & c\mathbb{E}_0[\mathbb{1}_{S_k > M_1}\mathbb{P}_0[S_k > M_2]] \\
& = & c\mathbb{P}_0[S_k > M_1]\mathbb{P}_0[S_k > M_2]
\end{array}$$

It follows that the quantity $c\mathbb{P}[S_k > M]$ is submultiplicative in $M > 0$, so by virtue of the subadditivity theorem, we obtain the existence of

$$\lim_{M\to+\infty} \frac{1}{M}\ln(c\mathbb{P}[S_k > M]) \ = \ \inf_{M>0} \frac{1}{M}\ln(c\mathbb{P}[S_k > M])$$

and by consequence the convergence presented in the lemma. The finitness and positivity of the opposite of the limit can then be deduced from (14).

■

We have now to verify that $\bar{\lambda}_k$ and $\widetilde{\lambda}_k$ are equal, but this is just direct computations from the above lemma which is easily seen to imply, for any $0 < \eta < \widetilde{\lambda}_k$,

$$\begin{array}{rcl}
\mathbb{E}[\exp((\widetilde{\lambda}_k - \eta)S_k)] & < & +\infty \\
\mathbb{E}[\exp((\widetilde{\lambda}_k + \eta)S_k)] & = & +\infty
\end{array}$$

One can even go further by using the fact that in the subadditivity theorem the limit is also the infimum, because this proves that

$$\mathbb{E}[\exp(\widetilde{\lambda}_k S_k)] \ = \ +\infty$$

It is time to come to our second step consisting in another formulation of $\bar{\lambda}_k$. Even if the result is valid for continuous potential U on $[-k,k]$ (and hopefully for measurable and bounded ones), in a traditional way, we will first work under the extra hypothesis that U is of class $\mathcal{C}^2$ on $[-k,k]$ and only latter on will we use some continuity properties of S_k with respect to the potential to extend the results obtained.

Proposition 3.3 *At least under the above regularity assumption, the two constants λ_k and $\bar{\lambda}_k$ coincide.*

Proof:

The shortest way to prove this equality is to admit the analytic fact that there exists a function $\varphi \in \mathcal{C}^2(]-k,k[)$ such that $\varphi > 0$ (Perron-Frobenius property), $\lim_{x\to -k_+}\varphi(x) = 0 = \lim_{x\to k_-}\varphi(x)$, and verifying

$$\frac{1}{2}(\varphi'' + U'\varphi') \quad = \quad -\lambda_k \varphi \tag{16}$$

thanks to a variationnal treatment of the definition of λ_k.

There we have taken into account the regularity of the potential U. This property also shows that in the latter definition of S_k, the process Z can be replaced by the solution of the sde on the whole real line

$$\begin{cases} Z_0 &= \ 0 \\ dZ_t &= \ dB_t + \frac{1}{2}U'(Z_t)\,dt \end{cases}$$

where B is a standard Brownian motion.

Thus it appears that the process $(\varphi(Z_{t\wedge S_k})\exp(\lambda_k(t\wedge S_k)))_{t\geq 0}$ is a martingale. But considering $0<\eta<k$, it follows that for any $t\geq 0$,

$$\mathbb{E}[\varphi(Z_{t\wedge S_{k-\eta}})\exp(\lambda_k(t\wedge S_{k-\eta}))] \quad = \quad \varphi(0)$$

Then if we let t growing to infinity, we obtain

$$[\varphi(-k+\eta)\vee\varphi(k-\eta)]\mathbb{E}[\exp(\lambda_k S_{k-\eta})] \quad \geq \quad \mathbb{E}[\varphi(Z_{S_{k-\eta}})\exp(\lambda_k S_{k-\eta})] \quad = \quad \varphi(0)$$

and next passing to the limit as η goes to zero, there is no doubt that $\mathbb{E}[\exp(\lambda_k S_k)] = +\infty$, which means that $\lambda_k \geq \bar{\lambda}_k$.

To see the reciproque, note that the above considerations also imply that $\mathbb{E}[\exp(\lambda_k S_{k-\eta})] < +\infty$ (because $\varphi(-k+\eta)\wedge\varphi(k-\eta) > 0$, note that this could not have been true if we had considered for φ an eigenvector associated to another eigenvalue), which can be rewritten as

$$\forall\ k,\eta > 0, \qquad \mathbb{E}[\exp(\lambda_{k+\eta}S_k)] \quad < \quad +\infty$$

ie $\lambda_{k+\eta} \leq \bar{\lambda}_k$.

Meanwhile, by replacing μ_k by the nonnormalized measure of density $\exp(U)$ on $[-k,k]$ in the definition of $\widetilde{\lambda}_k$, it is quite clear that $\mathbb{R}_+^* \ni l \mapsto \lambda_l$ is nonincreasing and càg. But it is not difficult to be convinced that it is indeed continuous (for instance by a dilatation of space, one can bring all the problems (for different $l>0$) on a fixed interval, say the segment $[-1,1]$, and then the result follows from the uniform convergence of the corresponding potentials). Thus letting η going to 0 in the above bound, we can conclude that $\lambda_k \leq \bar{\lambda}_k$ and so to the validity of the proposition.

■

Remarks 3.4: It could seem that we have lied in the introduction by promising not to resort to analytic methods, in the implicit sense of not taking advantage of thus obtained properties (existence, unicity, regularity ...) of solutions of ordinary or partial differential equations. But the previous spectral caracterisation of $\bar{\lambda}_k$ will not be really needed for (3), and consequently for the large deviations of $T^{(\epsilon,k,+)}$ and $T^{(\epsilon,k,-)}$ in the strict anti-symmetrical situation; in that case it is just useful to make a link with the work of Gradinaru, Herrmann and Roynette [13]. Nevertheless, let us mention informally that more probabilist-oriented approachs are possible for the above bound $\bar{\lambda}_k \geq \lambda_k$, this will also show how it is related to some close subjects.

a) We could have used another large deviation theorem by a customary change of level: for any fixed $\eta > 0$, the distribution of S_k is not modified if we consider in its definition the process Z issued from 0 and whose evolution inside $]-k-\eta, k+\eta[$ is heuristically described by the sde $dZ_t = dB_t + \frac{1}{2}U'(Z_t)\,dt$, where B is a standard Brownian motion, but which is reflected on the boundary $\{-k-\eta, k+\eta\}$ (Neumann boundary condition). This process has the advantage to be strongly ergodic, with $\mu_{k+\eta}$ as reversible measure, so it is well-known that a principle of large deviations holds for its empirical measures

$$\xi_t \quad \stackrel{\text{def.}}{=} \quad \frac{1}{t}\int_0^t \delta_{Z_s}\,ds$$

for large $t > 0$ (see for instance the theorem 4.2.58 of [8]). Its upper bound says that for all closed subset $\mathbf{A}$ of the set of probabilities on $[-k-\eta, k+\eta]$, endowed with the weak topology, we have

$$\limsup_{t\to+\infty} \frac{1}{t}\ln(\mathbb{P}[\xi_t \in \mathbf{A}]) \quad \leq \quad -\inf_{m\in\mathbf{A}} \Lambda(m)$$

with the action functional Λ given on any probability m on $[-k-\eta, k+\eta]$ by

$$\Lambda(m) \quad = \quad \begin{cases} \mu_{k+\eta}((f')^2) & \text{, if } m \ll \mu_{k+\eta} \text{ and } f \stackrel{\text{def.}}{=} \sqrt{dm/d\mu_{k+\eta}} \text{ is absolutely continuous} \\ +\infty & \text{, otherwise} \end{cases}$$

where f' will always stand for the weak derivative of an absolutely continuous function f.

Meanwhile, taking into account some famous particularities of the trajectories of diffusions such as Z (which can be deduced from those of a Brownian motion, either via an available Girsanov formula because of the regularity of U, or more generally through the transformations of the previous section), we are assured of the a.s. equality between the sets

$$\{S_k > M\} \quad = \quad \{\xi_M([-k-\eta, -k] \sqcup [k, k+\eta]) = 0\}$$

for any given $M > 0$.

Thus we get that

$$-\bar{\lambda}_k \quad \leq \quad -\inf_{f\in\mathcal{A}_{k+\eta}} \mu_{k+\eta}((f')^2)$$

where

$$\begin{aligned} \mathcal{A}_{k+\eta} \quad &= \quad \{f \in \bar{\mathcal{H}}_{k+\eta} \,:\, f = 0 \text{ on } [-k-\eta, -k] \sqcup [k, k+\eta]\} \\ \bar{\mathcal{H}}_{k+\eta} \quad &= \quad \{f \in \mathcal{C}([-k-\eta, k+\eta]) \,:\, f \text{ is absolutely continuous and } \int f^2\,d\mu_{k+\eta} = 1\} \end{aligned}$$

Let us also note the related set

$$\mathcal{H}_k \quad = \quad \{f \in \mathcal{C}([-k, k]) \,:\, f \text{ is absolutely continuous, } f(-k) = 0 = f(k)\}$$

Its interest is that it is not very difficult to show that

$$\begin{aligned}\lambda_k &= \inf_{f\in\mathcal{H}_k\setminus\{0\}} \frac{\mu_k((f')^2)}{\mu_k(f^2)} \\ &= \inf_{f\in\mathcal{A}_{k+\eta}} \mu_{k+\eta}((f')^2)\end{aligned}$$

from where we deduce the announced inequality $\bar{\lambda}_k \geq \lambda_k$.

Indeed, DeBlassie [6] used this approach to prove the reverse inequality, by coming back to the original article of Donsker and Varadhan [10], where they showed the lower bound for the large deviations of empirical probabilities under less restrictive topological hypotheses than for usual principles, in particular allowing considerations of the support of generical measures.

b) For another alternative proof, let us come back to a process Z evolving as before inside the interval $]-k,k[$, absorbed in its boundary, but whose starting point will be indicated by the subscript of the expectation. We begin by noting that if $\lambda > 0$ verifies $\mathbb{E}_0[\exp(\lambda S_k)] < +\infty$ (ie $\lambda < \lambda_k$), then furthermore the mapping

$$\varphi_\lambda\,:\,[-k,k]\ni x \;\mapsto\; \mathbb{E}_x[\exp(\lambda S_k)] \geq\; 1$$

is bounded. This comes again from the strong Markov property, since as in the proof of lemma 3.2, we can prove that

$$\forall\, x\in[-k,k],\qquad \mathbb{E}_x[\exp(\lambda S_k)] \leq \frac{V(k)-V(-k)}{(V(k)-V(0))\wedge(V(0)-V(-k))}\mathbb{E}_0[\exp(\lambda S_k)]$$

Similar computations, with merely the simple Markov property, also show that the process $\exp(\lambda t)$ $\varphi_\lambda(Z_t)$ is a submartingale. Thus using a few Itô's stochastic calculus, we get that

$$\forall\, t\geq 0,\,\forall\, x\in[-k,k],\qquad \lambda\exp(\lambda t)\varphi_\lambda(x)+\exp(\lambda t)L(\varphi_\lambda)(x)\;\geq\; 0$$

where L is the operator $(\partial^2+U'\partial)/2$, at least if we forget about the regularity conditions.

Next we multiply these inequalities by $(\varphi_\lambda(x)-1)\exp(-\lambda t)$ and integrate them with respect to $\mu_k(dx)$, to obtain,

$$\begin{aligned}\lambda(\mu_k((\varphi_\lambda-1)^2)+\mu_k(\varphi_\lambda-1)) &\geq -\mu_k((\varphi_\lambda-1)L(\varphi_\lambda)) \\ &= -\mu_k((\varphi_\lambda-1)L(\varphi_\lambda-1)) \\ &= \mu_k[((\varphi_\lambda-1)')^2]\end{aligned}$$

But recall that in the infimum of the definition of λ_k, we could have considered all absolutely continuous functions which are converging to zero near both boundaries and whose weak derivative is in $\mathbb{L}^2(\mu_k)$. This is the case of $\varphi_\lambda-1$, thus via a straightforward application of Cauchy-Schwartz inequality, we get that

$$\lambda\left(1+\frac{1}{\sqrt{\mu_k((\varphi_\lambda-1)^2)}}\right)\;\geq\;\lambda_k$$

and to conclude, note that if λ grows to λ_k, then

$$\begin{aligned}\mu_k((\varphi_\lambda-1)^2) &\underset{\lambda\to(\lambda_k)_-}{\longrightarrow} \int \mathbb{E}_x[\exp(\lambda_k S_k)-1]^2\,\mu_k(dx) \\ &= +\infty\end{aligned}$$

where the last equality comes from the fact that

$$\forall\, x \in]-k,k[, \qquad \mathbb{E}_x[\exp(\lambda_k S_k)] \;=\; +\infty$$

which itself can be deduced as before from the same property but only for $x = 0$.

These computations also suggest a posteriori that if the following convergences exist

$$\forall\, x \in]-k,k[, \qquad \varphi(x) \;=\; \lim_{\lambda\to(\lambda_k)_-} \frac{\varphi_\lambda(x)}{\varphi_\lambda(0)}$$

then the limiting object φ should be a minimizer in the definition of λ_k, ie an eigenvector as in (16).

Thus along the lines of the above development, it has appeared that $-\lambda_k$ is the largest eigenvalue of the autoadjoint extension $\bar{L}_k$ with Dirichlet boundary condition of the diffusion pregenerator $L_k = (\partial^2 + U'\partial)/2$ acting on $\mathcal{C}_c^2(]-k,k[)$. Alternatively, this operator $\bar{L}_k$ is associated to the Dirichlet form $(\mathcal{D}_k, \mathcal{E}_k)$, where

$$\begin{array}{rrcl} & \mathcal{D}_k &=& \{f \in \mathcal{H}_k \,:\, f' \in \mathbb{L}^2(\mu_k)\} \\ \forall\, f \in \mathcal{D}_k, & \mathcal{E}_k(f,f) &=& \mu_k((f')^2) \end{array}$$

But there is still another possible description of λ_k, through Schrödinger operators. It just corresponds to a change of reference measure, which should now rather be l_k, the Lebesgue measure on $[-k,k]$. More precisely, consider the isometric mapping

$$\begin{array}{rcl} g_k \,:\, \mathbb{L}^2(l_k) & \to & \mathbb{L}^2(\mu_k) \\ f & \mapsto & \exp(-U/2)f \end{array}$$

then by conjugacy, the operator L_k is transformed in

$$\forall\, f \in \mathcal{C}_c^2(]-k,k[), \qquad \widetilde{L}_k(f) \;=\; g_k^{-1} L_k(g_k f) \;=\; \frac{1}{2}\left(\partial^2 f - \left[\left(\frac{U'}{2}\right)^2 + \frac{U''}{2}\right] f\right)$$

Then considering the Schrödinger operator $\widehat{L}_k$ which is the autoadjoint extension of $\widetilde{L}_k$ corresponding to Dirichlet boundary condition, we see that its largest eigenvalue is also $-\lambda_k$. This can be rewritten in way of positive closed forms, or equivalently,

$$\lambda_k \;=\; \frac{1}{2} \inf_{f\in\mathcal{C}_c^1(]-k,k[)} \frac{l_k((f')^2 + [(U'/2)^2 + U''/2]f^2)}{l_k(f^2)}$$

Nevertheless, the Dirichlet formulation relative to $(\mathcal{D}_k, \mathcal{E}_k)$ seems the more interesting, since it will be the only one conserving a meaning for more general potential U.

Besides, let us now come to the end of the proof of theorem 3.1, by extending the above identification to the situation of continuous potentials U.

As one can guess, we will proceed by approximation: let $(U_n)_{n\geq 0}$ be a sequence of functions belonging to $\mathcal{C}^2(\mathbb{R})$, converging uniformly to a continuous extension of U on $[-k-1,k+1]$ for a fixed $k > 0$, and being uniformly continuous in $n \geq 0$. Such a family can be obtained by considering for instance regularisations by convolution and by using the uniform continuity of U on $[-k-1,k+1]$.

Then for any fixed small $0 < \eta < 1$, we can find a n_0 large enough such that for all $n \geq n_0$,

$$\begin{array}{rclcl} V(k-\eta) & \leq & V_n(k) & \leq & V(k+\eta) \\ V(-k-\eta) & \leq & V_n(-k) & \leq & V(-k+\eta) \\ (1-\eta)N(x) & \leq & N_n(x) & \leq & (1+\eta)N(x) \qquad \text{, for all } x \in [-k-1,k+1] \end{array}$$

(the n in subscript indicates that the corresponding notion are relative to U_n).

Thus we get for such $n \geq n_0$ that

$$(1-\eta)S_{k-\eta} \;\leq\; S_{k,n} \;\leq\; (1+\eta)S_{k+\eta}$$

from where it follows that

$$(1+\eta)^{-1}\bar{\lambda}_{k+\eta} \;\leq\; \liminf_{n\to+\infty} \bar{\lambda}_{k,n} \;\leq\; \limsup_{n\to+\infty} \bar{\lambda}_{k,n} \;\leq\; (1-\eta)^{-1}\bar{\lambda}_{k-\eta}$$

Next we are allowed to let η go to zero to obtain

$$\bar{\lambda}_{k_+} \;\leq\; \liminf_{n\to\infty} \bar{\lambda}_{k,n} \;\leq\; \liminf_{n\to\infty} \bar{\lambda}_{k,n} \;\leq\; \bar{\lambda}_{k_-}$$

(there is no doubt that $\bar{\lambda}_{k_+} \stackrel{\text{def.}}{=} \lim_{\eta\to 0_+} \bar{\lambda}_{k+\eta}$ and $\bar{\lambda}_{k_-} \stackrel{\text{def.}}{=} \lim_{\eta\to 0_+} \bar{\lambda}_{k-\eta}$ exist, since the quantity $\bar{\lambda}_k$ is clearly decreasing in $k > 0$).

On the other hand, the convergence of the $\lambda_{k,n}$ for large n is more direct and will be used below to finish the demonstration. Indeed, for any $n \geq 0$ and any function $f \in \mathcal{C}^1_{\mathrm{c}}(]-k,k[)$, we are assured of

$$\exp(-2\,\|U_n - U\|)\frac{\int (f')^2\,d\mu_k}{\int f^2\,d\mu_k} \;\leq\; \frac{\int (f')^2\,d\mu_{k,n}}{\int f^2\,d\mu_{k,n}} \;\leq\; \exp(2\,\|U_n - U\|)\frac{\int (f')^2\,d\mu_k}{\int f^2\,d\mu_k}$$

which implies at once the wanted convergence:

$$\lim_{n\to\infty} \lambda_{k,n} \;=\; \lambda_k$$

So using the identities $\bar{\lambda}_{k,n} = \lambda_{k,n}$, for $n \geq 0$, we deduce that

$$\bar{\lambda}_{k_+} \;\leq\; \lambda_k \;\leq\; \bar{\lambda}_{k_-}$$

But as before, the uniform continuity of U on say $[-k-1,k+1]$, shows that the mapping $]-k-1,k+1[\ni l \mapsto \lambda_l$ is continuous and taking into account that $]-k-1,k+1[\ni l \mapsto \bar{\lambda}_l$ has at most a denumerable number of discontinuities, it appears finally that

$$\lambda_k \;=\; \bar{\lambda}_k \;=\; \bar{\lambda}_{k_+} \;=\; \bar{\lambda}_{k_-}$$

Remark 3.5: Quite surprinsingly, the continuity of $]-k-1,k+1[\ni l \mapsto \bar{\lambda}_l$, or equivalently $]-k-1,k+1[\ni l \mapsto \widetilde{\lambda}_l$, is not easy to obtain directly. Resorting again to the infimum property of the subadditivity theorem occuring in the definition of $\widetilde{\lambda}_k$, one can indeed show that the latter mapping is càg. One of the main advantage of the introduction of the alternative description $]-k-1,k+1[\ni l \mapsto \lambda_l$ is to permit to go round this difficulty, even if it is not entirely satisfactory from a probabilistic point of view.

Besides in view of the previous results, one cannot keep from presenting the following statement:

Conjecture 3.6: Let Ω be a bounded smooth open and connected subset of a Riemannian manifold of dimension $d \in \mathbb{N}^*$. We denote by l_Ω the restriction of the Riemannian measure to this set. Let also $U : \Omega \to \mathbb{R}_+$ be a locally bounded potential which is assumed to be bounded in the neighborhood of at least one point in the frontier $\bar{\Omega} \setminus \Omega$ (this condition shall permit to the below process to get out of Ω). One can define a Dirichlet form $\mathcal{E}$ on the subdomain of $\mathbb{L}^2(\exp(-U)l_\Omega)$ constitued of the functions

f converging to zero near the boundary and admitting a weak gradient ∇f such that the next quantity is finite

$$\mathcal{E}(f,f) \stackrel{\text{def.}}{=} \int |\nabla f|^2 \exp(-U) dl_\Omega$$

By the general theory of Dirichlet forms [12], one can associate to this object a boundary absorbed diffusion $(X_t, \mathbb{P}_x)_{t\geq 0,\, x\in\bar{\Omega}}$ in the sense of Hunt processes.

Let $S = \inf\{t \geq 0 \,:\, X_t \in \partial\Omega\}$, then we believe that a large deviation principle similar to that of theorem 3.1 is taking place, with $-\lambda_k$ replaced by the largest eigenvalue of the autoadjoint operator associated to $\mathcal{E}$.

As we pointed it out at the beginning of this section, this result is known if U belongs to $\mathcal{C}^\infty(\bar{\Omega})$: DeBlassie [5], [6] and Kenig and Pipher [14] even do much better by giving expansions to any order of the probability $\mathbb{P}_x[S > M]$ for large M and fixed $x \in \Omega$, in terms of the eigenvalues and eigenvectors associated to the above operator with Dirichlet boundary condition. These authors are especially interested in the so-called h-conditioning of Brownian motion, which corresponds to the particular cases where U verifies $\triangle U + |\nabla U|^2/2 = 0$ and then the harmonic function h is just $\exp(U/2)$, but as they remarked, their computations can be generalized to diffusions with smooth coefficients, at least in Euclidian spaces $\mathbb{R}^d$, for $d \geq 1$. Even if their approachs are not entirely analytical, we really do believe that one should be able to go further in the probabilistic understandings of this phenomenon. Nevertheless proving or finding a counter-example to the above extension is out of the limited scope of this article. Note that even in the simpler context of the real line, we have not proved it entirely, since the measurable or the unbounded cases still resist, except for the special situation presented below.

So to finish, we look at exit times from a specified side, ie

$$T_{\pm k} \stackrel{\text{def.}}{=} \begin{cases} S_k & \text{, if } X_{S_k} = \pm k \\ +\infty & \text{, otherwise} \end{cases}$$

Indeed, these random variables will satisfy for their huge values exactly the same large deviation principle as that of S_k. This feature may seem strange at first view: why should an a priori non symmetrical process make no difference between its boundaries? In fact it does differentiate them, but only for relatively small values of the exit time (or at the level of precise large deviations for the huge ones, when one is also interested in the problem of evaluating the factor in front of the dominating exponential). At least this phenomenon is compatible with our heuristic remark about unpredictability, at the beginning of this section.

Thus our last objective here is to show:

Proposition 3.7 *Always under the assumption of continuity for U on $[-k,k]$, for any Borelian subset A of $[0,+\infty[$, we have*

$$-\lambda_k \inf(\underline{A}) \;\leq\; \liminf_{M\to+\infty} \frac{1}{M} \ln(\mathbb{P}[\frac{T_{\pm k}}{M} \in A]) \;\leq\; \limsup_{M\to+\infty} \frac{1}{M} \ln(\mathbb{P}[\frac{T_{\pm k}}{M} \in A]) \;\leq\; -\lambda_k \inf(\overline{A})$$

By symmetry of the formulation, it is sufficient to study the case of T_{-k} and in fact to show that

$$\lim_{M\to+\infty} \frac{1}{M} \ln(\mathbb{P}[+\infty > T_{-k} > M]) \;=\; -\lambda_k$$

Taking into account the trivial inequality

$$\mathbb{P}[+\infty > T_{-k} > M] \;\leq\; \mathbb{P}[S_k > M]$$

valid for any $M \geq 0$, we just have to be convinced that

$$\liminf_{M\to+\infty} \frac{1}{M} \ln(\mathbb{P}[+\infty > T_{-k} > M]) \quad \geq \quad -\lambda_k$$

To obtain this bound, it is natural to use a conditioning by the event $\{T_{-k} < T_k\}$ and according to the previous section, if we denote by an superscript † the notions relative to the unbounded potential

$$\forall\, x \in [-k, k[, \qquad U^\dagger(x) \quad = \quad U(x) + 2\ln\left(\int_x^k \exp(-U(y))\, dy\right)$$

we can write

$$\begin{aligned}
\mathbb{P}[+\infty > T_{-k} > M] &= \mathbb{P}[T_{-k} < T_k]\mathbb{P}[+\infty > T^\dagger_{-k} > M] \\
&= \mathbb{P}[T_{-k} < T_k]\mathbb{P}[S^\dagger_k > M] \\
&\geq \mathbb{P}[T_{-k} < T_k]\mathbb{P}[S^\dagger_{k-\eta} > M]
\end{aligned}$$

for any small $0 < \eta < k$.

But for $S^\dagger_{k-\eta}$ we can take advantage of the theorem 3.1, because $U^\dagger$ is continuous on $[-k+\eta, k-\eta]$, and we get

$$\lim_{M\to+\infty} \frac{1}{M} \ln(\mathbb{P}[S^\dagger_{k-\eta} > M]) \quad = \quad -\lambda^\dagger_{k-\eta}$$

If we show that indeed

$$\forall\, 0 < \eta < k, \qquad \lambda^\dagger_{k-\eta} \quad = \quad \lambda_{k-\eta} \tag{17}$$

then the above proposition will follow as usually by the convergence of $\lambda_{k-\eta}$ toward λ_k for small $\eta > 0$.

Also in a customary manner, we begin by treating the case of a potential U of class $\mathcal{C}^2$. Elementary computations show that in this situation,

$$\forall\, x \in [-k+\eta, k-\eta], \qquad \left(\frac{\partial U(x)}{2}\right)^2 + \frac{\partial^2 U(x)}{2} \quad = \quad \left(\frac{\partial U^\dagger(x)}{2}\right)^2 + \frac{\partial^2 U^\dagger(x)}{2}$$

This means that the spectral Schrödinger interpretations are the same for $\lambda^\dagger_{k-\eta}$ and $\lambda_{k-\eta}$, from where we deduce (17) in regular cases.

Now it remains to notice that the identity

$$\inf_{f\in\mathcal{C}^1_c(]-k+\eta,k-\eta[)} \frac{\int (f')^2 \exp(-U^\dagger) dl_{k-\eta}}{\int f^2 \exp(-U^\dagger) dl_{k-\eta}} \quad = \quad \inf_{f\in\mathcal{C}^1_c(]-k+\eta,k-\eta[)} \frac{\int (f')^2 \exp(-U) dl_{k-\eta}}{\int f^2 \exp(-U) dl_{k-\eta}}$$

is easily extended by an approximation procedure, from the $\mathcal{C}^2$-regular situation to any continuous potential.

4 A study case

We will treat here the large deviations as $\epsilon \to 0_+$ and for fixed $k > 0$, of the random variable $T^{(\epsilon,k)}$ seen in the introduction, in the almost symmetrical degenerate (in the sense that (1) admits several solutions) power potential cases expressed by (6), because its simplicity underlines the main ideas for the general situation.

Except for $T^{(\epsilon,k)}, T^{(\epsilon,k,+)}$ and $T^{(\epsilon,k,-)}$ with $\epsilon, k > 0$, we will use the notations introduced in the two previous sections. Thus, $\epsilon, \delta_-, \delta_+ > 0$ being fixed, we will be principaly interested in the diffusion X starting from 0, absorbed in $-\delta_-$ and δ_+ and constructed from the continuous potential

$$\forall\, x \in \mathbb{R}, \qquad U(x) \;=\; \begin{cases} a_+ x^\gamma & , \text{ if } x \geq 0 \\ a_- \left|x\right|^\gamma & , \text{ if } x < 0 \end{cases} \tag{18}$$

where $a_-, a_+ > 0$ and $0 < \gamma < 2$ are given.

Indeed, we will assume that δ_- and δ_+ depend on ϵ and more precisely that their (for the moment common) value is $k\epsilon^{2/\gamma}$, for any given $k > 0$. By their definition, it is quite clear that $T^{(\epsilon,k,-)}$, $T^{(\epsilon,k,+)}$ and $T^{(\epsilon,k)}$ coincide respectively with $T_{-\delta_-}(X)$, $T_{\delta_+}(X)$ and $T_{-\delta_-}(X) \wedge T_{\delta_+}(X)$.

As one can imagine, we will prove our result by rewriting everything in terms of the standard Brownian motion W appearing in (9). For instance, we note at once that

$$T^{(\epsilon,k)} \;=\; A(T_{v(-\delta_-)}(W) \wedge T_{v(\delta_+)}(W))$$

In order to exploit this formulation, we begin by expliciting the dependence of v, m and n in $\epsilon > 0$. In that respect, the following functions (which are just respective extensions of v_1, m_1 and n_1) are quite natural;

$$\begin{aligned} V \,:\, \mathbb{R} \ni y &\;\mapsto\; \int_0^y \exp(-U(z))\, dz \\ M \,:\, \mathbb{R} \ni y &\;\mapsto\; \exp(2U(y)) \\ N \,:\, \mathbb{R} \ni y &\;\mapsto\; M(V^{-1}(y)) \end{aligned}$$

because we are assured of

Lemma 4.1 *For any $\epsilon > 0$, we have for $-\delta_- \leq x \leq \delta_+$,*

$$\begin{aligned} v(x) &= \epsilon^{2/\gamma} V(\epsilon^{-2/\gamma} x) \\ m(x) &= \epsilon^{-2} M(\epsilon^{-2/\gamma} x) \\ n(x) &= \epsilon^{-2} N(\epsilon^{-2/\gamma} x) \end{aligned}$$

Proof:

This comes directly from the scaling property of U, for instance for v, one has for any $-\delta_- \leq x \leq \delta_+$,

$$\begin{aligned} v(x) &= \int_0^x \exp(-U(y)/\epsilon^2)\, dy \\ &= \int_0^x \exp(-U(y/\epsilon^{2/\gamma}))\, dy \\ &= \epsilon^{2/\gamma} \int_0^{\epsilon^{-2/\gamma} x} \exp(-U(y))\, dy \\ &= \epsilon^{2/\gamma} V(\epsilon^{-2/\gamma} x) \end{aligned}$$

■

These observations lead us to consider the new Brownian motion B defined by

$$\forall\, t \geq 0, \qquad B_t \;=\; \epsilon^{-2/\gamma} W_{\epsilon^{4/\gamma} t}$$

because we get

$$\begin{aligned} \epsilon^{-4/\gamma} T_{v(\delta_+)}(W) &= T_{\epsilon^{-2/\gamma} v(\delta_+)}(B) \\ &= T_{V(k)}(B) \end{aligned}$$

and similarly

$$\epsilon^{-4/\gamma} T_{v(-\delta_-)}(W) \;=\; T_{V(-k)}(B)$$

Thus it appears that

$$\begin{aligned} T^{(\epsilon,k)} &= \int_0^{T_{v(\delta_+)}(W) \wedge T_{v(-\delta_-)}(W)} n(W_t)\, dt \\ &= \epsilon^{-2} \int_0^{\epsilon^{4/\gamma}(T_{V(k)}(B) \wedge T_{V(-k)}(B))} N(B_{\epsilon^{-4/\gamma} t})\, dt \\ &= \epsilon^{4/\gamma - 2} S_k \end{aligned}$$

where the random variable S_k is independent of ϵ and is given by

$$S_k \;=\; \int_0^{T_{V(-k)}(B) \wedge T_{V(k)}(B)} N(B_t)\, dt$$

But as in the previous section, or by looking at the above computations with $\epsilon = 1$ in the other direction, we can reinterprete this quantity as the reaching time of the boundary for the process Z starting from 0, absorbed in $-k$ and k and whose evolution inside $]-k,k[$ is heuristically described by the sde $dZ_t = dB_t + \frac{1}{2} U'(Z_t)\, dt$, where B is a standard Brownian motion and U is our particular potential of interest.

The principle of large deviations (3) now follows directly from theorem 3.1 by just considering $\epsilon^{4/\gamma - 2}$ instead of M.

The convergence (4) is also quite clear with

$$\lambda_\infty \;\stackrel{\text{def.}}{=}\; \inf_{f \in \mathcal{C}_c^2(\mathbb{R})} \frac{\int (f'(x))^2 \, \exp(U(x)) dx}{\int f(x)^2 \, \exp(U(x)) dx}$$

because writing for $k > 0$,

$$\lambda_k \;=\; \inf_{f \in \mathcal{C}_c^2(]-k,k[)} \frac{\int (f'(x))^2 \, \exp(U(x)) dx}{\int f(x)^2 \, \exp(U(x)) dx}$$

we see that when we let k grow, via obvious identifications, we are simply increasing the set on which the infimum is taken and in the limit we obtain $\mathcal{C}_c^2(\mathbb{R}) = \cup_{k>0} \mathcal{C}_c^2(]-k,k[)$.

Returning to one of the spectral interpretations of latter section, we can look at λ_∞ as the smallest eigenvalue of the Schrödinger operator on the whole real line given by

$$\frac{1}{2}\left(-\partial^2 + a_{\mathrm{sign}(x)}^2 \frac{\gamma^2}{4} \left|x\right|^{2\gamma-2} + a_{\mathrm{sign}(x)} \frac{\gamma(\gamma-1)}{2} \left|x\right|^{\gamma-2} \right) \qquad , \text{ for } x \in \mathbb{R}$$

and this establishes (an outline of) the relation with the work of Gradinaru, Herrmann and Roynette [13], at least for $1 < \gamma < 2$.

But the interesting question is then to know if the constant $\lambda_\infty \geq 0$ is null or positive, and the answer is

Proposition 4.2 *In the previous setting, we have* $\lambda_\infty > 0 \iff \gamma \geq 1$.

Proof:

By analogy with λ_k, let us introduce another related notion for fixed $k, a > 0$,

$$\lambda_{k,a} \stackrel{\text{def.}}{=} \inf_{f \in \mathcal{C}_c^1([0,k[)} \frac{\int_0^k (f'(x))^2 \exp(ax^\gamma) dx}{\int_0^k f^2(x) \exp(ax^\gamma) dx}$$

Indeed, elementary manipulations enable one to make the constatation that

$$\lambda_{k,a_+} \wedge \lambda_{k,a_-} \leq \lambda_k \leq \lambda_{k,a_+} \vee \lambda_{k,a_-}$$

Thus we just have to see that for any fixed $a > 0$,

$$\forall\ 0 < \gamma < 2, \qquad \lim_{k \to +\infty} \lambda_{k,a} = \inf_{k>0} \lambda_{k,a} > 0 \iff \gamma \geq 1$$

But the Hardy's inequalities (cf for instance [17] or [2]) give explicit bounds for the above constants:

$$\frac{1}{4} L_{k,a}^{-1} \leq \lambda_{k,a} \leq L_{k,a}^{-1}$$

where

$$L_{k,a} = \sup_{0<x<k} \int_0^x \exp(-a(k-y)^\gamma)\, dy \int_x^k \exp(a(k-y)^\gamma)\, dy$$

The number which really matters for us is by consequence

$$\begin{aligned}
\sup_{k>0} L_{k,a} &= \sup_{k>0} \sup_{0<x<k} \int_x^k \exp(-ay^\gamma)\, dy \int_0^x \exp(ay^\gamma)\, dy \\
&= \sup_{x>0} \int_x^{+\infty} \exp(-ay^\gamma)\, dy \int_0^x \exp(ay^\gamma)\, dy
\end{aligned}$$

and in the last rhs we recognize the quantity appearing when one is wondering about the existence of a positive spectral gap for the classical reversible diffusion associated to the probability

$$\forall\ x \in \mathbb{R}, \qquad \mu(dx) = \left(\int_{\mathbb{R}} \exp(-a\,|y|^\gamma)\, dy \right)^{-1} \exp(-a\,|x|^\gamma)\, dx$$

and it is well-known that the condition $\sup_{k>0} L_{k,a} < +\infty$ is satisfied if and only if $\gamma \geq 1$.

To recover rapidly this result, it is sufficient to write the previous quantity as

$$\sup_{x>0} x^2 \int_1^{+\infty} \exp(-ax^\gamma y^\gamma)\, dy \int_0^1 \exp(ax^\gamma y^\gamma)\, dy$$

and to use the Laplace method for integrals with a small parameter to get

$$\begin{aligned}
\exp(ax^\gamma) \int_1^{+\infty} \exp(-ax^\gamma y^\gamma)\, dy &\underset{x \to +\infty}{\sim} \frac{1}{a\gamma x^\gamma} \\
\exp(-ax^\gamma) \int_0^1 \exp(ax^\gamma y^\gamma)\, dy &\underset{x \to +\infty}{\sim} \frac{1}{a\gamma x^\gamma}
\end{aligned}$$

■

5 Generalizations

As we alluded to it in the introduction, what is really meaningful for the project we have in mind is the asymptotic behavior of $T^{(\epsilon,k,+)}$ and $T^{(\epsilon,k,-)}$, not that of $T^{(\epsilon,k)}$, because in some sense they correspond to the parametrization of the non-null solutions of (1). Thus our duty here will be to get large deviation results for them.

First we consider drifts of the closed form given by (6) and as before we take $\delta(\epsilon) = \epsilon^{2/\gamma}$ in the definition of our stopping times of interest.

Note there is a situation where we can conclude at once; the truely anti-symmetric case where $a_+ = a_- > 0$, since then we a priori know that $T^{(\epsilon,k,+)}$ and $T^{(\epsilon,k,-)}$ have the same distribution, thus we can deduce the wanted principle from the identities

$$\forall\ A \in \mathcal{B}(\mathbb{R}_+), \qquad \mathbb{P}[T^{(\epsilon,k,-)} \in A] = \mathbb{P}[T^{(\epsilon,k,+)} \in A] \quad = \quad \frac{1}{2}\mathbb{P}[T^{(\epsilon,k)} \in A]$$

Nevertheless, the slightly non-antisymmetrical case (6) with $a_+ \neq a_-$ is not much more difficult in view of the results of section 3. Indeed, with the notations introduced there, we get by definition

$$\begin{array}{rcl} T^{(\epsilon,k,+)} & = & \epsilon^{4/\gamma-2}T_k \\ T^{(\epsilon,k,-)} & = & \epsilon^{4/\gamma-2}T_{-k} \end{array} \tag{19}$$

and with the help of proposition 3.7 we can conclude as before.

An alternative and suggestive way, after noting that

$$\lim_{\epsilon\to 0_+} \epsilon^{2-4/\gamma} \ln(\mathbb{P}[T^{(\epsilon,k,-)} < T^{(\epsilon,k,+)}]) \quad = \quad 0$$

(due to (10)) is to use, for what is concerning $T^{(\epsilon,k,-)}$, a conditioning by $\{T^{(\epsilon,k,-)} < T^{(\epsilon,k,+)}\}$ and to work directly from (12).

Indeed, in case where U is given by (18), one has a kind of scaling property for $\bar{U}$:

$$\forall\ x \in]-k\epsilon^{2/\gamma}, k\epsilon^{2/\gamma}[, \qquad \frac{\bar{U}(x)}{\epsilon^2} \quad = \quad U(x/\epsilon^{2/\gamma}) + 2\ln\left(\epsilon^{2/\gamma}\int_{x/\epsilon^{2/\gamma}}^{k} \exp(-U(y))\,dy\right)$$

which via the induced dependence on ϵ of the corresponding $\bar{v}$, $\bar{m}$ and $\bar{n}$, and the consideration of the Brownian motion $\bar{B}$ defined by

$$\forall\ t \geq 0, \qquad \bar{B}_t \quad = \quad \epsilon^{2/\gamma}W_{\epsilon^{-4/\gamma}t}$$

(the renormalisation is different from that of the previous section), leads to formulae similar to (19), but for the respective conditioned variables.

Now we come to treat the situation where the drift of X_ϵ only satisfies

$$\begin{cases} b(x) & \underset{x\to 0_+}{\sim} \quad a_+\frac{\gamma}{2}x^{\gamma-1} \\ b(x) & \underset{x\to 0_-}{\sim} \quad -a_-\frac{\gamma}{2}\left|x\right|^{\gamma-1} \end{cases}$$

with $a_+, a_- > 0$ and $0 < \gamma < 2$.

Proposition 5.1 *In the above relaxed setting, $T^{(\epsilon,k,+)}$ and $T^{(\epsilon,k,-)}$ still satisfies the same large deviation principle as (3).*

Proof:

The basic idea is to use stochastic comparisons. So let us start with a general remark: let b_1 and b_2 be smooth vector fields on $\mathbb{R}$ with bounded derivatives and satisfying

$$\forall\, x \in \mathbb{R}, \qquad b_1(x) \;\leq\; b_2(x)$$

We denote for $i = 1, 2$ and fixed $\epsilon > 0$, by $X_{i,\epsilon}$ the strong solution of the sde

$$\left\{ \begin{array}{rcl} X_{i,\epsilon}(0) & = & 0 \\ dX_{i,\epsilon}(t) & = & \epsilon dB_t + b_i(X_{i,\epsilon}(t))\, dt \end{array} \right.$$

with the same Brownian motion B. Then we are assured of

$$\mathbb{P}[\forall\, t \geq 0,\; X_{1,\epsilon}(t) \leq X_{2,\epsilon}(t)] \;=\; 1$$

and this obvious statement (but which is asking a little effort to be proved) leads to the fact that for any $l \geq 0$,

$$\mathbb{P}[T_l(X_{1,\epsilon}) \geq T_l(X_{2,\epsilon})] \;=\; 1$$

Through an usual approximation procedure, this result can be extended to less regular vector fields. Without entering here into all the details of such a formulation, let us just mention the application to our problem.

For $0 < \eta < 1$, we consider the vector fields $b_{\eta,+}$ and $b_{\eta,-}$ given by

$$\forall\, x \in \mathbb{R}, \qquad b_{\eta,\pm}(x) \;=\; \mathrm{sign}(x)(1 \pm \mathrm{sign}(x)\eta)a_{\mathrm{sign}(x)}\frac{\gamma}{2}\,|x|^{\gamma-1}$$

and the associated diffusions $X_{\epsilon,\eta,\pm}$ (eg for $X_{\epsilon,\eta,+}$, we have just replaced a_+ and a_- respectively by $(1+\eta)a_+$ and $(1-\eta)a_-$). More generally, any relative notion will be indicated by a subscript $\eta, \pm$. For instance, for any fixed $k > 0$, we are assured for all $\epsilon > 0$ small enough, of

$$\mathbb{P}[T_{\eta,+}^{(\epsilon,k,+)} \leq T^{(\epsilon,k,+)} \leq T_{\eta,-}^{(\epsilon,k,+)}] \;=\; 1$$

and thus we get for any $M > 0$,

$$-\lambda_{k,\eta,-}M \;\leq\; \liminf_{\epsilon\to 0_+} \epsilon^{\frac{2(2-\gamma)}{\gamma}} \ln(\mathbb{P}[T^{(\epsilon,k,+)} \leq M]) \;\leq\; \limsup_{\epsilon\to 0_+} \epsilon^{\frac{2(2-\gamma)}{\gamma}} \ln(\mathbb{P}[T^{(\epsilon,k,+)} \leq M]) \;\leq\; -\lambda_{k,\eta,+}M$$

In a traditional manner, we now take into account the convergences

$$\begin{array}{rcl} \lim_{\eta\to 0_+} \lambda_{k,\eta,-} & = & \lambda_k \\ \lim_{\eta\to 0_+} \lambda_{k,\eta,+} & = & \lambda_k \end{array}$$

to deduce

$$\lim_{\epsilon\to 0_+} \epsilon^{\frac{2(2-\gamma)}{\gamma}} \ln(\mathbb{P}[T^{(\epsilon,k,+)} \leq M]) \;=\; -\lambda_k M$$

from where follows easily the large deviation principle for $T^{(\epsilon,k,+)}$.

The corresponding one for $T^{(\epsilon,k,-)}$ is obtained in a similar way.

■

Remark 5.2: a) Already for $\gamma = 1$, Gradinaru, Herrmann and Roynette [13] show that (2) is no longer valid and must be replaced by

$$\forall\ (t,x) \in \triangle, \qquad \lim_{\epsilon \to 0_+} \epsilon^2 \ln(p_\epsilon(t,x)) = -\frac{1}{8t}\,(2\,|x| - t)^2$$

Thus for the values $0 < \gamma \leq 1$, our analysis of convenient exit times will not be sufficient in the present form. This comes from the fact that their order of large deviations is no more negligible with respect to that of a Freidlin-Wentzell type result. The case $\gamma = 1$ is especially stimulating for the reason that the above competiting orders of large deviations are equal, since $2(2-\gamma)/\gamma = 2$ and we still have $\lambda_\infty > 0$. Note that then the action functional for $p_\epsilon(t,x)$ not only depends on $(t,x) \in \triangle$ via $2\,|x|^2 - t$, which is as before the "exit time" of 0 for the solution of (1) passing through (t,x), but also on t. This problem shows that there is a lot of work ahead in this direction.

b) Even if the one-dimensional situation is far from being solved, let us mention that the multi-dimensional results mentioned in section 3 could lead to large deviation behaviors for the density of small random perturbations of degenerate dynamical systems on Euclidian space $\mathbb{R}^d$, with $d > 1$, but one will have to work out results for the exit couple formed of time and position.

References

[1] R. Bafico and P. Baldi. Small random perturbations of Peano phenomena. *Stochastics*, 6:279–292, 1982.

[2] S. G. Bobkov and F. Götze. Discrete isoperimetric and Poincaré-type inequalities. *Probab. Theory Related Fields*, 114(2):245–277, 1999.

[3] A.N. Borodin and P. Salminen. *Handbook of Brownian Motion - Facts and Formulae.* Probability and Its Applications. Birkhäuser, Basel, 1996.

[4] P. Carmona. The mean velocity of a Brownian motion in a random Lévy potential. *The Annals of Probability*, 25(4):1774–1788, 1997.

[5] R.D. DeBlassie. The lifetime of conditioned Brownian motion in certain Lipschitz domains. *Probab. Theory Related Fields*, 75(1):55–65, 1987.

[6] R.D. DeBlassie. Doob's conditioned diffusions and their lifetimes. *Ann. Probab.*, 16(3):1063–1083, 1988.

[7] A. Dembo and O. Zeitouni. *Large Deviations Techniques and Application.* Jones and Bartlett, 1993.

[8] J.-D. Deuschel and D.W. Stroock. *Large Deviations.* Pure and applied mathematics 137. Academic Press, 1989.

[9] G. Doetsch. *Introduction to the theory and application of the Laplace transformation.* Springer-Verlag, New York, 1974. Translated from the second German edition by Walter Nader.

[10] M. D. Donsker and S. R. S. Varadhan. Asymptotic evaluation of certain Markov process expectations for large time. III. *Comm. Pure Appl. Math.*, 29(4):389–461, 1976.

[11] M.I. Freidlin and A.D. Wentzell. *Random Perturbations of Dynamical Systems.* A Series of Comprehensive Studies in Mathematics, 260. Springer-Verlag, 1984.

[12] M. Fukushima, Y. Oshima, and M. Takeda. *Dirichlet Forms and Symmetric Markov Processes.* de Gruyter Studies in Mathematics 19. Walter de Gruyter, 1994.

[13] M. Gradinaru, S. Herrmann, and B. Roynette. A singular large deviations phenomenon. Preprint, 2000.

[14] C.E. Kenig and J. Pipher. The h-path distribution of the lifetime of conditioned Brownian motion for nonsmooth domains. *Probab. Theory Related Fields*, 82(4):615–623, 1989.

[15] P. Mandl. *Analytic Treatment of One-Dimensional Markov Processes.* Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band 151. Springer-Verlag, 1968.

[16] P. Mathieu. Hitting times and spectral gap inequalities. *Annales de l'Institut Henri Poincaré*, 33(4):437–465, 1997.

[17] B. Muckenhoupt. Hardy's inequality with weights. *Studia Mathematica*, XLIV:31–38, 1972.

[18] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion.* Grundlehren der mathematischen Wissenschaften, 293. Springer-Verlag, second edition, 1994.