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Strong stationary times for one-dimensional diffusions

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Abstract. A necessary and sufficient condition is obtained for the existence of strong stationary times for ergodic one-dimensional diffusions, whatever the initial distribution. The strong stationary times are constructed through intertwinings with dual processes, in the Diaconis–Fill sense, taking values in the set of segments of the extended line $\mathbb{R} \sqcup \{-\infty, +\infty\}$. They can be seen as natural Doob transforms of the extensions to the diffusion framework of the evolving sets of Morris–Peres. Starting from a singleton set, the dual process begins by evolving into true segments in the same way a Bessel process of dimension 3 escapes from 0. The strong stationary time corresponds to the first time the full segment $[-\infty, +\infty]$ is reached. The benchmark Ornstein–Uhlenbeck process cannot be treated in this way; it will nevertheless be seen how to use other strong times to recover its optimal exponential rate of convergence to equilibrium in the total variation sense.

Résumé. Une condition nécessaire et suffisante est obtenue pour l'existence de temps fort de stationnarité, quelque soit la condition initiale. Les temps forts de stationnarité sont construits par le biais d'entrelacements avec des processus duaux, au sens de Diaconis–Fill, prenant leurs valeurs dans l'ensemble des segments de la droite étendue $\mathbb{R} \sqcup \{-\infty, +\infty\}$. Ils peuvent être vus comme des transformées de Doob d'extensions au cadre diffusif des ensembles évoluants de Morris–Peres. Partant d'un singleton, le processus dual commence par évoluer en segments de la même manière qu'un processus de Bessel de dimension 3 s'échappe de 0. Le temps fort de stationnarité correspond au premier temps d'atteinte de $[-\infty, +\infty]$. Le processus d'Ornstein–Uhlenbeck ne peut pas être traiter de la sorte, il est toutefois possible d'utiliser d'autres temps forts pour retrouver son taux exponentiel optimal de convergence à l'équilibre en variation totale.

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1. Introduction

A strong stationary time τ is a stopping time relative to the filtration generated by an ergodic Markov process $(X_t)_{t\geq 0}$ (and possibly some independent randomness) which is such that τ and X_τ are independent and X_τ is distributed according to the underlying invariant probability distribution. They were first introduced by Aldous and Diaconis [1] in the context of finite Markov chains. Staying in the finite framework, Diaconis and Fill [12] developed the important tool of intertwining with absorbed Markov chains to construct strong stationary times. Intertwining of diffusions was also investigated by Rogers and Pitman [32] and Carmona, Petit and Yor [7], especially to deduce identities in law for particular processes. Recently, Pal and Shkolnikov [29] studied some conditions insuring that there exists an intertwining between two Markov semi-groups and their article also provides a welcome survey of applications of intertwining relations. Among other recent works in this direction, such as Lorek and Szekli [25] and Huillet and Martinez [19], the article [18] of Fill and Lyzinski is the closest to the present paper (see Remark 6). Our goal here is to come back to the investigation of strong stationary times through intertwining, but in the context of diffusions. We will also point out a relation with an extension to this continuous setting of the evolving sets of Morris and Peres [28]. More precisely, we are to be mainly concerned with one-dimensional diffusions, the simplest continuous framework and nevertheless already displaying some interesting features. Of course, extensions to multidimensional situations are more promising and challenging. They are outside the scope of this paper, which can be seen as only working out the preliminary steps in this direction that we hope to investigate in the future.

Consider the one-dimensional Markov generator given by

$$L := a\partial^2 + b\partial, \tag{1}$$

where $a > 0$ and b are two functions defined on $\mathbb{R}$. We won't be interested in regularity issues, so we assume that they are smooth and L can be interpreted as an operator from $\mathcal{C}_c^\infty(\mathbb{R})$ to itself. It is often convenient to extend it as a self-adjoint operator on a $\mathbb{L}^2$ space. Indeed, consider the two positive functions defined by

$$\begin{aligned} \forall x \in \mathbb{R}, \quad s(x) &:= \exp\left(-\int_0^x \frac{b(y)}{a(y)}\,dy\right), \\ m(x) &:= \frac{1}{2a(x)s(x)} \end{aligned} \tag{2}$$

and the associated *scale* and *speed* measures S and M characterized by

$$\begin{aligned} \forall x < y \in \mathbb{R}, \quad S[x,y] &= \int_x^y s(u)\,du, \\ M[x,y] &= \int_x^y m(u)\,du. \end{aligned}$$

The diffusion is said to be *positive recurrent* if

$$M(-\infty,\infty) < \infty \tag{3}$$

and if

$$\int_{-\infty}^0 M[y,0]S(dy) = +\infty \quad \text{and} \quad \int_0^\infty M[y,0]S(dy) = +\infty. \tag{4}$$

The first condition implies that the second can be replaced by

$$S(-\infty,0] = +\infty \quad \text{and} \quad S[0,\infty) = +\infty.$$

Under these conditions (3) and (4),

$$\forall x \in \mathbb{R}, \quad \pi(x) := \frac{m(x)}{M(-\infty,\infty)} \tag{5}$$

becomes the *stationary density function*. We will denote by the same symbol π the probability measure admitting the function π as density with respect to the Lebesgue measure. It is well-known (cf. for instance the Chapter 15 of the book of Karlin and Taylor [21]), and elementary to recover, that the operator L is symmetric in $\mathbb{L}^2(\pi)$, so we can consider the corresponding self-adjoint Friedrichs extension.

Let $X := (X_t)_{t\geq 0}$ be a diffusion process whose generator is L, in the sense of martingale problems. Assumption (4) prevents X from exploding in finite time (see for instance Theorem 3.2(3) of Chapter 6 of the book of Ikeda and Watanabe [20]). By self-adjointness of L, the probability π is reversible for X, this is always the case for one-dimensional positive recurrent diffusions (see e.g. Kent [22]). The reversibility of continuous processes can be equivalently characterized as follows: when a process $(X_t)_{0\leq t\leq T}$ starts from the reversible distribution π, it has the same law as the reversed process $(X_{T-t})_{0\leq t\leq T}$ (for càdlàg processes, the right and left limits and continuities of the trajectories have to be exchanged), see for instance Liggett [24].

The process X is a priori defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by X. For instance, Ω can be taken to be the set of continuous trajectories $\mathcal{C}([0,+\infty), (-\infty,+\infty))$ endowed with the σ-field and the filtration generated by the canonical coordinate process. But to allow for extra randomness, it is useful to enlarge the initial setting $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ into $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, (\bar{\mathcal{F}}_t)_{t\geq 0})$, preserving the fact that $X := (X_t)_{t\geq 0}$ is a continuous process starting from x_0, Markovian with respect to the filtration $(\bar{\mathcal{F}}_t)_{t\geq 0}$ and whose generator is L. This is often done by considering the tensor product of $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ with another probability space.

A random time τ taking values in $[0,\infty]$ is said to be a *stopping time*, if it is defined on a framework $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, (\bar{\mathcal{F}}_t)_{t\geq 0})$ as above and if it a stopping time with respect to the filtration $(\bar{\mathcal{F}}_t)_{t\geq 0}$, namely if

$$\forall t \geq 0, \quad \{\tau \leq t\} \in \bar{\mathcal{F}}_t.$$

From a practical point of view, it means that τ is constructed from X and from some independent randomness Y in such a way that for any $t \geq 0$ and in view of Y, to decide whether $\tau \leq t$ or not, it is sufficient to look at the trajectory $X_{[0,t]} := (X_s)_{s\in[0,t]}$.

The stopping time τ, taking values in $[0,\infty)$, is said to be *strong*, if τ and X_τ are independent. It is said to be a *strong stationary time*, if furthermore X_τ is distributed according to π.

Our main goal is to investigate the existence of strong stationary times for X. To state our first result, we need the following quantities

$$I_- := \int_{-\infty}^{0} S[x,0]M(dx) \quad \text{and} \quad I_+ := \int_0^{+\infty} S[0,x]M(dx),$$
$$I := \max(I_-, I_+).$$

The points $-\infty$ and ∞ are said to be *entrance boundaries* if, respectively, $I_- < +\infty$ and $I_+ < +\infty$. If X starts from an entrance boundary, it quickly moves into the interior of the state space but never return to this boundary, see for instance Karlin and Taylor [21]. This will be our main assumption, namely $I < +\infty$. Note that the role of 0 is irrelevant: it could be replaced by any other point of $\mathbb{R}$. But if we were looking for quantitative bounds, it should be chosen more carefully, maybe replacing it by x_0 in the case where X starts from the initial deterministic condition $X_0 = x_0$. For the next result, we allow any initial distribution for $\mathcal{L}(X_0)$.

Theorem 1. *Assume that X is positive recurrent. There exists a strong stationary time for X, whatever its initial distribution, if and only if $-\infty$ and $+\infty$ are entrance boundaries for X.*

Remark 2. *Despite the deliberate choice made in this paper not to get involved in optimal regularity questions, let us mention that the natural framework for the previous result is that of general one-dimensional diffusions (see for instance Chapter 15 of the book of Karlin and Taylor [21]): the generator is no longer described by (1), but under the form $L = \frac{d}{dM}\frac{d}{dS}$, where M and S are still the speed and scale measures, which may no longer admit densities with respect to the Lebesgue measure, contrary to our regular setting. We expect Theorem 1 to be still true in this context.*

Remark 3. *Instead of the whole line $\mathbb{R}$, we could have considered the half-line $\mathbb{R}_+$ with usual reflection at 0. Similar notions can be introduced in this context and the arguments can be adapted to show that the corresponding Theorem 1 is valid, where I is replaced by I_+. Cheng and Mao [9] shows that the assumption $I_+ < +\infty$ is equivalent to several conditions, among which are the strong ergodicity of X and the fact that the essential spectrum of L is empty and that the sum of the inverses of its non-zero eigenvalues is finite. This amounts to saying that the associated centered Green operator (which is the inverse of the generator on the space of functions whose mean with respect to the invariant measure vanishes) has a finite trace. In the recent paper [27], the result of Cheng and Mao [9] was extended, by showing that the condition $I < +\infty$ is equally equivalent to the finiteness of the trace of the centered Green operator associated to L, and this is indeed true for quantum graphs admitting only a finite number of splitting vertices and a finite number of infinite rays. Namely, for regular and reversible diffusions on these state spaces, to have only entrance boundaries is equivalent to the finiteness of the trace of the centered Green operator. We also believe this is necessary and sufficient for the process to admit strong stationary times, whatever the initial condition, similarly to Theorem 1 for $\mathbb{R}$.*

We believe that the finiteness of the trace of the centered Green operator is always a sufficient condition for the existence of strong stationary times, but the necessity of this property cannot be true in full generality: consider a probability π on a general measurable space and let L be the generator acting on functions $f \in \mathbb{L}^2(\pi)$ by $L[f] := \pi[f]\mathbb{1} - f$. A strong stationary time, whatever the initial distribution, is given by the first jump. The spectrum of L consists of 0 (with multiplicity 1) and of -1 with multiplicity the dimension of $\{f \in \mathbb{L}^2(\pi) : \pi[f] = 0\}$. So if the latter dimension is infinite, we get a counter-example to the necessity condition outside the framework of one-dimensional diffusions.

As announced at the beginning of this section, a strong stationary time will be constructed through duality via intertwining relations. More precisely, let

$$E^* := \{(x, y) : x, y \in [-\infty, +\infty], x \leq y\} \setminus \{(-\infty, -\infty), (+\infty, +\infty)\},$$
$$\mathring{E}^* := \{(x, y) \in \mathbb{R}^2 : x < y\}$$

be the interior of E^* and $D^* := \{(x, x) : x \in \mathbb{R}\} \subset E^*$ be the diagonal of $\mathbb{R}^2$. Consider the Markov kernel Λ from E^* to $\mathbb{R}$ defined by

$$\forall (x, y) \in E^*, \forall A \in \mathcal{B}(\mathbb{R}), \quad \Lambda((x, y), A) := \begin{cases} \delta_x(A), & \text{if } y = x, \\ \frac{\pi([x,y] \cap A)}{\pi([x,y])}, & \text{otherwise}, \end{cases}$$

where $\mathcal{B}(\mathbb{R})$ stands for the set of Borel sets from $\mathbb{R}$.

Transposing to the diffusion setting the program described by Diaconis and Fill [1] for finite Markov chains, we are looking for a diffusion generator L^* on E^* satisfying the *intertwining relation* $\Lambda L = L^* \Lambda$, in the sense that at least on $E^* \setminus (D^* \sqcup \{(-\infty, +\infty)\})$,

$$\forall f \in \mathcal{C}_c^\infty(\mathbb{R}), \quad \Lambda[L[f]] = L^*[\Lambda[f]]. \tag{6}$$

Here is one solution that will be derived later in the paper: on $\mathring{E}^*$,

$$\begin{aligned} L^* &:= \left(\sqrt{a(y)}\partial_y - \sqrt{a(x)}\partial_x\right)^2 + \left(a'(x)/2 - b(x)\right)\partial_x + \left(a'(y)/2 - b(y)\right)\partial_y \\ &\quad + 2\frac{\sqrt{a(x)}\pi(x) + \sqrt{a(y)}\pi(y)}{\pi([x, y])}\left(\sqrt{a(y)}\partial_y - \sqrt{a(x)}\partial_x\right), \end{aligned} \tag{7}$$

while on $\mathbb{R} \times \{+\infty\}$,

$$L^* := \left(\sqrt{a(x)}\partial_x\right)^2 + \left(a'(x)/2 - b(x)\right)\partial_x - 2\frac{\sqrt{a(x)}\pi(x)}{\pi([x, +\infty))}\sqrt{a(x)}\partial_x \tag{8}$$

and on $\{-\infty\} \times \mathbb{R}$,

$$L^* := \left(\sqrt{a(y)}\partial_y\right)^2 + \left(a'(y)/2 - b(y)\right)\partial_y + 2\frac{\sqrt{a(y)}\pi(y)}{\pi((-\infty, y])}\sqrt{a(y)}\partial_y. \tag{9}$$

Formally, (8) and (9) are obtained by respectively replacing y by $+\infty$ and x by $-\infty$ in (7). Such extensions of (7) will be called *natural* in the sequel.

We put a Dirichlet condition at $(-\infty, +\infty)$, insuring that it is an absorbing point.

It is not necessary to make precise the boundary condition on the diagonal D^*, because it will be shown in Section 2 to be an entrance boundary:

Proposition 4. *For any $x_0 \in \mathbb{R}$, there is a continuous Markov process $Z^* := (Z_t^*)_{t \geq 0}$ starting from (x_0, x_0), whose generator is L^* (in the sense of martingale problems) and satisfying for all $t > 0$, $Z_t^* \in E^* \setminus D^*$. The law of this* *process is unique if we impose that after the possibly finite time*

$$\tau^* := \inf\{t \geq 0 \colon Z_t^* = (-\infty, +\infty)\}, \tag{10}$$

Z^* *stays at position* $(-\infty, +\infty)$ (*i.e. if we consider the minimal process*).

The generator L^* defined in (7) is not the unique one satisfying (6). As we will see later, this relation is also true if L^* is replaced by

$$\begin{aligned} L_1^* &:= \bigl(\sqrt{a(y)}\partial_y + \sqrt{a(x)}\partial_x\bigr)^2 + \bigl(a'(x)/2 - b(x)\bigr)\partial_x + \bigl(a'(y)/2 - b(y)\bigr)\partial_y \\ &\quad + 2\frac{\sqrt{a(y)}\pi(y) - \sqrt{a(x)}\pi(x)}{\pi([x, y])}\bigl(\sqrt{a(y)}\partial_y + \sqrt{a(x)}\partial_x\bigr) \end{aligned} \tag{11}$$

(on $\mathring{E}^*$ and its natural extensions on $\mathbb{R} \times \{+\infty\}$ and $\{-\infty\} \times \mathbb{R}$). For this operator, D^* is not an entrance boundary: an associated process starting on D^* stays in D^*. This is related to the fact that on D^*, L_1^* can be written using only vector fields tangential to D^* (note also that the mapping $E^* \setminus D^* \ni (x, y) \mapsto (\sqrt{a(y)}\pi(y) - \sqrt{a(x)}\pi(x))/\pi([x, y])$ can be naturally extended into a symmetric and smooth function on $\mathbb{R}^2$).

There are other generators satisfying (6), e.g. the elliptic operator

$$\begin{aligned} L_{1/2}^* &:= a(y)\partial_y^2 + a(x)\partial_y^2 + \bigl(a'(x)/2 - b(x)\bigr)\partial_x + \bigl(a'(y)/2 - b(y)\bigr)\partial_y \\ &\quad + 2\frac{1}{\pi([x, y])}\bigl(a(y)\pi(y)\partial_y - a(x)\pi(x)\partial_x\bigr) \end{aligned} \tag{12}$$

(on $\mathring{E}^*$ and its natural extensions on $\mathbb{R} \times \{+\infty\}$ and $\{-\infty\} \times \mathbb{R}$). One would have remarked that $L_{1/2}^* = (L^* + L_1^*)/2$, and more generally for any $\alpha \in (0, 1)$, the generator $L_\alpha^* := (1 - \alpha)L^* + \alpha L_1^*$ satisfies (6) and is elliptic. But as it will be seen in Remark 15 at the end of the next section, these generators lead to strong stationary times which are larger than those obtained from L^*.

The generator L^* defined in (7) has another interest: it is related via a Doob transform to the continuous equivalent of the *evolving sets* introduced by Morris and Peres [28] for denumerable Markov chains. Consider the generator given by

$$\tilde{L} := \bigl(\sqrt{a(y)}\partial_y - \sqrt{a(x)}\partial_x\bigr)^2 + \bigl(a'(x)/2 - b(x)\bigr)\partial_x + \bigl(a'(y)/2 - b(y)\bigr)\partial_y \tag{13}$$

(on $\mathring{E}^*$ and its natural extensions on $\mathbb{R} \times \{+\infty\}$ and $\{-\infty\} \times \mathbb{R}$). It is explained in Appendix B how to associate Markov processes $(\tilde{X}_t, \tilde{Y}_t)_{t \geq 0}$ to $\tilde{L}$, so that the segment-valued process $([\tilde{X}_t, \tilde{Y}_t])_{t \geq 0}$ can be seen as a continuous evolving set in $\mathbb{R}$.

Next define the mapping h on E^* by

$$\forall z = (x, y) \in E^*, \quad h(z) := \pi\bigl([x, y]\bigr). \tag{14}$$

It will be checked in Lemma 9 of the next section that $\tilde{L}[h] = 0$ on $\mathring{E}^*$. Then L^* is the *Doob transform* of $\tilde{L}$ through h:

$$\begin{aligned} L^*[\cdot] &= \frac{1}{h}\tilde{L}[h\cdot] \\ &= \tilde{L}[\cdot] + \tilde{\Gamma}\bigl[\ln(h), \cdot\bigr], \end{aligned}$$

where $\tilde{\Gamma}$ is the *squared field operator* associated to $\tilde{L}$: for any smooth functions f, g defined on $\mathring{E}^*$,

$$\tilde{\Gamma}[f, g] := \tilde{L}[fg] - f\tilde{L}[g] - g\tilde{L}[f].$$

Let us now come back to a diffusion process X as in Theorem 1 and denote by m_0 its initial distribution. Consider the probability m_0^* defined on E^* by $m_0^* := \int \delta_{(x,x)} m_0(dx)$, so that $m_0^*\Lambda = m_0$. In general it is not the only probability on E^* satisfying this relation, for instance if $m_0 = \Lambda(z, \cdot)$, with $z \in \mathring{E}^*$, it seems more appropriate to choose $m_0^* := \delta_z$. The strong stationary time constructed in Proposition 5 below does depend on the choice of m_0^*, but in this paper we will not consider the important question of finding the best possible choice for m_0^* (the next section will show how to construct a process Z^* starting from any initial distribution on E^*, indeed Proposition 4 presented the most difficult cases). As it is explained by Diaconis and Fill [12] in the finite setting, the relations $m_0^* \Lambda = m_0$ and (6) should enable coupling X with the process Z^*, defined similarly as in Proposition 4 but with m_0^* as initial distribution, in such a way that for any $t \geq 0$, the conditional law of X_t knowing the trajectory $Z^*_{[0,t]}$ is given by $\Lambda(Z_t^*, \cdot)$. The extension to the positive recurrent one-dimensional diffusion case turned out to be quite tricky and will be developed in Section 4 (unfortunately the results of Pal and Shkolnikov [29] cannot be applied straightforwardly). Let us admit this technical point for the time being. A convenient feature of this coupling is that it can be obtained by starting with a trajectory X and by constructing Z^* from X and independent randomness. More precisely, for any $t \geq 0$, the piece of trajectory $Z^*_{[0,t]}$ is constructed from $X_{[0,t]}$ and independent randomness. Thus any stopping time τ with respect to the filtration generated by the process Z^* is also a stopping time for X. This is important, because the previous conditional property extends to any finite stopping time τ with respect to the filtration generated by the process Z^*:

$$\mathcal{L}\big(X_\tau | Z^*_{[0,\tau]}\big) = \Lambda\big(Z^*_\tau, \cdot\big), \tag{15}$$

where the l.h.s. is the conditional law of X_τ knowing the trajectory $Z^*_{[0,\tau]}$. In particular if we consider the stopping time τ^* defined in (10) and if we impose conditions such that this Z^*-stopping time is a.s. finite, then it is a strong stationary time for X. Indeed, the above considerations show that τ^* is a stopping time for X. Next note that X_{τ^*} is independent from $Z^*_{[0,\tau^*]}$, because according to (15), $\mathcal{L}(X_{\tau^*} | Z^*_{[0,\tau^*]}) = \Lambda((-\infty, +\infty), \cdot) = \pi$ does not depend on $Z^*_{[0,\tau^*]}$. It follows that τ^* is strong because it is measurable with respect to $Z^*_{[0,\tau^*]}$. Finally it is a strong stationary time for X, since from the above identity, $\mathcal{L}(X_{\tau^*}) = \pi$.

Up to the construction of the intertwining, these few standard arguments provide the direct implication in Theorem 1:

Proposition 5. *If $I < +\infty$, then the random time τ^* defined in (10) is a.s. finite and by consequence it is a strong stationary time for the positive recurrent diffusion X.*

Remark 6. *Fill and Lyzinski [18] recently studied the existence of strong stationary times via the duality of diffusion processes (see also Cheng and Mao [8], up to the construction of the intertwining coupling). They consider a diffusion X on [0,1], starting from 0 and they assume that 1 is either reflecting or entrance. Next they construct a diffusion dual Y^* on [0,1], starting from 0 and for which 1 is absorbing or exit, according to the behavior of X at 1 (see also [13], which deals with strong quasi-stationary times for finite birth and death process, but whose formalism is adapted to treat diffusion processes starting from the boundary). A strong stationary time is given by the first time Y^* reaches 1. Up to a rescaling of the state space and at least in the entrance situation, our dual process Z^* is then equal to $(0, Y^*)$. It should be seen as the segment-valued dual process $([0, Y_t^*])_{t\geq 0}$. That the left boundary of this segment is fixed and equal to the left boundary 0 of the state space comes directly from the fact that X starts from 0. To deal with more general initial distributions, we will consider segment-valued dual processes whose both boundaries are moving (except when one of the boundaries, $-\infty$ or $+\infty$, of the state space is attained, it will then remains fixed), this explains why our dual process will be 2-dimensional. We will see that it is better for our purposes that these motions of the left and right boundaries are strongly correlated (they will share the same driving Brownian motion), so from a theoretical point of view, one could come back to a one-dimensional dual process. The same technique can be applied to birth and death processes on $\mathbb{Z}$ for which ∞ and $+\infty$ are entrance boundaries. In certain symmetric situations, it is possible to come back directly to a one-dimensional dual, see Remark 37.*

Remark 7. *As it is mentioned in Diaconis and Fill [12], a special case of construction of a strong stationary time dates back to Dubins [15]: consider on $[0, A]$, with $A > 0$, a Brownian motion X reflected in 0 and A and starting from 0. Let τ_1 be the first time it hits $A/2$, next let τ_2 be the first time after τ_1 that $A/4$ or $3A/4$ are reached and call Y_2 this point. If τ_n and Y_n have been constructed for some $n \geq 2$, let τ_{n+1} be the first time after τ_n that $Y_n \pm A/2^{n+1}$ is reached and call this point Y_{n+1}. The limit $\tau := \lim_{n\to\infty} \tau_n$ exists a.s. and is a strong stationary time for X.*

Proposition 5 opens the way to a quantitative study of the convergence to equilibrium for X in the separation sense. Let us recall that the *separation discrepancy* $\mathfrak{s}(\nu,\pi)$ between two probability measures ν and π defined on the same state space E is given by

$$\mathfrak{s}(\nu,\pi) := \operatorname*{ess\,sup}_{\pi} 1 - \frac{d\nu}{d\pi},$$

where the essential supremum is taken with respect to π and where $\frac{d\nu}{d\pi}$ is the Radon–Nikodym derivative of the absolutely continuous part of ν with respect to π. Strictly speaking, the separation discrepancy is not a distance because it is not symmetric in its arguments. The computations of Aldous and Diaconis [1] show that for any strong stationary time τ for X, we have

$$\forall t \geq 0, \quad \mathfrak{s}\big(\mathcal{L}(X_t),\pi\big) \leq \mathbb{P}[\tau > t]. \tag{16}$$

Thus Proposition 5 enables us to get upper bounds on the speed of convergence of X toward its equilibrium π in the separation sense, by studying the speed of absorption at $(-\infty,+\infty)$ of Z^*. The inequalities (16) may be equalities for all times $t \geq 0$ and such times τ are then stochastically minimal among all strong stationary times. They are called sharp stationary times in Diaconis and Fill [12] (in the finite setting). The proof of the converse implication in Theorem 1 will rely on the fact that for initial distributions of X of the form $\Lambda((-\infty,x),\cdot)$ and $\Lambda((x,+\infty),\cdot)$, with $x \in \mathbb{R}$, the random time τ^* defined in (10) is indeed a sharp stationary time.

When is the condition $I < +\infty$ of Proposition 5 satisfied? It is convenient to consider the case of *Langevin diffusions*, where $a \equiv 1$ and $b = -U'$, where $U : \mathbb{R} \to \mathbb{R}$ is a smooth potential. In dimension 1 and up to shrinking the state space $\mathbb{R}$ to an open interval (through a smooth transformation), it is not really a restriction. The invariant measure π admits a density proportional to $\exp(-U)$. An application of Fubini's formula shows that the condition $I < +\infty$ can be written as

$$\max\left(\int_{-\infty}^{0} \pi\big((-\infty,x)\big)\frac{1}{\pi(x)}\,dx, \int_{0}^{+\infty} \pi\big((x,+\infty)\big)\frac{1}{\pi(x)}\,dx\right) < +\infty. \tag{17}$$

The l.h.s. is bounded below by

$$\max\left(\sup_{y\leq 0}\pi\big((-\infty,y)\big)\int_{y}^{0}\frac{1}{\pi(x)}\,dx, \sup_{y\geq 0}\pi\big((y,+\infty)\big)\int_{0}^{y}\frac{1}{\pi(x)}\,dx\right)$$

and if 0 was chosen to be the median of π (up to a translation there is not lack of generality in this choice), the previous quantity is the inverse of the spectral gap of L in $\mathbb{L}^2(\pi)$ up to a factor 4 (see e.g. Bobkov and Götze [6]). So at least for Langevin diffusions, the existence of a strong stationary time, whatever the initial distribution, implies a positive spectral gap. As it appears in the example below and as it can be expected from Remark 3, this is far from being a sufficient condition.

Example 8. *If for $|x|$ large enough, we have $U(x) = |x|^\alpha + V(x)$, with $\alpha > 0$ and $V : \mathbb{R} \to \mathbb{R}$ a bounded function, then condition (17) is satisfied if and only if $\alpha > 2$ (whereas the existence of a spectral gap is equivalent to $\alpha \geq 1$). For this example, the eigenvalues of $-L$ are not known, but we deduce from Remark 3 that the sum of their inverses (except 0) is finite. The expressions given in (7) and (13) for the generators L^* and $\tilde{L}$ can be simplified to*

$$L^* = (\partial_y - \partial_x)^2 - \big(\ln(\pi)\big)'(x)\partial_x - \big(\ln(\pi)\big)'(y)\partial_y + 2\frac{\pi(x)+\pi(y)}{\pi([x,y])}(\partial_y - \partial_x),$$

$$\tilde{L} = (\partial_y - \partial_x)^2 - \big(\ln(\pi)\big)'(x)\partial_x - \big(\ln(\pi)\big)'(y)\partial_y$$

(*with similar reductions for L_1^* and $L_{1/2}^*$ in* (11) *and* (12)).

In particular, the important case of the Ornstein–Uhlenbeck process is not covered by Proposition 5. Does it mean that the previous approach is useless in this situation? Indeed, it is possible to get around this difficulty by considering strong times τ where the distribution of X_τ is close to the invariant probability π. This possibility is presented in Section 2.5 of Diaconis and Fill [12] in the framework of finite state spaces. Put in practice in Section 5, this technique will enable us to recover good quantitative bounds on the convergence of the Ornstein–Uhlenbeck process toward the Gaussian distribution in the total variation sense.

The discrete analogues of one-dimensional diffusions are the birth and death processes. In this context, the strong stationary duality have been used to investigate perfect sampling by Fill [16], cut-off phenomenon by Diaconis and Saloff-Coste [14] and fastest mixing by Fill and Kahn [17]. It would be interesting to extend these consequences to the diffusion framework.

Let us just give a glimpse of why it could also be interesting to investigate the multidimensional situation. Let X be a hypoelliptic diffusion taking values in a smooth manifold M of dimension (strictly) larger than 1. Assume that it is possible to construct a process Z^* taking values in the set E^* of singletons and non-empty open subsets of M and which is intertwined with X through the Markov kernel Λ from E^* to M given by

$$\forall z \in E^*, \quad \Lambda(z, \cdot) := \begin{cases} \delta_x(\cdot), & \text{if } z = \{x\}, \\ \frac{\lambda(\cdot \cap z)}{\lambda(z)}, & \text{if } z \text{ is a non-empty subset of } M, \end{cases}$$

where λ is a σ-finite measure on M giving positive weights to all non-empty open subsets (for instance λ could be the invariant measure for X, but it could also be a more tractable measure). Then we would have at our disposal the following representation of the time marginal laws of X for all $t \geq 0$,

$$\forall x \in M, \quad \mathcal{L}(X_t)(dx) = \int \Lambda(z, dx) \mathcal{L}\big(Z_t^*\big)(dz),$$

from which absolute continuity and regularity properties can be deduced. It would be instructive to begin with a simple instance of X satisfying Hörmander's conditions and to see which features could be deduced for corresponding processes Z^*, especially in small times. Entrance boundary properties of singletons analogous to that presented in Proposition 4 would be particularly desirable.

The paper is constructed on the following plan. In the next section we investigate the dual process Z^*, making a link with the Bessel process of dimension 3 and we prove Proposition 4. Explosion times and Proposition 5 are the subject of Section 3. Section 4 ends the proof of Theorem 1, providing the missing details about the coupling of X with Z^* and showing the converse implication. The last section and Appendix A are devoted to the counter-example of the benchmark Ornstein–Uhlenbeck process, giving us the opportunity to see why it is interesting to consider more general strong times than strong stationary times. Appendix B presents some details about the analogy with the Morris–Peres evolving sets, through a Liggett duality.

2. Description of the dual process

We study here the solutions of the stochastic differential equations associated with the generator L^* given by (7), (8) and (9).

We begin by verifying the assertion made in the introduction about the relation between L^* and $\tilde{L}$ defined in (13).

Lemma 9. *Let h be the function introduced in* (14). *On $\mathring{E}^*$ we have $\tilde{L}[h] = 0$ and for any $F \in \mathcal{C}^\infty(\mathring{E}^*)$,*

$$\forall z \in \mathring{E}^*, \quad L^*[F](z) = \frac{1}{h} \tilde{L}[hF](z).$$

These properties extend to $\mathbb{R} \times \{+\infty\}$ and $\{-\infty\} \times \mathbb{R}$, up to the natural modifications.

Proof. For $(x, y) \in \mathring{E}^*$, we have

$$\partial_x h(x, y) = -\pi(x) \quad \text{and} \quad \partial_y h(x, y) = \pi(y),$$

so that

$$\big(\sqrt{a(y)}\partial_y - \sqrt{a(x)}\partial_x\big)^2 h(x,y) = \big(\sqrt{a(y)}\partial_y - \sqrt{a(x)}\partial_x\big)\big(\sqrt{a(y)}\pi(y) + \sqrt{a(x)}\pi(x)\big)$$
$$= \sqrt{a(y)}\partial_y\big(\sqrt{a(y)}\pi(y)\big) - \sqrt{a(x)}\partial_x\big(\sqrt{a(x)}\pi(x)\big).$$

Taking into account that

$$\pi' = \frac{b-a'}{a}\pi,$$

we get that

$$\big(\sqrt{a(y)}\partial_y - \sqrt{a(x)}\partial_x\big)^2 h(x,y) = \left(b(y) - \frac{a'(y)}{2}\right)\pi(y) - \left(b(x) - \frac{a'(x)}{2}\right)\pi(x)$$
$$= -\big(a'(x)/2 - b(x)\big)\partial_x h(x,y) - \big(a'(y)/2 - b(y)\big)\partial_y h(x,y),$$

namely $\tilde{L}[h] = 0$. In the same way one shows that $\tilde{L}[h] = 0$ on $(\mathbb{R} \times \{+\infty\}) \sqcup (\{-\infty\} \times \mathbb{R})$.

By definition of $\tilde{\Gamma}$, we observe that for any $F \in \mathcal{C}^\infty(\mathring{E}^*)$ and any $z \in \mathring{E}^*$,

$$\frac{1}{h}\tilde{L}[hF](z) = \frac{1}{h}\big(h\tilde{L}[F] + F\tilde{L}[h] + \tilde{\Gamma}[h,F]\big)$$
$$= \tilde{L}[F] + \frac{1}{h}\tilde{\Gamma}[h,F].$$

A direct computation shows that for any $F, G \in \mathcal{C}^\infty(\mathring{E}^*)$ and any $z = (x,y) \in \mathring{E}^*$,

$$\tilde{\Gamma}[G,F] = 2\big(\sqrt{a(y)}\partial_y - \sqrt{a(x)}\partial_x\big)G(x,y)\big(\sqrt{a(y)}\partial_y - \sqrt{a(x)}\partial_x\big)F(x,y).$$

Applying this formula with $G = h$, we obtain that $L^*[F] = \frac{1}{h}\tilde{L}[hF]$, as announced. Again these considerations extend without difficulty to $(\mathbb{R} \times \{+\infty\}) \sqcup (\{-\infty\} \times \mathbb{R})$. ☐

Remark 10. *Similar computations are valid for the generators given by* (11) *and* (12). *Indeed, they are respectively the Doob transforms through h of the generators defined by*

$$L_1 := \big(\sqrt{a(y)}\partial_y + \sqrt{a(x)}\partial_x\big)^2 + \big(a'(x)/2 - b(x)\big)\partial_x + \big(a'(y)/2 - b(y)\big)\partial_y \tag{18}$$

and

$$L_{1/2} := a(y)\partial_y^2 + a(x)\partial_y^2 + \big(a'(x)/2 - b(x)\big)\partial_x + \big(a'(y)/2 - b(y)\big)\partial_y \tag{19}$$

(*on $\mathring{E}^*$ and their natural extensions on $\mathbb{R} \times \{+\infty\}$ and $\{-\infty\} \times \mathbb{R}$*). *Essentially relying on the fact that $\partial_x\partial_y h = 0$, one deduces $L_1[h] = 0$ from $\tilde{L}[h] = 0$ and $L_{1/2}[h] = 0$ from $L_{1/2} = (\tilde{L} + L_1)/2$.*

Note that the generator L^* described in (7) expands into

$$L^* = a(x)\partial_x^2 + a(y)\partial_y^2 - 2\sqrt{a(x)}\sqrt{a(y)}\partial_x\partial_y$$
$$+ \big(a'(x) - b(x)\big)\partial_x + \big(a'(y) - b(y)\big)\partial_y$$
$$+ 2\frac{\sqrt{a(x)}\pi(x) + \sqrt{a(y)}\pi(y)}{\pi([x,y])}\big(\sqrt{a(y)}\partial_y - \sqrt{a(x)}\partial_x\big).$$

It follows that on $\mathring{E}^*$, the stochastic differential equation for $Z^* = (X^*, Y^*)$ associated to (7) can be written as

$$dX_t^* = \left(a'(X_t^*) - b(X_t^*) - 2\frac{\sqrt{a(X_t^*)}\pi(X_t^*) + \sqrt{a(Y_t^*)}\pi(Y_t^*)}{\pi([X_t^*, Y_t^*])}\sqrt{a(X_t^*)}\right)dt - \sqrt{2a(X_t^*)}\,dB_t,$$
$$dY_t^* = \left(a'(Y_t^*) - b(Y_t^*) + 2\frac{\sqrt{a(X_t^*)}\pi(X_t^*) + \sqrt{a(Y_t^*)}\pi(Y_t^*)}{\pi([X_t^*, Y_t^*])}\sqrt{a(Y_t^*)}\right)dt + \sqrt{2a(Y_t^*)}\,dB_t,$$

where $B = (B_t)_{t\geq 0}$ is a standard (one dimensional) Brownian motion. Starting from an initial condition in $\mathring{E}^*$, the regularity of the coefficients and standard results (see for instance the book of Ikeda and Watanabe [20]) show that the solution Z^* exists and is unique up to the explosion time $\tau^\dagger$ (a.s. with respect to B). This stopping time for Z^* is defined by

$$\tau^\dagger := \min(\tau_1, \tau_2, \tau_3), \tag{20}$$

where

$$\tau_1 := \lim_{r\to+\infty} \inf\{t \geq 0 : X_t^* < -r\},$$
$$\tau_2 := \lim_{r\to+\infty} \inf\{t \geq 0 : Y_t^* > r\},$$
$$\tau_3 := \lim_{r\to+\infty} \inf\{t > 0 : Y_t^* - X_t^* < 1/r\}.$$

Of course, we have $\tau^\dagger \leq \tau^*$, where τ^* is defined in (10). The next result shows that τ_3 plays no role.

Lemma 11. *Let Z^* start from an initial condition in $\mathring{E}^*$. Then a.s. $h(Z_t^*)$ converges as t goes to $\tau^\dagger$ toward a positive quantity. In particular $\tau^\dagger = \min(\tau_1, \tau_2)$ and Z^* can exit $\mathring{E}^*$ only through* $(\mathbb{R} \times \{+\infty\}) \sqcup (\{-\infty\} \times \mathbb{R}) \sqcup \{(-\infty, +\infty)\}$.

Proof. According to Lemma 9, we have on $\mathring{E}^*$, $L^*[1/h] = \tilde{L}(\mathbb{1})/h = 0$, where $\mathbb{1}$ is the function always taking the value 1 on $\mathring{E}^*$. It follows that the process $M = (M_t)_{t\geq 0}$, defined by

$$\forall t \geq 0, \quad M_t := \frac{1}{h(Z^*_{\tau^\dagger \wedge t})},$$

is a local martingale. Since it is furthermore positive, it is a positive supermartingale and thus must converge as t goes to infinity (see for instance Theorem 28 page 24 of the book of Dellacherie and Meyer [11]). The announced results follow. □

We can now obtain the equivalent of Proposition 4 but for initial conditions in $\mathring{E}^*$.

Proposition 12. *For any $z_0 \in \mathring{E}^*$, there is a continuous Markov process $Z^* := (Z_t^*)_{t\geq 0}$ starting from z_0 and whose generator is L^*. The law of this process is unique if we impose that after the possibly finite time τ^*, defined as in* (10), *Z^* stays at position $(-\infty, +\infty)$. Furthermore for all $t \geq 0$, $Z_t^* \in E^* \setminus D^*$.*

Proof. According to the previous arguments, we already have the existence and uniqueness of Z^* up to the time $\tau^\dagger$. If $\tau^\dagger = +\infty$, the construction is over. If $\tau^\dagger < +\infty$, we deduce from Lemma 11 that either $\tau_1 = \tau^\dagger < +\infty$, or $\tau_2 = \tau^\dagger < +\infty$. We only consider the first case, the second can be treated in the same way. By the required continuity of the trajectories, we must have $Z^*_{\tau^\dagger} = (X^*_{\tau_1}, +\infty)$, where $X^*_{\tau_1} \in [-\infty, +\infty)$. We first consider the case where $X^*_{\tau_1} \neq -\infty$. By the assumption on the form of L^* on $\mathbb{R} \times \{+\infty\}$, Z^* must stay there after time $\tau^\dagger$. Let us denote for any $t \geq 0$, $X_t^\dagger := X^*_{\tau_1+t}$. The process $X^\dagger$ must be (and is constructed as) a solution of the one-dimensional stochastic differential equation

$$dX_t^\dagger = \left(a'(X_t^\dagger) - b(X_t^\dagger) - 2\frac{a(X_t^\dagger)\pi(X_t^\dagger)}{\pi([X_t^\dagger, +\infty))}\right)dt - \sqrt{2a(X_t^\dagger)}\,dB_t$$

(where $B=(B_t)_{t\geq 0}$ is a standard Brownian motion), starting from $X^*_{\tau_1}$. Due to the regularity of the coefficients, there is no difficulty in getting existence and uniqueness of the solution up to the time

$$\bar{\tau}:=\lim_{r\to+\infty}\inf\{t\geq 0: |X^\dagger_t|>r\}.$$

As in the proof of Lemma 11, the process $M^\dagger=(M^\dagger_t)_{t\geq 0}$, defined by

$$\forall t\geq 0,\quad M^\dagger_t:=\frac{1}{h((X^\dagger_{\bar{\tau}\wedge t},+\infty))},$$

is a positive local martingale. From its convergence we deduce that

$$\lim_{t\to\bar{\tau}-}X^\dagger_t=-\infty$$

and it follows that $\tau^\dagger+\bar{\tau}=\tau^*$. Note that this identity is trivial if $X^*_{\tau_1}=-\infty$. The analogue result is satisfied in the situation $\tau_2=\tau^\dagger$. Thus the law of $Z^*_{[0,\tau^*)}$ is uniquely determined and since we impose that $Z^*_t=(-\infty,+\infty)$ for $t\geq\tau^*$ (by continuity for $t=\tau^*$), the law of the whole process Z^* is also uniquely determined. The fact that $Z^*_t\in E^*\setminus D^*$ for all $t\geq 0$ is obvious from the previous martingale arguments. □

For $z_0\in\mathring{E}^*$, designate by $\mathbb{P}_{z_0}$ the law on the set of trajectories $\mathcal{C}(\mathbb{R}_+,E^*)$ of Z^* starting from z_0 and constructed as above. One way to construct $\mathbb{P}_{z_0}$ for $z_0=(x_0,x_0)\in D^*$, is to consider for $\varepsilon,\varepsilon'>0$, $\mathbb{P}_{x_0-\varepsilon,x_0+\varepsilon'}$ and to let ε,ε' go to zero. To make clearer the convergence, we will consider a transformation of $\mathcal{C}(\mathbb{R}_+,E^*)$ so that all the difficulties are encapsulated into a Bessel process of dimension 3.

Here is how it appears: under $\mathbb{P}_{z_0}$ for some $z_0\in\mathring{E}^*$, consider

$$\varsigma:=2\int_0^{\tau^*}\left(\sqrt{a(X^*_s)}\pi(X^*_s)+\sqrt{a(Y^*_s)}\pi(Y^*_s)\right)^2 ds\in(0,+\infty] \tag{21}$$

(with the convention $\sqrt{a(\pm\infty)}\pi(\pm\infty)=0$), and the time change $(\theta_t)_{t\in[0,\varsigma]}$ defined by

$$\forall t\in[0,\varsigma],\quad 2\int_0^{\theta_t}\left(\sqrt{a(X^*_s)}\pi(X^*_s)+\sqrt{a(Y^*_s)}\pi(Y^*_s)\right)^2 ds=t. \tag{22}$$

We are interested in the process $R:=(R_t)_{t\geq 0}$ given by

$$\forall t\geq 0,\quad R_t:=h(Z^*_{\theta_{t\wedge\varsigma}}). \tag{23}$$

Proposition 13. *Under $\mathbb{P}_{z_0}$ with $z_0\in\mathring{E}^*$, R has the law of a Bessel process of dimension* 3 *starting from $h(z_0)\in(0,1)$ and stopped at* 1. *In particular ς is distributed as the first hitting time of* 1 *for this process.*

Proof. We begin by computing $L^*[h]$: in view of Lemma 9 we have on $\mathring{E}^*$,

$$\begin{aligned}L^*[h]&=\frac{1}{h}\tilde{L}[h^2]\\&=\frac{1}{h}\big(2h\tilde{L}[h]+\tilde{\Gamma}[h,h]\big)\\&=\frac{1}{h}\tilde{\Gamma}[h,h].\end{aligned}$$

Taking into account the stochastic differential equations satisfied by the coordinates X^* and Y^* of Z^*, Itô's formula give us

$$dh(Z^*_t)=\frac{1}{h(Z^*_t)}\tilde{\Gamma}[h,h](Z^*_t)\,dt+\left(\sqrt{2a(X^*_t)}\pi(X^*_t)+\sqrt{2a(Y^*_t)}\pi(Y^*_t)\right)dB_t.$$

In Lemma 9 we have already seen that

$$\forall z=(x,y)\in \mathring{E}^*,\quad \tilde{\Gamma}[h,h](z)=2\big(\sqrt{a(x)}\pi(x)+\sqrt{a(y)}\pi(y)\big)^2.$$

Classical stochastic time change calculus (cf. for instance Chapter 5 of the book [31] of Revuz and Yor) then shows that the process R satisfies for $t<\varsigma$:

$$dR_t=\frac{d\theta_t}{dt}\frac{1}{h(Z^*_{\theta_t})}\tilde{\Gamma}[h,h]\big(Z^*_{\theta_t}\big)\,dt+\sqrt{\frac{d\theta_t}{dt}}\sqrt{\tilde{\Gamma}[h,h]\big(Z^*_{\theta_t}\big)}\,dW_t,$$

where $W=(W_t)_{t\geq 0}$ is a standard Brownian motion. From the definition of the time change $(\theta_t)_{t\in[0,\varsigma)}$, we have

$$\forall t\in[0,\varsigma),\quad \frac{d\theta_t}{dt}=\frac{1}{\tilde{\Gamma}[h,h](Z^*_{\theta_t})},$$

so we end up with

$$dR_t=\frac{1}{R_t}\,dt+dW_t.$$

One recognizes the stochastic differential equation characterizing the Bessel process of dimension 3 (see e.g. Chapter 11 of the book [31] of Revuz and Yor). Since Z^* is stopped when it reaches $(-\infty,+\infty)$, namely when $h(Z^*)$ hits 1, R is stopped when it reaches 1, which shows the assertions of the proposition. □

Here is a first consequence of the previous result:

Corollary 14. *We have almost surely,*

$$\lim_{t\to\tau^*-}X^*_t=-\infty,$$
$$\lim_{t\to\tau^*-}Y^*_t=+\infty.$$

Proof. From (21) and (22), we get that as t converges to $\varsigma-$, θ_t converges to τ^*-. It follows that

$$\lim_{t\to\tau^*-}h\big(Z^*_t\big)=\lim_{t\to\varsigma-}R_t=1.$$

Recalling the definition of h given in (14), this is possible if and only if the limits described in the above corollary take place. □

There is no difficulty in letting a Bessel process of dimension 3 start from 0. The idea behind the proof of Proposition 4 is to come back to this situation by distorting the state space of Z^*, so that the length of segment-valued process becomes a Bessel process. This transformation of martingale problem associated to L^*, suggested by Proposition 13, enables us to solve it when the starting point z_0 is on the diagonal. It also appears that the solution is the limit of the unique (in law) Markov processes associated to L^* and starting from points off the diagonal but converging to z_0. Conversely, any solution of the martingale problem associated to L^* and starting from z_0 can be approximated in this way (because of the requirements that the trajectories are continuous and that the process never comes back to the diagonal), feature which leads to the uniqueness of this solution.

Proof of Proposition 4. First we remark that it is sufficient to show that for any $x_0\in\mathbb{R}$, there is a continuous Markov process $(Z^*_t)_{t\in[0,\tau^\dagger)}$, where $\tau^\dagger$ is defined as in (20), starting from (x_0,x_0), living in $E^*\setminus D^*$ for $t\in(0,\tau^\dagger)$, and whose generator is L^*. Furthermore, we will check that the law of this process is unique. Indeed, the proof of Proposition 12 could next be used again to uniquely extend $(Z^*_t)_{t\in[0,\tau^\dagger)}$ into $(Z^*_t)_{t\geq 0}$. This observation brings us back to the martingale problem associated to the initial condition (x_0,x_0) and to the restriction of the generator L^* to $\mathring{E}^*$.

But we begin by replacing (x_0, x_0) by $(x_0-\varepsilon, x_0+\varepsilon')$, with $\varepsilon, \varepsilon' > 0$, and we consider the time change described in (22). This amounts to replacing the generator L^* by $\widehat{L} := (1/\tilde{\Gamma}(h,h))L^*$. Or, equivalently, we could apply the following transformation to the trajectories

$$\big(Z^*_t\big)_{t\in[0,\tau^\dagger)} \mapsto (\widehat{Z}_t)_{t\in[0,\varsigma)} := \big(Z^*_{\theta_t}\big)_{t\in[0,\varsigma)},$$

where

$$\varsigma := 2\int_0^{\tau^\dagger} \Big(\sqrt{a\big(X^*_s\big)}\pi\big(X^*_s\big) + \sqrt{a\big(Y^*_s\big)}\pi\big(Y^*_s\big)\Big)^2\, ds \in (0,+\infty]$$

and $(\theta_t)_{t\in[0,\varsigma)}$ is defined as in (22). The reverse mapping is given by

$$(\widehat{Z}_t)_{t\in[0,\varsigma)} \mapsto \big(Z^*_t\big)_{t\in[0,\tau^\dagger)} := (\widehat{Z}_{\vartheta_t})_{t\in[0,\tau^\dagger)},$$

where

$$\tau^\dagger := \frac{1}{2}\int_0^{\varsigma} \Big(\sqrt{a(\widehat{X}_s)}\pi(\widehat{X}_s) + \sqrt{a(\widehat{Y}_s)}\pi(\widehat{Y}_s)\Big)^{-2}\, ds \in (0,+\infty]$$

and

$$\forall t\in[0,\tau^\dagger),\quad \frac{1}{2}\int_0^{\vartheta_t} \Big(\sqrt{a(\widehat{X}_s)}\pi(\widehat{X}_s) + \sqrt{a(\widehat{Y}_s)}\pi(\widehat{Y}_s)\Big)^{-2}\, ds = t.$$

The stochastic differential equation for $(\widehat{Z}_t)_{t\in[0,\varsigma)} = (\widehat{X}_t, \widehat{Y}_t)_{t\in[0,\varsigma)}$ associated to $\widehat{L}$ on $\mathring{E}^*$ is given by

$$\forall t\in[0,\varsigma),\quad \begin{cases} d\widehat{X}_t = b_1(\widehat{X}_t, \widehat{Y}_t)\, dt + \sigma_1(\widehat{X}_t, \widehat{Y}_t)\, dB_t, \\ d\widehat{Y}_t = b_2(\widehat{X}_t, \widehat{Y}_t)\, dt + \sigma_2(\widehat{X}_t, \widehat{Y}_t)\, dB_t, \end{cases}$$

where for any $(x,y)\in\mathring{E}^*$,

$$\begin{aligned} b_1(x,y) &:= \frac{a'(x)-b(x)}{2(\sqrt{a(y)}\pi(y)+\sqrt{a(x)}\pi(x))^2} - \frac{\sqrt{a(x)}}{(\sqrt{a(y)}\pi(y)+\sqrt{a(x)}\pi(x))\pi([x,y])}, \\ b_2(x,y) &:= \frac{a'(y)-b(y)}{2(\sqrt{a(y)}\pi(y)+\sqrt{a(x)}\pi(x))^2} + \frac{\sqrt{a(y)}}{(\sqrt{a(y)}\pi(y)+\sqrt{a(x)}\pi(x))\pi([x,y])}, \\ \sigma_1(x,y) &:= -\frac{\sqrt{a(x)}}{\sqrt{a(y)}\pi(y)+\sqrt{a(x)}\pi(x)}, \\ \sigma_2(x,y) &:= \frac{\sqrt{a(y)}}{\sqrt{a(y)}\pi(y)+\sqrt{a(x)}\pi(x)}. \end{aligned}$$

Finally we consider the transformation Ψ of the state space $\mathring{E}^*\sqcup D^*$ given by

$$\mathring{E}^*\sqcup D^* \ni (x,y) \mapsto \big(h(x,y), \mu(x,y)\big),$$

where $\mu(x,y)$ is the middle point of $[x,y]$ when $\mathbb{R}$ is endowed with the Riemannian structure for which $a\partial^2$ is the Laplace–Beltrami operator. More prosaically, $\mu(x,y)$ is defined as the unique point in $[x,y]$ such that

$$\int_x^{\mu(x,y)} \frac{1}{\sqrt{a(u)}}\, du = \int_{\mu(x,y)}^y \frac{1}{\sqrt{a(u)}}\, du.$$

Its main interest is that

$$\forall (x,y)\in\mathring{E}^*,\quad \sqrt{a(y)}\partial_y\mu(x,y) - \sqrt{a(x)}\partial_x\mu(x,y) = 0,$$

because

$$\partial_x \mu(x,y) = \frac{1}{2}\sqrt{\frac{a(\mu(x,y))}{a(x)}} \quad \text{and} \quad \partial_y \mu(x,y) = \frac{1}{2}\sqrt{\frac{a(\mu(x,y))}{a(y)}}. \tag{24}$$

It is not difficult to see that Ψ is a smooth diffeomorphism from $\mathring{E}^* \sqcup D^*$ to its image. Denote $(R_t, S_t)_{t\in[0,\varsigma)} := (\Psi(\widehat{Z}_t))_{t\in[0,\varsigma)}$. From Proposition 13 and (24) we deduce that the stochastic differential equation satisfied by $\widehat{Z}_{t\in[0,\varsigma)}$ is transformed into

$$\forall t \in [0,\varsigma), \quad \begin{cases} dR_t = \frac{1}{R_t}\,dt + dW_t, \\ dS_t = \beta(R_t, S_t)\,dt, \end{cases} \tag{25}$$

where $W = (W_t)_{t\geq 0}$ is a standard Brownian motion and where the mapping β is defined on $\Psi(\mathring{E}^* \sqcup D^*)$ by

$$\begin{aligned} \beta\big(\Psi(x,y)\big) &:= \left(b_1\partial_x\mu + b_2\partial_y\mu + \frac{1}{2}\big[\sigma_1^2\partial_x^2\mu + \sigma_2^2\partial_y^2\mu + \sigma_1\sigma_2\partial_x\partial_y\mu\big]\right)(x,y) \\ &= \frac{\sqrt{a(\mu(x,y))}}{8(\sqrt{a(y)}\pi(y) + \sqrt{a(x)}\pi(x))^2}\left(\frac{a'(x) - 2b(x)}{\sqrt{a(x)}} + \frac{a'(y) - 2b(y)}{\sqrt{a(y)}}\right), \end{aligned}$$

for any $(x,y) \in \mathring{E}^* \sqcup D^*$. This function is clearly smooth on its domain. So the resolution of (25) is quite obvious. The initial condition is $(R_0, S_0) = \Psi((x_0 - \varepsilon, x_0 + \varepsilon'))$. Then one solves the autonomous stochastic differential equation satisfied by $R := (R_t)_{t\geq 0}$. The solution R is defined for all $t \geq 0$, and as it was more precisely seen in Proposition 13, it is a Bessel process of dimension 3 starting from $h(x_0 - \varepsilon, x_0 + \varepsilon') > 0$. The trajectory R being constructed, it remains to investigate the ordinary differential equation $\frac{dS_t}{dt} = \beta(R_t, S_t)$, starting from S_0. Since β is smooth, it gives us a solution, up to the possible explosion time ς when $(R_{\varsigma-}, S_{\varsigma-})$ reaches the boundary of $\Psi(\mathring{E}^* \sqcup D^*)$. From the form of Ψ, the time ς is necessarily the first time when either $\widehat{X}$ explodes to $-\infty$ or $\widehat{Y}$ explodes to $+\infty$, where $(\widehat{X}_t, \widehat{Y}_t) := \Psi^{-1}(R_t, S_t)$ for $t \in [0,\varsigma)$, as wanted.

These observations are also valid if $R_0 = 0$ and enable the construction of $\mathbb{P}_{(x_0,x_0)}$ by reversing the previous transformations, starting from the initial condition $(R_0, S_0) = (0, x_0)$. It is also seen to be the limit of $\mathbb{P}_{(x_0-\varepsilon,x_0+\varepsilon')}$ as $\varepsilon, \varepsilon' > 0$ converge to zero. In the last sentence, the weak convergence of the probability measures is with respect to the uniform convergence of the trajectories over compact time intervals, when the state space E^* is endowed with a bounded distance compatible with its natural topology (inherited from that of the compact set $[-\infty, +\infty]^2$). This continuity property and the requirements made on $\mathbb{P}_{(x_0,x_0)}$ in Proposition 4 enable us to be convinced of its uniqueness. Indeed, consider $\mathbb{P}$ another probability on the set of trajectories $\mathcal{C}(\mathbb{R}_+, E^*)$ satisfying the same properties. On $\mathcal{C}(\mathbb{R}_+, E^*)$ consider the natural time-shift maps Θ_t, for $t \geq 0$: if $(Z_s^*)_{s\geq 0}$ stands for the canonical coordinate process, we have $Z_s^*(\Theta_t) = Z_{t+s}^*$ for all $t, s \geq 0$. Let F be a bounded and continuous functional on $\mathcal{C}(\mathbb{R}_+, E^*)$. By the Markov property we must have that for any $t > 0$,

$$\mathbb{E}\big[F(\Theta_t)\big] = \int \mathbb{E}_z[F]\,m_t(dz),$$

where m_t is the law of Z_t^* under $\mathbb{P}$. By the requirements that Z_t^* belongs to $\mathbb{E}^* \setminus D^*$ a.s. under $\mathbb{P}$ and that $\mathbb{P}$ is a solution to the martingale problem associated to L^*, the expectations $\mathbb{E}_z$ in the r.h.s. are relative to the laws constructed in Proposition 12. By the continuity of the trajectories and the assumption that $Z_0^* = (x_0, x_0)$ under $\mathbb{P}$, m_t converges weakly to the Dirac mass at (x_0, x_0) as t goes to 0_+. Thus $\lim_{t\to 0_+} \int \mathbb{E}_z[F]m_t(dz) = \mathbb{E}_{(x_0,x_0)}[F]$ by the continuity of $z \mapsto \mathbb{P}_z$ at (x_0, x_0). On the other hand, by the dominated convergence theorem, $\lim_{t\to 0_+} \mathbb{E}[F(\Theta_t)] = \mathbb{E}[F]$. Thus $\mathbb{E}[F] = \mathbb{E}_{(x_0,x_0)}[F]$ for all bounded and continuous functional F on $\mathcal{C}(\mathbb{R}_+, E^*)$. This is sufficient to insure that $\mathbb{P} = \mathbb{P}_{(x_0,x_0)}$ and ends the proof of Proposition 4. □

Remark 15. *Proposition 12 and its proof are also valid for the generators defined in (11) and (12) and more generally for the generators $L_\alpha^* := (1-\alpha)L^* + \alpha L_1^*$, where $\alpha \in [0,1]$. But Proposition 4 is not true for L_1^*: as it was mentioned in the introduction, due to the regularity of the coefficients of L_1^*, the unique solutions $\mathbb{P}_{1,z}$ for the corresponding* *martingale problem can be directly constructed for all the initial conditions $z \in E^*$ and the mapping $z \mapsto \mathbb{P}_z$ is continuous. Unfortunately, starting from $z \in D^*$, the process cannot escape from D^*, except by possibly exploding at one of its two ends (Lemma 11 is not helping to prevent this event: $h(Z_t^*)$ remains null). Indeed in this degenerate situation one may have to add the two absorbing points $(-\infty, -\infty)$ and $(+\infty, +\infty)$ to the state space E^*.*

This problem is not encountered by the generators L_α^*, for $\alpha \in [0, 1)$, to which the above considerations (corresponding to the case $\alpha = 0$) can be extended. Let us put a corresponding index α to all the objects we have considered so far when L^* is replaced by L_α^*. For instance we introduce the generator $\tilde{L}_\alpha := (1-\alpha)\tilde{L} + \alpha L_1$ and we compute that its squared field operator $\tilde{\Gamma}_\alpha$ satisfies

$$\begin{aligned}\forall z = (x, y) \in \mathring{E}^*, \quad \tilde{\Gamma}_\alpha[h, h](z) &= 2\big(\sqrt{a(y)}\pi(y) + \sqrt{a(x)}\pi(x)\big)^2 - 8\alpha\sqrt{a(x)a(y)}\pi(x)\pi(y) \\ &= \tilde{\Gamma}[h, h](z) - 8\alpha\sqrt{a(x)a(y)}\pi(x)\pi(y).\end{aligned} \tag{26}$$

It leads us to replace (21), (22) and (23) respectively by

$$\varsigma_\alpha := \int_0^{\tau^*} \tilde{\Gamma}_\alpha[h, h]\big(Z_s^*\big)\, ds \in (0, +\infty], \tag{27}$$

$$\forall t \in [0, \varsigma_\alpha], \quad \int_0^{\theta_{\alpha,t}} \tilde{\Gamma}_\alpha[h, h]\big(Z_s^*\big)\, ds = t$$

and

$$\forall t \geq 0, \quad R_{\alpha,t} := h\big(Z^*_{\theta_{\alpha,t\wedge\varsigma_\alpha}}\big).$$

The interest of the latter process is that under $\mathbb{P}_{\alpha,z_0}$, it is again a square process of dimension 3 starting from $h(z_0)$ and stopped at 1. The proof is identical to that of Proposition 13.

But from (26), we get that for any $z \in E^* \setminus D^*$, the quantity $\tilde{\Gamma}_\alpha[h, h](z)$ is non-increasing in $\alpha \in [0, 1)$ (it is decreasing when $z \in \mathring{E}^*$). It follows from (27), that for any fixed $z_0 \in \mathring{E}^* \sqcup D^*$, if $\alpha_1 < \alpha_2 \in [0, 1)$, then the law of τ^* under $\mathbb{P}_{\alpha_1,z_0}$ is strictly larger than the law of τ^* under $\mathbb{P}_{\alpha_2,z_0}$, with respect to the usual stochastic ordering of laws on $\mathbb{R}_+ \sqcup \{+\infty\}$. Hence among all the generators L_α^* for $\alpha \in [0, 1)$, $L^* = L_0^*$ leads to the dual process Z^* to be the fastestly absorbed at $(-\infty, +\infty)$ and thus is the most adequate for our purpose of constructing relatively small stationary times for L. In words, the mirror-symmetry coupling of the Brownian motions at the boundary of the evolving segment is optimal and the identical coupling is the worst (being utterly useless for the evolving segments starting from a singleton). The heuristic intuition behind the mirror-symmetry coupling being the best one is that the underlying Brownian motion should help the boundaries to expand the segment, since the drift is taking care of the segment not shrinking too much.

3. Explosion times

Our main objective here is to prove the finiteness assertion of Proposition 5. The arguments are based on comparisons with some appropriate diffusions on half-lines.

Consider $Z^* = (X^*, Y^*)$ the process described in Proposition 4 (for some fixed $x_0 \in \mathbb{R}$) and constructed in the previous section. We are interested in the (total) explosion time τ^* defined in (10) and our main task is to show that it is almost surely finite if $I < +\infty$. So let us consider the (partial) explosion times

$$\tau^- := \inf\big\{t \geq 0 : X_t^* = -\infty\big\},$$

$$\tau^+ := \inf\big\{t \geq 0 : Y_t^* = +\infty\big\}.$$

For our purpose it is sufficient to show the following result (recall that I_- and I_+ were defined just above the statement of Theorem 1).

Proposition 16. *If $I_+ < +\infty$, then τ^+ is a.s. finite.*

Indeed, by symmetry it will follow that if $I_- < +\infty$, then τ^- is a.s. finite, so that $\tau^* = \tau^- \vee \tau^+ < +\infty$ a.s. if $I < +\infty$.

The proof of Proposition 16 relies on the comparison of Y^* with a diffusion $U := (U_t)_{t\geq 0}$ taking values in $\mathbb{R}_+ \sqcup \{+\infty\}$, reflected at 0, absorbed at $+\infty$ and whose generator on $(0, +\infty)$ is $a\partial^2 - (b - a' + 2ak')\partial$, where k is the mapping $\mathbb{R} \ni x \mapsto \ln(\pi((-\infty, x]))$. More precisely, we take for U the solution of the stochastic differential equation

$$dU_t = \big(a'(U_t) - b(U_t) + 2a(U_t)k'(U_t)\big)\, dt + \sqrt{2a(U_t)}\, dB_t + dl_t(U), \tag{28}$$

up to the explosion time $\tau(U) = \inf\{t \geq 0 : U_t = +\infty\}$, where $(l_t(U))_{t\geq 0}$ is the local time of U at 0 and where $B = (B_t)_{t\geq 0}$ is the same standard Brownian motion as the one driving the s.d.e. satisfied by Y^*

$$dY_t^* = \left(a'\big(Y_t^*\big) - b\big(Y_t^*\big) + 2\frac{\sqrt{a(X_t^*)}\pi(X_t^*) + \sqrt{a(Y_t^*)}\pi(Y_t^*)}{\pi([X_t^*, Y_t^*])}\sqrt{a\big(Y_t^*\big)}\right) dt + \sqrt{2a\big(Y_t^*\big)}\, dB_t,$$

for $t \leq \tau^+$ (with the natural modification of the drift term if $X_t^* = -\infty$). The interest is that the quantity $[\sqrt{a(X_t^*)}\pi(X_t^*) + \sqrt{a(Y_t^*)}\pi(Y_t^*)]\sqrt{a(Y_t^*)}/\pi([X_t^*, Y_t^*]) - a(Y_t^*)k'(Y_t^*)$ is non-negative and even positive for $0 < t < \tau^-$. So if U and Y^* started from the same initial condition $u_0 \in (0, +\infty)$, then U stays below Y^* up to the time

$$T := \inf\{t \geq 0 : U_t = 0\}$$

and this is true whatever the behavior of X^*:

Lemma 17. *For all $t \in [0, T]$, we have $U_t \leq Y_t^*$.*

As usual, this assertion has to be understood a.s., but not to burden the presentation, this is assumed to be implicit from now on. Note also that after the time T, the local time $(l_t(U))_{t\geq 0}$ starts to play a role and U can end being above Y^*.

Proof. This kind of comparison result is standard, see for instance Section 1 of Chapter 6 of the book of Ikeda and Watanabe [20]. Nevertheless, we find more illuminating to present a simple and direct proof than to check their assumptions via localizing arguments.

It is convenient to first transform $\mathbb{R} \sqcup \{-\infty, +\infty\}$ via the mapping A given by

$$\forall u \in \mathbb{R}_+ \sqcup \{+\infty\}, \quad A(u) := \int_0^u \frac{1}{\sqrt{2a(v)}}\, dv.$$

Next we consider the processes $\tilde{U} := (\tilde{U}_t)_{t\geq 0} := (A(U_t))_{t\geq 0}$ and $\tilde{Y} := (\tilde{Y}_t)_{t\geq 0} := (A(Y_t^*))_{t\geq 0}$. Owing to Itô's formula, for $t \in [0, T \wedge \tau^+)$, they satisfies respectively the s.d.e.

$$d\tilde{U}_t = f(\tilde{U}_t)\, dt + dB_t,$$
$$d\tilde{Y}_t = \big(f(\tilde{Y}_t) + S_t\big)\, dt + dB_t,$$

where

$$\forall u \in \mathbb{R}, \quad f(u) := \big(a'(u) - b(u) + 2a(u)k'(u)\big)A'(u) + a(u)A''(u)$$

and $S := (S_t)_{t\geq 0}$ is the previsible process given by

$$\begin{aligned}\forall t \geq 0, \quad S_t &:= \sqrt{2}\frac{\sqrt{a(X_t^*)}\pi(X_t^*) + \sqrt{a(Y_t^*)}\pi(Y_t^*)}{\pi([X_t^*, Y_t^*])} - \sqrt{2a\big(Y_t^*\big)}k'\big(Y_t^*\big) \\ &\geq \sqrt{2}\frac{\sqrt{a(Y_t^*)}\pi(Y_t^*)}{\pi([X_t^*, Y_t^*])} - \sqrt{2a\big(Y_t^*\big)}\frac{\pi(Y_t^*)}{\pi((-\infty, Y_t^*])}\end{aligned}$$

$$= \sqrt{2a(Y_t^*)}\frac{\pi(Y_t^*)\pi((-\infty, X_t^*))}{\pi([X_t^*, Y_t^*])\pi((-\infty, Y_t^*])}$$
$$\geq 0.$$

As already mentioned, what is important is this non-negativity of S. Consider

$$\sigma := \inf\{t \in [0, T) : \tilde{U}_t > \tilde{Y}_t\},$$

with the usual convention that $\sigma := +\infty$ if the set in the r.h.s. is empty. We proceed by contradiction: assume that $\sigma < T$ (and in particular σ is finite). Necessarily we also have $\sigma < \tau^+$, because $\tilde{Y}_t = A(+\infty) \geq A(U_t) = \tilde{U}_t$ for all $t \geq \tau^+$. By continuity $\tilde{U}_\sigma = \tilde{Y}_\sigma$ and we consider two cases:

- If $\sigma < \tau^-$, then $S_\sigma > 0$, thus there exists $\varepsilon > 0$ such that for $s \in [\sigma, \sigma + \varepsilon]$, $f(\tilde{Y}_s) + S_s - f(\tilde{U}_s) > 0$. From the above s.d.e. we deduce that for all $\varepsilon' \in (0, \varepsilon]$,

$$\tilde{Y}_{\sigma+\varepsilon'} - \tilde{U}_{\sigma+\varepsilon'} = \int_\sigma^{\sigma+\varepsilon'} f(\tilde{Y}_s) + S_s - f(\tilde{U}_s)\, ds > 0$$

  and this contradicts the definition of σ. It follows that $\tilde{U}_t \leq \tilde{Y}_t$ for all $t \in [0, T)$ and by continuity this is also true for $t = T$.
- If $\sigma \geq \tau^-$: for $t \geq \tau^-$, $S_t = 0$, so $(\tilde{U}_t)_{\tau^- \leq t \leq T\wedge\tau^+}$ and $(\tilde{Y}_t)_{\tau^- \leq t \leq T\wedge\tau^+}$ follow the same s.d.e. whose coefficients are regular. Since $\tilde{U}_\sigma = \tilde{Y}_\sigma$, the local uniqueness of the solution of their s.d.e. implies that $\tilde{U}$ and $\tilde{Y}$ keep on being equal for some time after σ and this is again contradictory with the definition of σ. □

The advantage of the process U is that its explosion time $\tau(U)$ is well-understood, as we deduce from Theorem 3.2 of Chapter 6 of the book of Ikeda and Watanabe [20] the following criterion:

Proposition 18. *The explosion time $\tau(U)$ is finite almost surely if and only if $I_+ < +\infty$.*

Proof. The most convenient way to exploit Section 3 of Chapter 6 of the book of Ikeda and Watanabe [20] seems to symmetrize U: consider the functions $\widehat{a}$ and $\widehat{b}$ defined by

$$\forall x \in \mathbb{R}, \quad \begin{cases} \widehat{a}(x) := \begin{cases} a(x), & \text{if } x \geq 0, \\ a(-x), & \text{if } x < 0, \end{cases} \\ \widehat{b}(x) := \begin{cases} a'(x) - b(x) + 2a(x)k'(x), & \text{if } x \geq 0, \\ -\widehat{b}(-x), & \text{if } x < 0, \end{cases} \end{cases}$$

to which we associate the operator

$$\widehat{L} := \widehat{a}\partial^2 + \widehat{b}\partial.$$

Since $\widehat{a}$ is continuous and positive and $\widehat{b}$ is measurable and locally bounded, we can use Theorem 3.3 of Chapter 4 of the book of Ikeda and Watanabe [20] and usual localization procedures to obtain, for any given starting point $v \in \mathbb{R}$, the existence and uniqueness of the solution $V := (V_t)_{0\leq t\leq\tau(V)}$ of the s.d.e. associated to v and $\widehat{L}$:

$$\begin{cases} V_0 = v, \\ dV_t = \widehat{b}(V_t)\, dt + \sqrt{2\widehat{a}(V_t^*)}\, dB_t, \end{cases}$$

up to the explosion time $\tau(V) := \inf\{t \geq 0 : \lim_{s\to t-} |V_s| = +\infty\}$, and where $B = (B_t)_{t\geq 0}$ is a standard Brownian motion. Tanaka's formula (e.g. Chapter 6 of the book [31] of Revuz and Yor) enables us to see that $(|V_t|)_{0\leq t<\tau(V)}$ coincides in law with the process $(U_t)_{0\leq t<\tau(U)}$ starting from $|v|$. Formally, if Theorem 3.2(3) of Chapter 6 of the book of Ikeda and Watanabe [20] is applied (take $c = 0$ there), we get that the a.s. finiteness of $\tau(U) = \tau(V)$ (independently of the initial condition) is equivalent to

$$\int_0^{+\infty} \exp\left(-\int_0^x \frac{\widehat{b}(y)}{\widehat{a}(y)}\, dy\right) \int_0^x \exp\left(\int_0^z \frac{\widehat{b}(u)}{\widehat{a}(u)}\, du\right) \frac{dz}{\widehat{a}(z)}\, dx < +\infty. \tag{29}$$

Taking into account the expressions for $\widehat{a}$ and $\widehat{b}$, we compute that

$$\forall x \in \mathbb{R}_+, \quad \int_0^x \frac{\widehat{b}(y)}{\widehat{a}(y)}\,dy = \int_0^x \frac{a'(y) - b(y) + 2a(y)k'(y)}{a(y)}\,dy$$
$$= \ln\left(\frac{a(x)}{a(0)}\right) + \ln\big(s(x)\big) + 2\big(k(x) - k(0)\big),$$

so that the l.h.s. of (29) is proportional to the quantity

$$\int_0^{+\infty}\left(\int_0^x \big(\pi\big((-\infty, y]\big)\big)^2 S(dy)\right)\frac{1}{(\pi((-\infty, x]))^2}\pi(dx).$$

Since $0 < \pi((-\infty, 0]) \le \pi((-\infty, x]) \le 1$ for $x \in \mathbb{R}_+$, the finiteness of the previous expression is equivalent to that of I_+. The only problem is that the coefficients $\widehat{a}$ and $\widehat{b}$ were required to be of class $\mathcal{C}^1$ by Ikeda and Watanabe. But one can check directly in Section 3 of Chapter 6 of their book [20] that the proof extends to the situation where the lack of regularity is restricted to 0, where $\widehat{a}$ is assumed to be continuous and positive and $\widehat{b}$ locally bounded. Alternatively, one can come back to the smooth situation in the following way. Define

$$\forall x \in \mathbb{R} \sqcup \{-\infty, +\infty\}, \quad \tau(V, x) := \inf\{t \ge 0 : V_t = x\}.$$

As a consequence of the Markov property and of the symmetry of V, the fact that $\tau(V)$ is finite a.s., whatever the initial point, is equivalent to

$$\begin{cases} \mathbb{P}[\tau(V, -2) \wedge \tau(V, 2) < +\infty | V_0 = 0] = 1, \\ \mathbb{P}[\tau(V, +\infty) < \tau(V, 1) | V_0 = 2] > 0. \end{cases} \tag{30}$$

The Girsanov transformation enables us to see that the first of these conditions is true as soon as $\widehat{a}$ is continuous and positive on $[-2, 2]$ and $\widehat{b}$ is bounded on $[-2, 2]$. The second condition is not affected by modifications of $\widehat{a}$ and $\widehat{b}$ in $(-2, 2)$. So we can first apply Theorem 3.2(3) of Chapter 6 of the book of Ikeda and Watanabe [20] to symmetric smoothings of $\widehat{a}$ and $\widehat{b}$ in $(-2, 2)$ (this does not change the condition $I_+ < +\infty$ either) and next deduce the same conclusion for the original process V via (30). $\square$

Now we have at our disposal all the ingredients necessary to the proof of Proposition 16. So let us assume that $I_+ < +\infty$.

We begin by defining the following stopping times.

$$\tilde{\sigma}_0 := \inf\big\{t \ge 0 : Y_t^* \ge 1\big\}.$$

By Corollary 14 we already know that $\lim_{t\to+\infty} Y_t^* = +\infty$ so that $\tilde{\sigma}_0$ is finite a.s. Next consider

$$\widehat{\sigma}_0 := \inf\big\{t > \tilde{\sigma}_0 : Y_t^* = +\infty \text{ or } Y_t^* = 0\big\}.$$

Since $I_+ < +\infty$, we deduce from Lemma 17 and Proposition 18 that $\widehat{\sigma}_0$ is finite a.s. and we have either $Y^*_{\widehat{\sigma}_0} = +\infty$ or $Y^*_{\widehat{\sigma}_0} = 0$. Indeed knowing the trajectory $Z^*_{[0,\tilde{\sigma}_0]}$, the conditional probability that $Y^*_{\widehat{\sigma}_0} = +\infty$ is bounded below by $\mathbb{P}[\tau(U) < \tau(U, 0) | U_0 = Y^*_{\tilde{\sigma}_0}]$, where $\tau(U, 0) = \inf\{t \ge 0 : U_t = 0\}$. Since the mapping $\mathbb{R}_+ \ni x \mapsto \mathbb{P}[\tau(U) < \tau(U, 0) | U_0 = x]$ is non-decreasing, we get that

$$\mathbb{P}\big[Y^*_{\widehat{\sigma}_0} = +\infty | \mathcal{F}^*_{\tilde{\sigma}_0}\big] \ge p_* := \mathbb{P}\big[\tau(U) < \tau(U, 0) | U_0 = 1\big] > 0,$$

where $\mathcal{F}^*_\sigma$ will designate the σ-field associated to the stopping time σ in the filtration generated by the process Z^*: more explicitly $\mathcal{F}^*_\sigma$ is generated by the piece of trajectory $Z^*_{[0,\sigma]}$ (see e.g. Chapter 1 of the book [31] of Revuz and

Yor). It follows that $\mathbb{P}[Y^*_{\widehat{\sigma}_0} = +\infty] \geq p_*$. If $Y^*_{\widehat{\sigma}_0} = +\infty$, we set $N = 0$ and otherwise the value of the random variable N will be defined later on in the procedure. Indeed, if $Y^*_{\widehat{\sigma}_0} = 0$, we consider

$$\tilde{\sigma}_1 := \inf\{t > \widehat{\sigma}_0 : Y^*_t = 1\},$$
$$\widehat{\sigma}_1 := \inf\{t > \tilde{\sigma}_1 : Y^*_t = +\infty \text{ or } Y^*_t = 0\}.$$

These stopping times are again a.s. finite (still conditionally on $Y^*_{\widehat{\sigma}_0} = 0$). If $Y^*_{\widehat{\sigma}_1} = +\infty$, we set $N = 1$. Note that as before,

$$\mathbb{P}\big[Y^*_{\widehat{\sigma}_1} = +\infty | \mathcal{F}^*_{\widehat{\sigma}_0}, Y^*_{\widehat{\sigma}_0} = 0\big] \geq p_*.$$

The construction goes on similarly: if for some $n \in \mathbb{N}$, $\widehat{\sigma}_n$ has been defined, we set $N = n$ if $Y^*_{\widehat{\sigma}_n} = +\infty$ and the procedure stops. Otherwise, namely if $Y^*_{\widehat{\sigma}_n} = 0$, we consider the a.s. finite random times

$$\tilde{\sigma}_{n+1} := \inf\{t > \widehat{\sigma}_n : Y^*_t = 1\},$$
$$\widehat{\sigma}_{n+1} := \inf\{t > \tilde{\sigma}_{n+1} : Y^*_t = +\infty \text{ or } Y^*_t = 0\}$$

and we set $N = n + 1$ if $Y^*_{\widehat{\sigma}_{n+1}} = +\infty$. The previous arguments show that

$$\mathbb{P}\big[Y^*_{\widehat{\sigma}_{n+1}} = +\infty | \mathcal{F}^*_{\widehat{\sigma}_n}, Y^*_{\widehat{\sigma}_n} = 0\big] \geq p_*.$$

The validity of this property for all $n \in \mathbb{Z}^+_0$ implies that N is stochastically bounded below by a geometric random variable of parameter $1 - p_* < 1$:

$$\forall n \in \mathbb{Z}^+_0, \quad \mathbb{P}[N \geq n] \leq (1 - p_*)^n.$$

In particular, N is a.s. finite as well as $\tau^+ = \widehat{\sigma}_N$. This ends the proof of Proposition 16 and the finiteness assertion of Proposition 5. As explained in the introduction, this implies that τ is a strong stationary time for X, once X and Z^* are intertwined through Λ.

4. Intertwining

In the two previous sections, the process Z^* has been studied in some details. It is time now to check that it can be intertwined with the initial one-dimensional positive recurrent diffusion X.

We begin by verifying that the commutation relation (6) is satisfied with L^* defined by (7), (8) and (9).

Lemma 19. *For any $f \in \mathcal{C}^2(\mathbb{R})$ such that f and $L[f]$ belong to $\mathbb{L}^1(\pi)$, we have*

$$\forall z \in E^* \setminus \big(D^* \sqcup \{(-\infty, +\infty)\}\big), \quad \Lambda\big[L[f]\big](z) = L^*\big[\Lambda[f]\big](z).$$

Proof. A priori there are three situations to be considered $z \in \mathring{E}^*$, $z \in \{-\infty\} \times \mathbb{R}$ and $z \in \mathbb{R} \times \{+\infty\}$. We are to deal only with the first case, the other ones being similar (and even easier). So let $f \in \mathcal{C}^2(\mathbb{R})$ be given (the integrability assumptions are needed only for $z \in \{-\infty\} \times \mathbb{R}$ and $z \in \mathbb{R} \times \{+\infty\}$ to insure the integrability of f and $L[f]$ with respect to π on semi-infinite intervals). For $z := (x, y) \in \mathbb{R}^2$ with $x < y$, we have

$$\Lambda[f](z) = \frac{1}{h(x, y)} \int_x^y f(u)\pi(du),$$

where h was defined in (14). Taking into account Lemma 9, we get that

$$L^*\big[\Lambda[f]\big](z) = \frac{1}{h(z)} \tilde{L}[F](z),$$

where $\tilde{L}$ was given in (13) and where F is the function defined on E^* by

$$\forall (x', y') \in E^*, \quad F(x', y') := \int_{x'}^{y'} f(u)\pi(du). \tag{31}$$

For $(x, y) \in \mathring{E}^*$, $\partial_x F(x, y) = -\pi(x) f(x)$ and $\partial_y F(x, y) = \pi(y) f(y)$, so that we get that

$$\begin{aligned} \tilde{L}[F](x, y) &= \left(\sqrt{a(y)}\partial_y - \sqrt{a(x)}\partial_x\right)\left(\sqrt{a(y)}\pi(y) f(y) + \sqrt{a(x)}\pi(x) f(x)\right) \\ &\quad - \left(a'(x)/2 - b(x)\right)\pi(x) f(x) + \left(a'(y)/2 - b(y)\right)\pi(y) f(y) \\ &= \sqrt{a(y)}\partial_y\left(\sqrt{a(y)}\pi(y) f(y)\right) - \sqrt{a(x)}\partial_x\left(\sqrt{a(x)}\pi(x) f(x)\right) \\ &\quad - \left(a'(x)/2 - b(x)\right)\pi(x) f(x) + \left(a'(y)/2 - b(y)\right)\pi(y) f(y) \\ &= a(y)\pi(y)\partial_y f(y) - a(x)\pi(x)\partial_y f(x) - g(x) f(x) + g(y) f(y), \end{aligned} \tag{32}$$

where g is the function defined by

$$\forall x \in \mathbb{R}, \quad g(x) := \sqrt{a(x)}\partial_x\left(\sqrt{a(x)}\pi(x)\right) + \left(a'(x)/2 - b(x)\right)\pi(x).$$

Recalling the definition of π given in (5), we compute that g vanishes identically, so that we obtain

$$\forall (x, y) \in \mathring{E}^*, \quad L^*\left[\Lambda[f]\right](x, y) = \frac{1}{h(x, y)}\left(a(y)\pi(y)\partial_y f(y) - a(x)\pi(x)\partial_y f(x)\right).$$

We turn now to the computation of $\Lambda[L[f]]$ on $\mathring{E}^*$. Note that L can be factorized into

$$L\cdot = a\exp\left(\ln(s)\right)\partial\left(\exp\left(-\ln(s)\right)\partial\cdot\right) = \frac{1}{\pi}\partial(a\pi\partial\cdot).$$

It follows that for all $f \in \mathcal{C}^2(\mathbb{R})$ and $(x, y) \in \mathring{E}^*$,

$$\begin{aligned} \int_x^y L[f](u)\pi(du) &= \int_x^y \partial(a\pi\partial f)(u)\,du \\ &= a(y)\pi(y) f'(y) - a(x)\pi(x) f'(x). \end{aligned}$$

The wanted commutation relation follows at once, on $\mathring{E}^*$. □

Remark 20. *If in the above proof $\tilde{L}$ is replaced by the generators L_1 or $L_{1/2}$ defined respectively by* (18) *and* (19) (*on $\mathring{E}^*$ and their natural extensions on $\mathbb{R} \times \{+\infty\}$ and by $\{-\infty\} \times \mathbb{R}$*), *the same computations are still valid. Indeed, in* (32) *the cross differentiation $\partial_x\partial_y$ vanishes, meaning that on $\mathring{E}^*$ and for the function F defined in* (31),

$$\tilde{L}[F] = L_1[F] = L_{1/2}[F]$$

(*simpler considerations are also valid on $\{-\infty\} \times \mathbb{R} \sqcup \mathbb{R} \times \{+\infty\}$*). *The commutation relations $\Lambda L = L_1^*\Lambda$ and $\Lambda L = L_{1/2}^*\Lambda$ for the generators L_1^* and $L_{1/2}^*$* (*defined respectively in* (11) *and* (12)) *are then also true, because these operators are the h-transforms of L_1 and $L_{1/2}$, as it was mentioned in Remark* 10. *This justifies the assertions made after Proposition* 4 *in the Introduction.*

Even if some of the subsequent developments could be extended to these generators, recall that their interest is limited, due to the observations made in Remark 15.

We are now going to lift the commutation relation of Lemma 19 to the level of the corresponding semi-groups. More precisely, let $(P_t)_{t\geq 0}$ be the semi-group associated to L. From a probabilistic point of view, it is constructed in the following way. For any $x \in \mathbb{R}$, consider $(X_t)_{t\in\mathbb{R}}$ the solution starting from x of the s.d.e.

$$dX_t = b(X_t)\,dt + \sqrt{2a(X_t)}\,dB_t, \tag{33}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion. Then for any $t \geq 0$ and any bounded and continuous mapping f on $\mathbb{R}$, we have

$$P_t[f](x) = \mathbb{E}_x\big[f(X_t)\big].$$

The semi-group $(P_t^*)_{t\geq 0}$ can be constructed similarly. For $z \in E^* \cap \mathbb{R} \times \mathbb{R}$, consider the process Z^* starting from z defined in Proposition 4 or 12, depending if $z \in D^*$ or not (if $z = (-\infty, +\infty)$, Z^* stays forever at $(-\infty, +\infty)$). For $z \in \{-\infty\} \times \mathbb{R}$ or $z \in \mathbb{R} \times \{+\infty\}$, Z^* is constructed as explained in the proof of Proposition 12. Then for any $t \geq 0$ and any bounded and continuous mapping f on E^*, we take

$$P_t^*[f](z) = \mathbb{E}_z\big[f\big(Z_t^*\big)\big].$$

Proposition 21. *Assume that X is positive recurrent. Then for all $T \geq 0$ and all bounded and continuous function f on $\mathbb{R}$, we have*

$$\forall z \in E^*, \quad \Lambda\big[P_T[f]\big](z) = P_T^*\big[\Lambda[f]\big](z).$$

Formally, writing $P_t = \exp(tL)$ and $P_t^* = \exp(tL^*)$, the deduction of these commutation relations from their infinitesimal version given in Lemma 19 may seem clear. Nevertheless a direct rigorous justification does not seem so obvious (see Remark 23 below). We found it preferable to follow a recurrent idea in the study of semi-groups à la Bakry [3] and Ledoux [23].

Proof of Proposition 21. It consists in investigating the evolution of

$$[0, T] \ni t \mapsto P_t^*\big[\Lambda\big[P_{T-t}[f]\big]\big],$$

for given $T > 0$ and first for $f \in \mathcal{C}_c^\infty(\mathbb{R})$.

We begin by recalling how to exploit the martingale property of Z^*. A function defined on E^* is said to be $\mathcal{C}^2$ if it is continuous on E^* and if it is $\mathcal{C}^2$ on $\mathring{E}^*$, on $\{-\infty\} \sqcup \mathbb{R}$ and on $\mathbb{R} \sqcup \{+\infty\}$. Similarly, a continuous function defined on $\mathbb{R}_+ \times E^*$ is said to be $\mathcal{C}^{1,2}$ if it is $\mathcal{C}^1$ with respect with the first variable in $\mathbb{R}_+$ and $\mathcal{C}^2$ with respect to the second variable in E^*, the corresponding partial derivatives being continuous on $\mathbb{R}_+ \times \mathring{E}^*$, on $\mathbb{R}_+ \times (\{-\infty\} \times \mathbb{R})$ and on $\mathbb{R}_+ \times (\mathbb{R} \times \{+\infty\})$. Denote by $\mathcal{C}_b^{1,2}(\mathbb{R}_+ \times E^*)$ the set of such functions F which are furthermore bounded, as well as the mapping $\mathbb{R}_+ \times (E^* \setminus (D^* \sqcup \{(-\infty, +\infty)\})) \ni (t, z) \mapsto \partial_t F(t, z) + L^*[F(t, \cdot)](z)$. Let us prove that for any $z \in E^* \setminus D^*$, $t \geq 0$ and $F \in \mathcal{C}_b^{1,2}(\mathbb{R}_+ \times E^*)$,

$$\mathbb{E}_z\big[F\big(t \wedge \tau^*, Z_{t\wedge\tau^*}^*\big)\big] = F(0, z) + \mathbb{E}_z\bigg[\int_0^{t\wedge\tau^*} \partial_s F\big(s, Z_s^*\big) + L^*\big[F(s, \cdot)\big]\big(Z_s^*\big)\, ds\bigg]. \tag{34}$$

First we treat the case where $z = (x, y) \in \mathring{E}^*$ and we replace τ^* by $\tau^\dagger$ which was defined in (20). Indeed, for $n \in \mathbb{N}$ large enough, say $n \geq n_0$, where $n_0 \in \mathbb{N}$ is such that $y - x > 1/n_0$, consider

$$\begin{aligned}
\tau_1(n) &:= \inf\big\{t \geq 0 : X_t^* < -n\big\},\\
\tau_2(n) &:= \inf\big\{t \geq 0 : Y_t^* > n\big\},\\
\tau_3(n) &:= \inf\big\{t \geq 0 : Y_t^* - X_t^* < 1/n\big\},\\
\tau^\dagger(n) &:= \min\big(\tau_1(n), \tau_2(n), \tau_3(n)\big),
\end{aligned}$$

where $Z^* = (X_t^*, Y_t^*)_{t\geq 0}$. The sequence $(\tau^\dagger(n))_{n\geq n_0}$ is a localizing sequence for Z^* on the random time interval $[0, \tau^\dagger)$, in the sense that

$$\tau^\dagger = \lim_{n\to\infty} \tau^\dagger(n)$$

and for any $F \in \mathcal{C}_{\mathrm{b}}^{1,2}(\mathbb{R}_+ \times E^*)$, we can write

$$\forall t \geq 0, \quad F\big(t \wedge \tau^\dagger(n), Z^*_{t\wedge\tau^\dagger(n)}\big) = F(0,z) + \int_0^{t\wedge\tau^\dagger(n)} \partial_s F\big(s, Z^*_s\big) + L^*\big[F(s,\cdot)\big]\big(Z^*_s\big)\,ds + M_t,$$

where for any $n \geq n_0$, the process $(M_{t\wedge\tau^\dagger(n)})_{t\geq 0}$ is a martingale starting from 0.

So taking expectations, we end up with

$$\mathbb{E}_z\big[F\big(t \wedge \tau^\dagger(n), Z^*_{t\wedge\tau^\dagger(n)}\big)\big] = F(0,z) + \mathbb{E}_z\bigg[\int_0^{t\wedge\tau^\dagger(n)} \partial_s F\big(s, Z^*_s\big) + L^*\big[F(s,\cdot)\big]\big(Z^*_s\big)\,ds\bigg].$$

Our boundedness and continuity assumptions on F enable us to use the bounded convergence theorem to get

$$\mathbb{E}_z\big[F\big(t \wedge \tau^\dagger, Z^*_{t\wedge\tau^\dagger}\big)\big] = F(0,z) + \mathbb{E}_z\bigg[\int_0^{t\wedge\tau^\dagger} \partial_s F\big(s, Z^*_s\big) + L^*\big[F(s,\cdot)\big]\big(Z^*_s\big)\,ds\bigg]. \tag{35}$$

Recall from Lemma 11 that if $\tau^\dagger < +\infty$, then $Z^*_\dagger$ belongs to $\{(-\infty,+\infty)\} \sqcup (\{-\infty\} \times \mathbb{R}) \sqcup (\mathbb{R} \times \{+\infty\})$. Note also that if $\tau^\dagger < +\infty$ and $Z^*_{\tau^\dagger} = (-\infty,+\infty)$, then $\tau^* = \tau^\dagger$, so

$$\mathbb{E}_z\big[F\big(t \wedge \tau^\dagger, Z^*_{t\wedge\tau^\dagger}\big)\mathbb{1}_{\{\tau^\dagger \leq t, Z^*_{\tau^\dagger}=(-\infty,+\infty)\}}\big] = \mathbb{E}_z\big[F\big(t \wedge \tau^*, Z^*_{t\wedge\tau^*}\big)\mathbb{1}_{\{\tau^\dagger \leq t, Z^*_{\tau^\dagger}=(-\infty,+\infty)\}}\big].$$

Thus to prove (34), taking into account the strong Markov property at time $t \wedge \tau^\dagger$ (which is true by construction of Z^*), it is sufficient to see that for all $z \in (\{-\infty\} \times \mathbb{R}) \sqcup (\mathbb{R} \times \{+\infty\})$,

$$\mathbb{E}_z\big[F\big(t \wedge \tau^*, Z^*_{t\wedge\tau^*}\big)\big] = F(0,z) + \mathbb{E}_z\bigg[\int_0^{t\wedge\tau^*} \partial_s F\big(s, Z^*_s\big) + L^*\big[F(s,\cdot)\big]\big(Z^*_s\big)\,ds\bigg]. \tag{36}$$

This is immediate, following a localization procedure similar to that leading to (35).

Since $(-\infty,+\infty)$ is absorbing, if $\tau^* < t$ we can write for $F \in \mathcal{C}_{\mathrm{b}}^{1,2}(\mathbb{R}_+ \times E^*)$,

$$\begin{aligned} F\big(t, Z^*_t\big) &= F\big(t, (-\infty,+\infty)\big) \\ &= F\big(\tau^*, (-\infty,+\infty)\big) + \int_{\tau^*}^t \partial_s F\big(s, (-\infty,+\infty)\big)\,ds \\ &= F\big(\tau^*, Z^*_{\tau^*}\big) + \int_{\tau^*}^t \partial_s F\big(s, Z^*_s\big)\,ds, \end{aligned}$$

so that, recalling the Dirichlet condition for L^* at $(-\infty,+\infty)$, (36) can be transformed into

$$\mathbb{E}_z\big[F\big(t, Z^*_t\big)\big] = F(0,z) + \mathbb{E}_z\bigg[\int_0^t \partial_s F\big(s, Z^*_s\big) + L^*\big[F(s,\cdot)\big]\big(Z^*_s\big)\,ds\bigg],$$

namely in semi-group notations,

$$P^*_t\big[F(t,\cdot)\big](z) = F(0,z) + \int_0^t P^*_s\big[\partial_s F(s,\cdot) + L^*\big[F(s,\cdot)\big]\big](z)\,ds. \tag{37}$$

Let $T > 0$ and $f \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R})$ be fixed, we want to apply the previous considerations with the function F defined on $[0,T] \times E^*$ by

$$\forall (t,z) \in [0,T] \times E^*, \quad F(t,z) := \Lambda\big[P_{T-t}[f]\big](z).$$

Since $\mathbb{R}_+ \times \mathbb{R} \ni (t,x) \mapsto P_t[f](x)$ is well-known to be smooth, it is clear that F is $\mathcal{C}^{1,2}$. Furthermore, recall that the semi-group $(P_t)_{t\geq 0}$ can be extended into a self-adjoint continuous semi-group on $\mathbb{L}^2(\pi)$, whose generator is the

Friedrichs extension of L on $\mathbb{L}^2(\pi)$. It follows that the relation $\partial_t P_t[f] = LP_t[f]$ is satisfied in the usual sense and in $\mathbb{L}^2(\pi)$ and we get

$$\forall t \in [0, T], \forall z \in E^*, \quad \partial_t F(t, z) = -\Lambda\big[L\big[P_{T-t}[f]\big]\big](z). \tag{38}$$

Since the mapping $\mathbb{R} \ni x \mapsto P_{T-t}[f](x)$ is $\mathcal{C}^2$ and

$$\pi\big[\big|P_{T-t}[f]\big|\big] \le \pi\big[|f|\big],$$
$$\pi\big[\big|L\big[P_{T-t}[f]\big]\big|\big] = \pi\big[\big|P_{T-t}\big[L[f]\big]\big|\big] \le \pi\big[\big|L[f]\big|\big],$$

we are in position to apply Lemma 19 (with f replaced by $P_{T-t}[f]$) to get that in the r.h.s. of (38), we can replace $\Lambda[L[P_{T-t}[f]]](z)$ by $L^*[\Lambda[P_{T-t}[f]]](z)$, at least for $z \in E^* \setminus (D^* \sqcup \{(-\infty, +\infty)\})$. Thus we get that

$$\forall t \in [0, T], \forall z \in E^* \setminus \big(D^* \sqcup \big\{(-\infty, +\infty)\big\}\big), \quad \partial_t F(t, z) + L^*\big[F(t, \cdot)\big](z) = 0.$$

This relation is also true for $z = (-\infty, +\infty)$. Indeed, due to the fact that X is positive recurrent, we get

$$\forall t \ge 0, \quad F\big(t, (-\infty, +\infty)\big) = \pi\big[P_t[f]\big] = \pi[f],$$

so that

$$\partial_t F\big(t, (-\infty, +\infty)\big) = 0. \tag{39}$$

In particular it is licit to apply (37) (for $t \in [0, T]$) to get

$$\forall z \in E^*, \quad P_T^*\big[F(T, \cdot)\big](z) = F(0, z),$$

which is just the conclusion stated in the proposition, at least for $f \in \mathcal{C}_c^\infty(\mathbb{R})$. To extend it to any bounded and continuous function f, note that for any fixed $T \ge 0$ and $z \in E^*$, the mappings

$$\mathcal{R} \ni A \mapsto \Lambda\big[P_T[\mathbb{1}_A]\big](z) \quad \text{and} \quad \mathcal{R} \ni A \mapsto P_T^*\big[\Lambda[\mathbb{1}_A]\big](z)$$

($\mathcal{R}$ stands for the σ-algebra of Borel subsets of $\mathbb{R}$) define two probability measures. Because they coincide on every $f \in \mathcal{C}_c^\infty(\mathbb{R})$, they must be equal. □

Remark 22. *The assumption that X is positive recurrent is really necessary for the previous result. Indeed, there exists generators L satisfying* (3) *but not* (4). *For the associated semi-group $(P_t)_{t\ge 0}$, for any time $T > 0$ and any point $x \in \mathbb{R}$ we have $P_T[\mathbb{1}](x) < 1$. As a consequence, for any $T > 0$ and $z \in E^*$, $\Lambda[P_T[\mathbb{1}]](z) < 1$, while by construction $P_T^*[\Lambda[\mathbb{1}]](z) = P_T^*[\mathbb{1}](z) = 1$.*

In the above proof, the positive recurrence of X is encapsulated in (39).

Remark 23. *When F doesn't depend on the time variable,* (37) *can be written in the familiar form*

$$\partial_t P_t^*[F] = L^*\big[P_t^*[F]\big].$$

But from an analytical point of view, it is not clear a priori in which Banach space one should interpret this evolution equation to deduce the semi-group $(P_t^*)_{t\ge 0}$ from L^*. If we were to work with the elliptic generator $L_{1/2}^*$ defined in (12), *there is a natural $\mathbb{L}^2$ Hilbert setting. Indeed, let $\tilde\eta$ be the σ-finite measure on $\mathbb{R}$ whose density with respect to the Lebesgue measure is* $\exp(-c)$. *The generator $L_{1/2}$ given in* (19) *is then symmetric with respect to the measure η which coincides on E^* with the restriction of $(\tilde\eta + \delta_{-\infty} + \delta_{+\infty})^{\otimes 2}$. Since $L_{1/2}^*$ corresponds to the h-transform of $L_{1/2}$, it is symmetric relatively to the measure ν admitting h^2 as density with respect to η. Thus the relations $P_{1/2,t}^* = \exp(tL_{1/2}^*)$, for $t \ge 0$, could be given a meaning in $\mathbb{L}^2(\nu)$. Heuristically, the intertwining between L and L^* can be seen as "weak conjugation relation" between them, so we can expect that L^* is equally reversible with respect to some σ-finite measure on E^*. Unfortunately we have not been able to find it and in addition we have no idea about* *possible quasi-invariant measures of L^*. Nevertheless, we believe that this subject really deserves to be investigated further, especially from a quantitative point of view. An initiation of this program in a very particular case is presented in the next section.*

Proposition 21 is the main technical point to get the intertwined coupling of X with Z^*. Indeed, we can follow the construction of Diaconis and Fill [12] by applying it to skeletons of X with Z^*. Passing to the limit in the latter approximations will enable us to justify the arguments given before the statement of Proposition 5 in the introduction.

Assume m_0 and m_0^* are two probability measures, respectively on $\mathbb{R}$ and E^*, such that $m_0^*\Lambda = m_0$. We want to construct an intertwining of X with Z^* whose initial distribution is described by $\eta_0(dx, dz^*) := m_0^*(dz^*)\Lambda(z^*, dx)$ (in particular the laws of X_0 and Z_0^* are respectively m_0 and m_0^*). For fixed $N \in \mathbb{N}$, define a discrete time Markov chain $(\bar{X}^{(N)}_{n2^{-N}}, \bar{Z}^{(N,*)}_{n2^{-N}})_{n\in\mathbb{Z}_+}$, intertwined through Λ, in the following way: its initial distribution is η_0 and its transition kernel $Q^{(N)}$ is given by

$$Q^{(N)}\big(\big(x, z^*\big), d\big(\tilde{x}, \tilde{z}^*\big)\big) := P_{2^{-N}}(x, d\tilde{x}) P^*_{2^{-N}}\big(z^*, d\tilde{z}^*\big) \frac{\Lambda(\tilde{z}^*, d\tilde{x})}{\triangle_{2^{-N}}(z^*, d\tilde{x})}$$

(from $(x, z^*) \in \mathbb{R}_+ \times E^*$ to the infinitesimal neighborhood $d(\tilde{x}, \tilde{z}^*)$ of $(\tilde{x}, \tilde{z}^*) \in \mathbb{R}_+ \times E^*$), where the last ratio is the Radon–Nikodym derivative of the measure $\Lambda(\tilde{z}^*, d\tilde{x})$ with respect to $\triangle_{2^{-N}}(z^*, d\tilde{x}) := (P^*_{2^{-N}}\Lambda)(z^*, d\tilde{x}) = (\Lambda P_{2^{-N}})(z^*, d\tilde{x})$. One would have remarked that due to Propositions 4 and 12, for any $z^* \in E^*$, $\triangle_{2^{-N}}(z^*, \cdot)$ is equivalent to the Lebesgue measure. So except if $\tilde{z}^*$ corresponds to a singleton, we have $\Lambda(\tilde{z}^*, \cdot) \ll \triangle_{2^{-N}}(z^*, \cdot)$. But $P^*_{2^{-N}}(z^*, d\tilde{z}^*)$-a.s. $\tilde{z}^*$ does not correspond to a singleton, so $Q^{(N)}$ is indeed a transition kernel (not only a sub-Markovian kernel, if the mass of the singular part of $\Lambda(\tilde{z}^*, \cdot)$ with respect to $\triangle_{2^{-N}}(z^*, \cdot)$ was missing). The computations of Diaconis and Fill [12] can then be adapted to this setting, because of the structure of the initial distribution and of Proposition 21, to show that the Markov chain $(\bar{X}^{(N)}_{n2^{-N}}, \bar{Z}^{(N,*)}_{n2^{-N}})_{n\in\mathbb{Z}_+}$ thus constructed satisfies the following properties:

$$\big(\bar{X}^{(N)}_{n2^{-N}}\big)_{n\in\mathbb{Z}_+} \text{ and } (X_{n2^{-N}})_{n\in\mathbb{Z}_+} \text{ have the same law,} \tag{40}$$

$$\big(\bar{Z}^{(N,*)}_{n2^{-N}}\big)_{n\in\mathbb{Z}_+} \text{ and } \big(Z^*_{n2^{-N}}\big)_{n\in\mathbb{Z}_+} \text{ have the same law,} \tag{41}$$

$$\forall m \in \mathbb{Z}_+, \text{ the conditional law of } \bar{X}^{(N)}_{m2^{-N}} \text{ knowing } \bar{Z}^{(N,*)}_0, \bar{Z}^{(N,*)}_{2^{-N}}, \ldots, \bar{Z}^{(N,*)}_{m2^{-N}} \text{ is } \Lambda\big(\bar{Z}^{(N,*)}_{m2^{-N}}, \cdot\big), \tag{42}$$

$$\begin{aligned}&\forall m \in \mathbb{Z}_+, \text{ the conditional law of } \big(\bar{Z}^{(N,*)}_0, \bar{Z}^{(N,*)}_{2^{-N}}, \ldots, \bar{Z}^{(N,*)}_{m2^{-N}}\big) \text{ knowing } \big(\bar{X}^{(N)}_{n2^{-N}}\big)_{n\in\mathbb{Z}_+}\\ &\quad \text{only depends on } \bar{X}^{(N)}_0, \bar{X}^{(N)}_{2^{-N}}, \ldots, \bar{X}^{(N)}_{m2^{-N}}.\end{aligned} \tag{43}$$

Next we embed the Markov chain $(\bar{X}^{(N)}_{n2^{-N}}, \bar{Z}^{(N,*)}_{n2^{-N}})_{n\in\mathbb{Z}_+}$ into the (time-inhomogeneous) Markov process $(\bar{X}^{(N)}, \bar{Z}^{(N,*)}) := (\bar{X}^{(N)}_t, \bar{Z}^{(N,*)}_t)_{t\in\mathbb{R}_+}$, by taking

$$\forall t \geq 0, \quad \big(\bar{X}^{(N)}_t, \bar{Z}^{(N,*)}_t\big) := \big(\bar{X}^{(N)}_{\lfloor t2^N\rfloor 2^{-N}}, \bar{Z}^{(N,*)}_{\lfloor t2^N\rfloor 2^{-N}}\big),$$

where $\lfloor\cdot\rfloor$ stands for the integer part.

Proposition 24. *The sequence of the laws of $(\bar{X}^{(N)}, \bar{Z}^{(N,*)})$, for $N \in \mathbb{N}$, on the Skorokhod space $\mathbb{D}(\mathbb{R}_+, \mathbb{R} \times E^*)$, is relatively compact. We can thus extract a subsequence converging to a probability measure $\mathbb{P}$ which is necessarily supported by the set of continuous trajectories. Under $\mathbb{P}$ the canonical coordinate process $(\bar{X}_t, \bar{Z}^*_t)_{t\in\mathbb{R}_+}$ is a coupling of X with Z^* satisfying for all $t \in \mathbb{R}_+$,*

$$\text{the conditional law of } \bar{X}_t \text{ knowing } \bar{Z}^*_{[0,t]} \text{ is } \Lambda\big(\bar{Z}^*_t, \cdot\big), \tag{44}$$

$$\text{the conditional law of } \bar{Z}^*_{[0,t]} \text{ knowing } \bar{X} \text{ depends only on } \bar{X}_{[0,t]}. \tag{45}$$

Proof. Using traditional properties of the Skorokhod topology on the Polish space $\mathbb{D}(\mathbb{R}_+, \mathbb{R} \times E^*)$ (see for instance the book [5] of Billingsley), we deduce from (40) and (41) that the laws of $\bar{X}^{(N)}$ and $\bar{Z}^{(N,*)}$ converge respectively toward those of X and Z^*. This observation implies without difficulty the first three assertions of the above proposition. For the last two ones, note that as consequences of (42) and (43), we have

$$\forall t \geq 0, \text{ the conditional law of } \bar{X}_t^{(N)} \text{ knowing } \bar{Z}_{[0,t]}^{(N,*)} \text{ is } \Lambda\big(\bar{Z}_t^{(N,*)}, \cdot\big),$$

$$\forall t \geq 0, \text{ the conditional law of } \bar{Z}_{[0,t]}^{(N,*)} \text{ knowing } \bar{X}^{(N)} \text{ depends only on } \bar{X}_{[0,t]}^{(N)}.$$

The deduction of (44) and (45) is then a standard exercise on conditional expectations: use on one hand that the σ-algebra generated by $\xi_{[0,t]}$, where $t \geq 0$ and ξ is either $\bar{X}$ or $\bar{Z}^*$, is the same as that generated by mappings of the form $F(\xi_{t_1}, \ldots, \xi_{t_r})$, where $r \in \mathbb{N}$, $t_1, \ldots, t_r$ are dyadic numbers satisfying $0 \leq t_1 < \cdots < t_r \leq t$ and F is a bounded and continuous function on either $\mathbb{R}^r$ or $(E^*)^r$, and on the other hand that such mappings are $\mathbb{P}$-a.s. continuous. □

Remark 25. *Pal and Shkolnikov* [29] *investigated the existence of intertwinings between diffusion semi-groups whose generators are appropriately linked by a Markov kernel. Unfortunately the assumptions of their Theorem* 3 *do not cover our situation, essentially due to the lack of ellipticity of Z^*. The intertwining of L with $L^*_{1/2}$ (defined in* (12)) *is more amenable to their conditions, after looking at X through the chart $\mathbb{R} \ni s \mapsto \int_0^s a^{-1/2}(u)\,du$ (and correspondingly for Z^*). Nevertheless it would still remain to check their boundary conditions. In the approach presented above, we escaped the corresponding delicate description of what happens to the intertwined process (X, Z^*) when X enters in contact with one of the boundaries of Z^* by resorting to the computations of Diaconis and Fill* [12] *applied to the skeleton chains. A similar approach is discussed in Fill and Lyzinski* [18], *for diffusions starting from the boundary.*

Proposition 24 enables us to prove the direct part of Theorem 1 along the arguments given before Proposition 4. To end this section, we show the converse implication, by considering the diffusion X whose initial distribution is π conditioned to be on $\mathbb{R}_-$ (namely $\Lambda((-\infty, 0), \cdot)$, the cases where the initial distribution is $\Lambda((-\infty, x), \cdot)$ or $\Lambda((x, +\infty), \cdot)$, for some $x \in \mathbb{R}$, can be treated similarly).

In this situation the process Z^* has the form $(-\infty, Y^*)$, where Y^* is the solution starting from 0 of the s.d.e.

$$dY_t^* = \left(a'\big(Y_t^*\big) - b\big(Y_t^*\big) + 2\frac{\sqrt{a(Y_t^*)}\pi(Y_t^*)}{\pi((-\infty, Y_t^*])}\sqrt{a\big(Y_t^*\big)}\right)dt + \sqrt{2a\big(Y_t^*\big)}\,dB_t.$$

Because $X^* \equiv -\infty$, this evolution of Y^* is the same as that of the process U defined in (28), except that U is forced to stay non-negative. From Corollary 14, we know a priori that $\lim_{t\to+\infty} Y_t^* = +\infty$. So there is a random time after which the evolutions of Y^* and U are the same. As in Section 3 with Proposition 18, one can conclude that the boundary $+\infty$ will be reached by Y^* in finite time (a.s.) if and only if $I_+ < +\infty$. The hitting time of $+\infty$ by Y^* is indeed the random time τ^* defined in (10). If we assume that X admits a strong stationary time and if we show that such a strong stationary time is stochastically larger than τ^*, we would then get that $I_+ < +\infty$. Symmetrically we would prove that the existence of a strong stationary time for X starting from $\Lambda((0, +\infty), \cdot)$ implies that $I_- < +\infty$ and the converse part of Theorem 1 will be shown. Thus according to (16), it remains to check that

Lemma 26. *Under the previous assumption on X, we have*

$$\forall t \geq 0, \quad \mathfrak{s}\big(\mathcal{L}(X_t), \pi\big) = \mathbb{P}\big[\tau^* > t\big].$$

The following arguments are an adaptation to our setting of Remark 2.39 of Diaconis and Fill [12].

Proof. Consider the intertwining of X and $Z^* = (-\infty, Y^*)$ obtained in Proposition 24. It follows that for all $t \geq 0$,

$$\mathcal{L}(X_t) = \mathbb{E}\big[\Lambda\big(\big(-\infty, Y_t^*\big), \cdot\big)\big].$$

In particular, we get that

$$\mathfrak{s}\big(\mathcal{L}(X_t), \pi\big) = \sup_{x\in\mathbb{R}} \mathbb{E}\left[1 - \frac{d\Lambda((-\infty, Y_t^*), \cdot)}{d\pi}(x)\right]$$
$$= 1 - \inf_{x\in\mathbb{R}} \mathbb{E}\left[\frac{d\Lambda((-\infty, Y_t^*), \cdot)}{d\pi}(x)\right].$$

The above Radon–Nikodym derivative is easy to compute: for all $x \in \mathbb{R}$,

$$\frac{d\Lambda((-\infty, Y_t^*), \cdot)}{d\pi}(x) = \frac{1}{\pi((-\infty, Y_t^*))}\mathbb{1}_{(-\infty, Y_t^*)}(x).$$

Note that the r.h.s. is non-increasing as a function of $x \in \mathbb{R}$, so the same is true of the expression $\mathbb{E}[\frac{d\Lambda((-\infty,Y_t^*),\cdot)}{d\pi}(x)]$ and we get

$$\mathfrak{s}\big(\mathcal{L}(X_t), \pi\big) = 1 - \lim_{x\to+\infty} \mathbb{E}\left[\frac{d\Lambda((-\infty, Y_t^*), \cdot)}{d\pi}(x)\right]$$
$$= 1 - \mathbb{P}\big[Y_t^* = +\infty\big]$$
$$= \mathbb{P}\big[Y_t^* < +\infty\big]$$
$$= \mathbb{P}\big[\tau^* > t\big]. \qquad \square$$

5. On the Ornstein–Uhlenbeck counter-example

In the study of convergence to equilibrium for diffusions, the Ornstein–Uhlenbeck process is a benchmark, in particular due to its Gaussian feature which enables explicit computations. Unfortunately, according to Example 8, it is just outside the domain of validity of the assumption $I < +\infty$ of Proposition 5. We will see here how the method can nevertheless be adapted to recover sharp information.

The Ornstein–Uhlenbeck process corresponds to the choice in (1) of $a \equiv 1$ and $b(x) = -x$, for all $x \in \mathbb{R}$. The associated reversible measure is the centered and standard Gaussian distribution γ whose density is given by $\gamma(x) = \exp(-x^2/2)/\sqrt{2\pi}$, for all $x \in \mathbb{R}$. A traditional integration by parts leads to

$$\gamma\big([x, +\infty)\big) \sim \frac{\gamma(x)}{x},$$

where $\sim$ means that the ratio of the l.h.s. to the r.h.s. converges to 1 as x goes to $+\infty$. It follows that the second integral of the l.h.s. of (17) is infinite. Theorem 1 then asserts that there exist initial distributions for which it is not possible to construct strong stationary times for the associated process X. Indeed, this is true as soon as the initial distribution m_0 has a compact support. To see it, let us recall how the law $\mathcal{L}(X_t)$ is easily computed in this situation: since X satisfies the s.d.e.

$$\forall t \geq 0, \quad dX_t = -X_t\,dt + \sqrt{2}\,dB_t$$

(where $B = (B_t)_{t\geq 0}$ is a standard Brownian motion), the variation of parameters method gives us:

$$X_t = \exp(-t)X_0 + \sqrt{2}\int_0^t \exp(s-t)\,dB_s.$$

It follows that $m_t := \mathcal{L}(X_t)$ is the convolution of $\mathcal{L}(\exp(-t)X_0)$ with $\gamma_{1-\exp(-2t)}$, the centered Gaussian distribution of variance $1 - \exp(-2t) = 2\int_0^t \exp(2(s-t))\,ds$. Thus if m_0 has compact support, we get that for any fixed $t > 0$, the separation discrepancy of m_t with γ is one:

$$\mathfrak{s}(m_t, \gamma) = \lim_{|x|\to+\infty} 1 - \frac{dm_t}{d\gamma}(x) = 1$$

(a similar reasoning, considering only the limit at $-\infty$ or $+\infty$, would lead to the same conclusion if the support of m_0 is bounded below or above: this enables us to include the initial distributions considered for the reverse part of Theorem 1). The bound (16) then implies that there is no strong stationary time for X.

To simplify the presentation, we will assume that the initial distribution is the Dirac mass at 0. We deduce from the above considerations that for any $t > 0$, $\mathcal{L}(X_t) = \gamma_{1-\exp(-2t)}$. In particular $\mathcal{L}(X_t)$ converges toward γ in total variation. Let us check that the exponential rate for this convergence is 2:

Lemma 27. *We have*

$$\lim_{t\to+\infty} \frac{1}{t} \ln\bigl(\|m_t - \gamma\|_{\mathrm{tv}}\bigr) = -2.$$

Proof. By one of the characterization of the total variation norm, we have for all $t > 0$,

$$\|m_t - \gamma\|_{\mathrm{tv}} = \int (f_t - 1)_+ \, d\gamma, \tag{46}$$

where f_t is the Radon–Nikodym derivative of m_t with respect to γ. We compute that

$$\forall x \in \mathbb{R}, \quad f_t(x) = \bigl(1 - e^{-2t}\bigr)^{-1/2} \exp\left(-\frac{e^{-2t}x^2}{2(1-e^{-2t})}\right)$$

and we deduce that

$$f_t(x) \geq 1 \quad \Longleftrightarrow \quad |x| \leq x_t := \sqrt{\bigl(1-e^{2t}\bigr)\ln\bigl(1-e^{-2t}\bigr)}.$$

The quantity x_t converges toward 1 when t goes to infinity. A simple expansion of the expression $f_t(x) - 1$ then leads to

$$\begin{aligned} \|m_t - \gamma\|_{\mathrm{tv}} &= 2\int_0^{x_t} f_t(x) - 1\gamma(dx) \\ &\sim e^{-2t} \int_0^1 1 - x^2\gamma(dx), \end{aligned}$$

for large $t > 0$. The announced result follows at once. $\square$

Remark 28. *The logarithmic Sobolev constant associated to L is* 4, *so starting from any initial distribution m_0 such that the relative entropy of m_t with respect to γ is finite for some $t \geq 0$, we get that the exponential rate of converge in the relative entropy sense is at least* 4. *Using next Pinsker's inequality, we recover that the above exponential rate of convergence in total variation is at least* 2. *For this traditional approach, see for instance the book* [2] *of Ané, Blachère, Chafaï, Fougères, Gentil, Malrieu, Roberto and Scheffer. The Ornstein–Uhlenbeck process is also critical for the use of the logarithmic Sobolev inequalities method, but it is in the "interior boundary" of the domain of application.*

Let us show how to recover this exponential rate 2 for the convergence in total variation by using strong (non-stationary) times. So the emphasis is on testing the method, not in the result itself. It will also enable us to illustrate on this example the directions suggested by Remark 23.

We begin by noting that the construction of the process $Z^* = (X^*, Y^*)$ made in Section 2 is still valid. By symmetry and since we are considering $Z_0^* = (0, 0)$, we have that $X^* = -Y^*$. It comes from the fact that Z^* and $(-Y^*, Y^*)$ satisfy the same well-posed martingale problem. The diffusion Y^* is given as the solution starting from 0 (which is an entrance boundary for Y^*) of the s.d.e.

$$\forall t > 0, \quad dY_t^* = \bigl(Y_t^* + g\bigl(Y_t^*\bigr)\bigr)\,dt + \sqrt{2}\,dB_t, \tag{47}$$

where as usual $B := (B_t)_{t\geq 0}$ is a standard Brownian motion, and where g is the mapping defined by

$$\forall y > 0, \quad g(y) := 2\frac{\gamma(y)}{\gamma([0, y])}. \tag{48}$$

The coupling of X and Y^* constructed in Section 4 is equally valid. We deduce that any stopping time for Y^* is a strong time for X. For any $M > 0$, we are particularly interested in the following stopping time

$$\tau_M^* := \inf\{t \geq 0 : Y_t^* = M\}.$$

It has the property that τ_M^* and $X_{\tau_M^*}$ are independent and that $X_{\tau_M^*}$ is distributed according to $\gamma_{[-M,M]}$, the conditioning of γ on the interval $[-M, M]$. The interest of the independence of the time and the position appears in the proof of the following result. It is a particular continuous analogue of the considerations about early stopping bounds on total variation via duality presented in Section 2.5 of Diaconis and Fill [12]. It could be extended to general one-dimensional diffusions by considering the first time the dual process covers a large segment.

Lemma 29. *For all $t \geq 0$ and $M > 0$, we have*

$$\|m_t - \gamma\|_{\text{tv}} \leq \mathbb{P}[\tau_M^* > t] + \|\gamma_{[-M,M]} - \gamma\|_{\text{tv}}.$$

Proof. An equivalent formulation to (46) of the total variation is given by

$$\|m_t - \gamma\|_{\text{tv}} = \frac{1}{2} \sup_{\|f\|_\infty = 1} \mathbb{E}[f(X_t)] - \gamma[f], \tag{49}$$

where the supremum is taken over all measurable functions f taking values in $[-1, 1]$.

Let $\mathcal{F}_{\tau_M^*}$ be the σ-field generated by the piece of trajectory of the intertwined process (X, Y^*) up to time τ_M^*. It is in fact generated by $X_{[0,\tau_M^*]}$ and some randomness independent from the whole trajectory X. Using the strong Markov property, we get for any function f as above,

$$\mathbb{E}[f(X_t) - \gamma[f]|\mathcal{F}_{\tau_M^*}] = P_{t-\tau_M^*\wedge t}[f](X_{\tau_M^*\wedge t}) - \gamma[f],$$

where $(P_t)_{t\geq 0}$ is the semi-group generated by L. Taking into account that $\sigma(\tau_M^*)$, the σ-field generated by τ_M^*, is included into $\mathcal{F}_{\tau_M^*}$ and that $X_{\tau_M^*}$ is independent from τ_M^* and distributed according to $\gamma_{[-M,M]}$, we get on the event $\{\tau_M^* \leq t\}$,

$$\begin{aligned}
\mathbb{E}[f(X_t) - \gamma[f]|\sigma(\tau_M^*)] &= \mathbb{E}[\mathbb{E}[f(X_t) - \gamma[f]|\mathcal{F}_{\tau_M^*}]|\sigma(\tau_M^*)] \\
&= \mathbb{E}[P_{t-\tau_M^*}[f](X_{\tau_M^*}) - \gamma[f]|\sigma(\tau_M^*)] \\
&= \mathbb{E}\left[\int P_{t-\tau_M^*}[f](x)\gamma_{[-M,M]}(dx) - \gamma[f]|\sigma(\tau_M^*)\right] \\
&\leq 2\|\nu_{t-\tau_M^*} - \gamma\|_{\text{tv}},
\end{aligned}$$

where for any $s \in [0, t]$, $\nu_{t-s} := \gamma_{[-M,M]}P_{t-s}$ is the law of X_{t-s}, when X is started from the initial distribution $\gamma_{[-M,M]}$. As a consequence of the Jensen inequality (relatively to the absolute value), it is well-known that the mapping

$$\mathbb{R}_+ \ni s \mapsto \|\nu_s - \gamma\|_{\text{tv}}$$

is non-increasing, so we have proved that

$$\mathbb{E}[f(X_t) - \gamma[f]|\sigma(\tau_M^*)]\mathbb{1}_{\{\tau_M^* \leq t\}} \leq 2\|\gamma_{[-M,M]} - \gamma\|_{\text{tv}}.$$

The announced result is a consequence of this bound, by writing

$$\begin{aligned}\mathbb{E}\big[f(X_t)-\gamma[f]\big] &= \mathbb{E}\big[\big(f(X_t)-\gamma[f]\big)\mathbb{1}_{\{\tau_M^*>t\}}\big]+\mathbb{E}\big[\big(f(X_t)-\gamma[f]\big)\mathbb{1}_{\{\tau_M^*\le t\}}\big]\\ &\le 2\mathbb{P}\big[\tau_M^*>t\big]+\mathbb{E}\big[\mathbb{E}\big[f(X_t)-\gamma[f]|\sigma\big(\tau_M^*\big)\big]\mathbb{1}_{\{\tau_M^*\le t\}}\big]\\ &\le 2\mathbb{P}\big[\tau_M^*>t\big]+2\|\gamma_{[-M,M]}-\gamma\|_{\mathrm{tv}},\end{aligned}$$

and of (49), by taking the supremum over all measurable functions f taking values in $[-1,1]$. □

The last term of the previous bound is immediate to evaluate:

Lemma 30. *For all $M>0$, we have*

$$\|\gamma_{[-M,M]}-\gamma\|_{\mathrm{tv}}\le\frac{\sqrt{2}}{\sqrt{\pi}M}\exp\big(-M^2/2\big).$$

Proof. One sees that

$$\forall x\in\mathbb{R},\quad \frac{d\gamma_{[-M,M]}}{d\gamma}(x)=\frac{1}{\gamma([-M,M])}\mathbb{1}_{[-M,M]}(x),$$

so coming back to (46), it appears that

$$\begin{aligned}\|\gamma_{[-M,M]}-\gamma\|_{\mathrm{tv}} &= \int_{-M}^{M}\frac{1}{\gamma([-M,M])}-1\,d\gamma\\ &=1-\gamma\big([-M,M]\big)\\ &=2\gamma\big((M,+\infty)\big)\\ &\le\frac{\sqrt{2}}{\sqrt{\pi}M}\exp\big(-M^2/2\big).\end{aligned}$$

□

In view of Lemma 29, it remains to study the queues of the distribution of τ_M^*. The first idea is to use a probabilistic approach via natural comparisons of Y^* with simpler processes. This is presented in the first appendix, where the weakness of this method is also explained. Indeed the efficient approach is via spectral considerations in the direction suggested by Remark 23.

In the above computations, only Y^* was needed, so X (starting from 0) was in fact intertwined with Y^*. It is convenient to adopt the corresponding notations. Let $L^\dagger$ be the generator of Y^*: it acts on functions $f\in\mathcal{C}_{\mathrm{c}}^\infty((0,+\infty))$ via

$$\forall y\in\mathbb{R}_+,\quad L^\dagger[f](y):=f''(y)+V'(y)f'(y),$$

where

$$\forall y\in\mathbb{R}_+,\quad V(y):=\frac{y^2}{2}+2\ln\big(\gamma\big([0,y]\big)\big).$$

So $L^\dagger$ factorizes under the form $\exp(-V)\partial\exp(V)\partial$, making it apparent that $L^\dagger$ is symmetric in $\mathbb{L}^2(\nu)$, where ν is the σ-finite measure on $\mathbb{R}$ whose density is $\exp(V)$. Thus $L^\dagger$ can be extended into its Friedrichs extension in $\mathbb{L}^2(\nu)$. We will denote $(P_t^\dagger)_{t\ge0}$ the associated semi-group. At least on functions of $\mathbb{L}^2(\nu)$ which are non-negative, it coincides with its probabilistic representation given on measurable and non-negative functions f by

$$\forall y\in\mathbb{R}_+,\quad P_t^\dagger[f](y)=\mathbb{E}_y\big[f\big(Y_t^*\big)\big],$$

where the y in index of the expectation indicates that Y^* starts from y. Besides, we designate by $\Lambda^\dagger$ the Markov kernel from $\mathbb{R}_+$ to $\mathbb{R}$ inherited from Λ:

$$\forall y \in \mathbb{R}_+, \forall A \in \mathcal{B}(\mathbb{R}), \quad \Lambda^\dagger(y, A) := \begin{cases} \delta_0(A), & \text{if } y = 0, \\ \frac{\gamma([-y,y]\cap A)}{\gamma([-y,y])}, & \text{otherwise.} \end{cases}$$

From the previous considerations, we deduce the intertwining relation

$$L^\dagger \Lambda^\dagger = \Lambda^\dagger L. \tag{50}$$

This weak conjugacy relation suggests that the spectral decomposition of $L^\dagger$ should be related to that of L. So let us recall the latter. Consider $(H_n)_{n\in\mathbb{Z}_+}$ the Hermite polynomials defined by

$$\forall n \in \mathbb{Z}_+, \forall x \in \mathbb{R}, \quad H_n(x) := (-1)^n \exp\bigl(x^2/2\bigr)\partial^n \exp\bigl(-x^2/2\bigr).$$

They form a orthogonal basis of $\mathbb{L}^2(\gamma)$ and diagonalize L (cf. e.g. the book of Ané et al. [2] or of Bakry, Gentil and Ledoux [4]):

$$\forall n \in \mathbb{Z}_+, \quad L[H_n] = -nH_n.$$

Note that H_n is even (respectively odd) if n is even (resp. odd). It follows that $\Lambda^\dagger[H_n] = 0$ if n is odd. Since $H_0 = \mathbb{1}$, we get that $\Lambda^\dagger[H_0] = \mathbb{1}$ and this function does not belong to $\mathbb{L}^2(\nu)$ because ν has an infinite mass. For the remaining Hermite polynomials, we have:

Lemma 31. *For all $n \in \mathbb{N}$, denote $H_{2n}^\dagger := \Lambda^\dagger[H_{2n}]$. This function belongs to $\mathbb{L}^2(\nu) \setminus \{0\}$, satisfies $L^\dagger H_{2n}^\dagger = -2nH_{2n}^\dagger$ and is given by*

$$\forall y > 0, \quad H_{2n}^\dagger(y) = -\frac{1}{\sqrt{2\pi}\gamma([0,y])} H_{2n-1}(y) \exp\bigl(-y^2/2\bigr).$$

Proof. Indeed, we compute that for any $n \in \mathbb{N}$ and $y > 0$,

$$\begin{aligned}
\Lambda^\dagger[H_{2n}](y) &= \frac{1}{\sqrt{2\pi}\gamma([-y,y])} \int_{-y}^{y} H_{2n}(x) \exp\bigl(-x^2/2\bigr)\,dx \\
&= \frac{1}{\sqrt{2\pi}\gamma([-y,y])} \int_{-y}^{y} \partial^{2n} \exp\bigl(-x^2/2\bigr)\,dx \\
&= \frac{1}{\sqrt{2\pi}\gamma([-y,y])} \bigl[\partial^{2n-1} \exp\bigl(-x^2/2\bigr)\bigr]_{-y}^{y} \\
&= \frac{2}{\sqrt{2\pi}\gamma([-y,y])} \partial^{2n-1} \exp\bigl(-y^2/2\bigr) \\
&= -\frac{1}{\sqrt{2\pi}\gamma([0,y])} H_{2n-1}(y) \exp\bigl(-y^2/2\bigr).
\end{aligned}$$

Thus recalling that for $y > 0$, $\nu(y) = (\gamma([0,y]))^2 \exp(y^2/2)$, we get that

$$\begin{aligned}
\nu\bigl[\bigl(H_{2n}^\dagger\bigr)^2\bigr] &= \frac{1}{2\pi} \int_0^{+\infty} H_{2n-1}^2(y) \exp\bigl(-y^2/2\bigr)\,dy \\
&= \frac{1}{\sqrt{2\pi}} \gamma\bigl[H_{2n-1}^2\bigr] \\
&= (2n-1)!
\end{aligned}$$

(taking into account that for any $n \in \mathbb{Z}_+$, $\gamma[H_n^2] = \sqrt{2\pi}n!$). In particular, $H_{2n}^\dagger$ belongs to $\mathbb{L}^2(\nu)$ for $n \in \mathbb{N}$.

The fact that $H_{2n}^{\dagger}$ is an eigenfunction associated to the eigenvalue $-2n$ is a consequence of (50) applied to H_{2n}. $\square$

Let η be the positive measure on $\mathbb{R}_+$ whose density is given by

$$\forall y > 0, \quad \eta(y) := y\gamma\big([0, y]\big).$$

It has an infinite weight, but it should nevertheless be seen as a quasi-invariant measure:

Lemma 32. *For all $t \geq 0$ and all measurable and non-negative function $f : \mathbb{R}_+ \to \mathbb{R}_+$, we have (in $\mathbb{R}_+ \sqcup \{+\infty\}$),*

$$\eta\big[P_t^{\dagger}[f]\big] = \exp(-2t)\eta[f].$$

Proof. Consider $H_2^{\dagger}$, from Lemma 31 we have for all $t \geq 0$, $P_t^{\dagger}[H_2^{\dagger}] = \exp(-2t)H_2^{\dagger}$. So for any $f \in \mathbb{L}^2(\nu)$,

$$\begin{aligned}\nu\big[H_2^{\dagger}P_t^{\dagger}[f]\big] &= \nu\big[P_t^{\dagger}\big[H_2^{\dagger}\big]f\big] \\ &= \exp(-2t)\nu\big[H_2^{\dagger}f\big].\end{aligned}$$

This is the identity announced in the lemma, at least for $f \in \mathbb{L}^2(\nu)$, as a consequence of the proportionality of the densities η and $\nu H_2^{\dagger}$:

$$\begin{aligned}\forall y > 0, \quad \nu(y)H_2^{\dagger}(y) &= -\big(\gamma\big([0, y]\big)\big)^2 \exp\big(-y^2/2\big)\frac{1}{\sqrt{2\pi}\gamma([0, y])}H_1(y)\exp\big(-y^2/2\big) \\ &= -\frac{1}{\sqrt{2\pi}}\gamma\big([0, y]\big)H_1(y) \\ &= -\frac{1}{\sqrt{2\pi}}\eta(y).\end{aligned}$$

The extension to all measurable and non-negative functions f comes from the representation of $P_t^{\dagger}$ as a probability kernel and from a usual application of the monotone class theorem. $\square$

This result readily shows that the queues of τ_M^* admits the exponential rate 2, at least for convenient initial distributions of Y_0^*:

Lemma 33. *Assume that the law m_0 of X_0 has a bounded density with respect to η. Then there exists $C > 0$ depending on m_0 such that*

$$\forall t \geq 0, \forall M > 0, \quad \mathbb{P}_{m_0}\big[\tau_M^* > t\big] \leq CM^2 \exp(-2t).$$

Proof. Denote f the Radon–Nikodym derivate of m_0 with respect to η and let $C > 0$ be an upper bound of f. We have

$$\begin{aligned}\mathbb{P}_{m_0}\big[\tau_M^* > t\big] &\leq \mathbb{P}_{m_0}\big[Y_t^* \in [0, M]\big] \\ &= \eta\big[f P_t^{\dagger}[\mathbb{1}_{[0,M]}]\big] \\ &\leq C\eta\big[P_t^{\dagger}[\mathbb{1}_{[0,M]}]\big] \\ &\leq C\exp(-2t)\eta\big([0, M]\big) \\ &\leq C\exp(-2t)\int_0^M y\gamma\big([0, y]\big)\,dy\end{aligned}$$

$$\begin{aligned}
&\le C\exp(-2t)\int_0^M y\,dy\\
&\le CM^2\exp(-2t). \qquad\square
\end{aligned}$$

We want to extend the previous bound to the case where m_0 is the Dirac mass at 0. To do so, first remark that we can restrict ourselves to $M>1$, because τ_M^* is increasing in M. Next fix $\sigma>0$ small enough such that $\mathbb{P}[\tau_1^*\le\sigma]\le 1/2$ and denote ξ the sub-probability which is the image by Y_σ^* of the restriction of $\mathbb{P}_0$ on $\{\tau_1^*>\sigma\}$, namely given by $\xi[f]=\mathbb{E}_0[f(Y_\sigma^*)\mathbb{1}_{\tau_1^*>\sigma}]$ for any bounded and measurable function f on $\mathbb{R}_+$. Its interest is:

Lemma 34. *We have for all $t\ge 0$ and for all $M>1$,*

$$\mathbb{P}_0\big[\tau_M^*>\sigma+t\big]\le 2\mathbb{P}_\xi\big[\tau_M^*>t\big].$$

Proof. This is a consequence of the strong Markov property applied to the stopping time $\sigma\wedge\tau_1^*$:

$$\mathbb{P}_0\big[\tau_M^*>\sigma+t\big]=\mathbb{E}_0\big[f\big(\sigma\wedge\tau_1^*,Y^*_{\sigma\wedge\tau_1^*}\big)\big],$$

where

$$\forall s\in[0,\sigma],\forall y\ge 0,\quad f(s,y):=\mathbb{P}_y\big[\tau_M^*>t+\sigma-s\big].$$

Note that the quantity $f(s,y)$ is non-decreasing in s and non-increasing in y. We deduce that

$$\begin{aligned}
\mathbb{E}_0\big[\mathbb{1}_{\{\tau_1^*<\sigma\}}f\big(\sigma\wedge\tau_1^*,Y^*_{\sigma\wedge\tau_1^*}\big)\big]&=\mathbb{E}_0\big[\mathbb{1}_{\{\tau_1^*<\sigma\}}f\big(\tau_1^*,1\big)\big]\\
&\le f(\sigma,1)\mathbb{P}_0\big[\tau_1^*<\sigma\big].
\end{aligned}$$

Since $\mathbb{P}_0[\tau_1^*<\sigma]\le 1/2$ and $f(\sigma,1)\le f(\sigma,y)$ for all $y\in[0,1]$, we get

$$\begin{aligned}
f(\sigma,1)\mathbb{P}_0\big[\tau_1^*<\sigma\big]&\le\mathbb{E}_0\big[\mathbb{1}_{\{\tau_1^*\ge\sigma\}}f\big(\sigma,Y_\sigma^*\big)\big]\\
&\le\mathbb{E}_0\big[\mathbb{1}_{\{\tau_1^*\ge\sigma\}}f\big(\sigma\wedge\tau_1^*,Y^*_{\sigma\wedge\tau_1^*}\big)\big].
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathbb{P}_0\big[\tau_M^*>\sigma+t\big]&\le 2\mathbb{E}_0\big[\mathbb{1}_{\{\tau_1^*\ge\sigma\}}f\big(\sigma,Y_\sigma^*\big)\big]\\
&=2\mathbb{P}_\xi\big[\tau_M^*>t\big]. \qquad\square
\end{aligned}$$

Thus to prove that there exists a constant $C>0$ such that

$$\forall t\ge\sigma,\forall M>1,\quad \mathbb{P}_0\big[\tau_M^*>t\big]\le CM^2\exp(-2t), \tag{51}$$

it remains to show that ξ admits a density with respect to η which is bounded above. This is not a priori obvious, because $\eta(y)$ is of order y^2 for small $y>0$. But it is true, essentially due to the behavior of the function g defined in (48) near 0_+.

Lemma 35. *There exists a constant $C>0$ such that*

$$\forall y>0,\quad \frac{d\xi}{d\eta}(y)\le C.$$

Proof. Consider the process $Y:=(Y_t)_{t\ge 0}$ starting from 0 and solution of the s.d.e.

$$\forall t\ge 0,\quad dY_t=\frac{2}{Y_t}\,dt+\sqrt{2}\,dB_t, \tag{52}$$

where $B = (B_t)_{t \geq 0}$ is a standard Brownian motion. Up to the change of time $\mathbb{R}_+ \ni t \mapsto t/2$, Y is a Bessel process of dimension 3. It follows (see the Section 1 of Chapter 11 of Revuz and Yor [31]), that there exists a constant $K > 0$ (depending on σ) such that the density χ of Y_σ has the form $Ky^2 \exp(-y^2/(4\sigma))$. In particular, we can find another constant $K' > 0$ such that

$$\forall y > 0, \quad \frac{\chi}{\eta}(y) \leq K'. \tag{53}$$

To compare with the law of Y^*_σ, we use the Girsanov's formula. More precisely, define the function φ on $\mathbb{R}_+$ by

$$\forall y > 0, \quad \varphi(y) := \frac{1}{2} \int_0^y u + 2\frac{\gamma(u)}{\gamma([0,u])} - \frac{2}{u} \, du.$$

Elementary computations show that these integrals are well-defined, because the integrand is equivalent to $u/3$ for $u > 0$ small. It also appears that $|\varphi|$, $|\varphi'|$ and $|\varphi''|$, as well $u \mapsto |\varphi'(u)/u|$ are bounded on $(0, \sigma]$. Since the s.d.e. satisfied by Y^* can be written

$$\forall t \geq 0, \quad dY^*_t = \left(\frac{2}{Y^*_t} + \varphi'\big(Y^*_t\big) \right) dt + \sqrt{2} \, dB_t,$$

Girsanov's formula (e.g. Chapter 8 of Revuz and Yor [31]) gives us

$$\forall y > 0, \quad \frac{\xi(y)}{\chi(y)} = \mathbb{E}_0 \left[\exp\left(\sqrt{2} \int_0^\sigma \varphi'(Y_s) \, dB_s - \int_0^\sigma \big(\varphi'(Y_s)\big)^2 \, ds \right) \mathbb{1}_{\tau_1(Y) > \sigma} | Y_\sigma = y \right],$$

where Y is the solution of (52) starting from 0 and $\tau_1(Y)$ is its hitting time of 1. To evaluate the latter conditional expectation, we write that

$$\sqrt{2} \int_0^\sigma \varphi'(Y_s) \, dB_s = \varphi(Y_\sigma) - \int_0^\sigma \frac{2}{Y_s} \varphi'(Y_s) + \varphi''(Y_s) \, ds,$$

which enables us to see that

$$\begin{aligned} &\mathbb{E}_0 \left[\exp\left(\sqrt{2} \int_0^\sigma \varphi'(Y_s) \, dB_s - \int_0^\sigma \big(\varphi'(Y_s)\big)^2 \, ds \right) \mathbb{1}_{\tau_1(Y) > \sigma} | Y_\sigma = y \right] \\ &\quad = \exp\big(\varphi(y)\big) \mathbb{E}_0 \left[\exp\left(- \int_0^\sigma \psi(Y_s) \, ds \right) \mathbb{1}_{\tau_1(Y) > \sigma} | Y_\sigma = y \right], \end{aligned}$$

where

$$\forall y > 0, \quad \psi(y) := \frac{2}{y} \varphi'(y) + \varphi''(y) + \big(\varphi'(y)\big)^2.$$

From our previous observations, $|\psi(y)|$ and $|\varphi(y)|$ are bounded for $y \in (0, \sigma]$. It follows that the function ξ/χ is bounded on $(0, \sigma]$. This also true on $(\sigma, +\infty)$, since ξ vanishes there. In conjunction with (53), it ends the proof of the lemma. □

Note that there is no difficulty in transforming (51) into

$$\forall t \geq 0, \forall M > 0, \quad \mathbb{P}_0\big[\tau^*_M > t\big] \leq C(1 \vee M)^2 \exp(-2t),$$

up to a change of the constant $C > 0$. Thus putting together all the previous results, we have proven that there exists a constant $C > 0$ such that for all $t \geq 0$ and all $M > 0$, we have

$$\|m_t - \gamma\|_{\text{tv}} \leq C(1 \vee M)^2 \exp(-2t) + \frac{\sqrt{2}}{\sqrt{\pi} M} \exp\big(-M^2/2\big).$$

One could try to minimize the r.h.s. in $M > 0$ for fixed $t \geq 0$, but it is sufficient to take $M = \sqrt{2t}$ to see that $\|m_t - \gamma\|_{\mathrm{tv}}$ converges exponentially fast to zero and that

$$\limsup_{t\to+\infty} \frac{1}{t} \ln\bigl(\|m_t - \gamma\|_{\mathrm{tv}}\bigr) \leq -2,$$

Lemma 27 shows that we have recovered the optimal rate, so that the approach via strong times is quite sharp.

Remark 36. *Denote by $\mathcal{H}$ the Hilbert space generated by the H_{2n} with $n \in \mathbb{N}$. The operator $\Lambda^\dagger$ is compact from $\mathcal{H}$ to $\mathbb{L}^2(\nu)$ and one to one. Indeed, this is an immediate consequence of*

$$\forall n, m \in \mathbb{N}, \quad \nu\bigl[H_{2n}^\dagger H_{2m}^\dagger\bigr] = \frac{1}{2\sqrt{2\pi n}} \gamma[H_{2n} H_{2m}],$$

which is shown as in the proof of Lemma 31. This leads us to introduce $\mathcal{G} := \Lambda^\dagger(\mathcal{H})$ and to check that $\mathcal{G}$ is dense in $\mathbb{L}^2(\nu)$. It is sufficient to see that any smooth mapping $F : \mathbb{R}_+ \to \mathbb{R}$ with compact support belongs to $\Lambda^\dagger(\mathcal{H})$, i.e. that we can find a measurable function $f : (0, +\infty) \to \mathbb{R}$ with $\int_0^{+\infty} f\, d\gamma = 0$, $\int_0^{+\infty} f^2\, d\gamma < +\infty$ and

$$\forall y > 0, \quad \frac{\int_0^y f\, d\gamma}{\gamma([0, y])} = F(y)$$

(we will then have $F = \Lambda^\dagger[\tilde{f}]$ where $\tilde{f}$ is the symmetrization of f, which belongs to $\mathcal{H}$). So just take

$$\forall x > 0, \quad f(x) := \partial\bigl(F(x)\gamma\bigl([0, x]\bigr)\bigr).$$

It follows that $(H_{2n}^\dagger)_{n\in\mathbb{N}}$ is an orthogonal Hilbertian basis of $\mathbb{L}^2(\nu)$ consisting of eigenvectors of $L^\dagger$. Thus the spectrum of $L^\dagger$ is $-2\mathbb{N}$. By self-adjointness, we deduce that

$$\forall t \geq 0, \forall f \in \mathbb{L}^2(\nu), \quad \bigl\|P_t[f]\bigr\|_{\mathbb{L}^2(\nu)} \leq \exp(-2t)\|f\|_{\mathbb{L}^2(\nu)}.$$

This could also have been used to recover the exponential rate 2 in (51) (just use the Cauchy–Schwarz inequality in Lemma 33, under the assumption that $dm_0/d\eta \in \mathbb{L}^2(\eta)$, and note in the proof of Lemma 35 that $d\chi/d\eta \in \mathbb{L}^2(\eta)$), nevertheless we find it more instructive to work with the quasi-stationary measure η.

In the same spirit as Remark 3, taking into account Theorem 3.3 of the recent preprint of Cheng and Mao [9], we could also have deduced that Y^* is non-explosive from the fact that the sum of the inverse of the eigenvalues of $-L^\dagger$ in $\mathbb{L}^2(\nu)$ is infinite, namely, according to the previous considerations, from $\sum_{n\in\mathbb{N}} 1/(2n) = \infty$. For more information on the eigentime identity, which states that certain reversible Markov processes are explosive if and only if the sum of the inverse of its eigenvalues is finite, we refer to the paper of Mao [26].

Let us end this section by mentioning that the duality between X and Y^* used above can be extended to a large class of symmetric diffusions X starting from 0.

Remark 37. *Assume that in (1), a is an even smooth function with $a(x) > 0$ for every $x \in R$, and that b is an odd smooth function. Then the scale function s and the speed function m defined in (2) are even. When the non-explosive diffusion X associated to L starts from 0, the process Z^* has the form $(-Y^*, Y^*)$, where Y^* is the solution starting from 0 of the s.d.e.*

$$\forall t > 0, \quad dY_t^* = \left(a'\bigl(Y_t^*\bigr) - b\bigl(Y_t^*\bigr) + \frac{2m(Y_t^*)a(Y_t^*)}{M(Y_t^*)}\right) dt + \sqrt{2a\bigl(Y_t^*\bigr)}\, dB_t,$$

where $B := (B_t)_{t\geq 0}$ is a standard Brownian motion and where M is given by

$$\forall y \geq 0, \quad M(y) := \int_0^y m(\xi)\, d\xi.$$

It is not difficult to see that 0 *is an entrance boundary for* Y^*, *which by consequence is positive for positive times. The processes* X *and* Y^* *are intertwined through the kernel* $\Lambda^\dagger$ *from* $\mathbb{R}_+$ *to* $\mathbb{R}$ *defined by*

$$\forall y \in \mathbb{R}_+, \forall A \in \mathcal{B}(\mathbb{R}), \quad \Lambda^\dagger(y, A) := \begin{cases} \delta_0(A), & \text{if } y = 0, \\ \frac{\int_{[-y,y]\cap A} m(x)\,dx}{2M(y)}, & \text{otherwise.} \end{cases}$$

The construction of the bivariate process (X, Y^*) *does not require* $M(\infty) < \infty$.

The considerations of Section 2 also apply to this situation. In particular, define

$$\forall t \geq 0, \quad \theta_t := \inf\left\{u \geq 0 : 8\int_0^u a\big(Y_s^*\big)m^2\big(Y_s^*\big)\,ds = t\right\}.$$

According to Proposition 13, the process $(R_t)_{t\geq 0} := (2M(Y^*_{\theta_t}))_{t\geq 0}$ *is a* 3*-dimensional Bessel process starting from* 0.

Consider the usual Brownian motion $B := (B_t)_{t\geq 0}$, *where* $a \equiv 1/2$ *and* $b \equiv 0$. *Then we get* $s \equiv 1 \equiv m$ *and*

$$\forall t \geq 0, \quad R_t = 2Y^*_{t/8}.$$

So using the homogeneity of the 3*-dimensional Bessel process,* Y^* *is itself a* 3*-dimensional Bessel process. Thus we recover the fact that the Brownian motion is intertwined with the* 3*-dimensional Bessel process (starting from* 0*) through the link*:

$$\forall y \in \mathbb{R}_+, \forall A \in \mathcal{B}(\mathbb{R}), \quad \Lambda^\ddagger(y, A) := \begin{cases} \delta_0(A), & \text{if } y = 0, \\ \frac{\int_{[-y,y]\cap A} dx}{2y}, & \text{otherwise.} \end{cases}$$

This result is due to Pitman [30], who furthermore gave an explicit construction of an intertwining dual $Y := (Y_t)_{t\geq 0}$:

$$\forall t \geq 0, \quad Y_t := 2\max\big(B_s : s \in [0, t]\big) - B_t.$$

It would be interesting to extend such path constructions, in order to avoid the considerations of Section 4.

Appendix A: A probabilistic estimate on queues of τ_M^*

In the previous section we have seen that is important to upper bound quantities like $\mathbb{P}[\tau_M^* > t]$ and we obtained nice estimates via spectral considerations. We were lucky because the spectral decomposition of L is explicit in the Ornstein–Uhlenbeck example. In general a probabilistic approach is more flexible, even if in the example at hand we did not succeed in recovering the optimal rate using this method. Let us nevertheless present this approach. At the end we will see another interplay between probability and spectral theories.

The basic idea is to compare Y^* with the simpler process $Y := (Y_t)_{t\geq 0}$ starting from 0 and solution of the s.d.e.

$$\forall t \geq 0, \quad dY_t = Y_t\,dt + \sqrt{2}\,dB_t, \tag{54}$$

where $B := (B_t)_{t\geq 0}$ is a Brownian motion. We then define for all $M > 0$,

$$\tau_M := \inf\big\{t \geq 0 : |Y_t| = M\big\}.$$

Lemma 38. *The law of* τ_M^* *is stochastically dominated by that of* τ_M.

Proof. Recall the following behaviors of the mapping g defined in (48): as y goes to 0_+, $g(y) \sim 2/y$ and as y goes to $+\infty$, $g(y) \ll 1/y$. So we can define

$$a := \inf\big\{y > 0 : g(y) = 1/y\big\}.$$

We first compare Y^* and Y up to the time τ_a^*. Let $\tilde{Y}$ be an independent copy of Y: it starts from 0 and is a solution of the s.d.e.

$$\forall t \geq 0, \quad d\tilde{Y}_t = \tilde{Y}_t \, dt + \sqrt{2} \, d\tilde{B}_t,$$

where $\tilde{B} := (\tilde{B}_t)_{t \geq 0}$ is a Brownian motion independent from B. Consider the process $\widehat{Y} := (\widehat{Y}_t)_{t \geq 0}$ given by

$$\forall t \geq 0, \quad \widehat{Y}_t := \sqrt{Y_t^2 + \tilde{Y}_t^2}.$$

Simple Itô's computations lead to the fact that $\widehat{Y}$ is the solution starting from 0 of the s.d.e.

$$\forall t \geq 0, \quad d\widehat{Y}_t = \left(\widehat{Y}_t + \frac{1}{\widehat{Y}_t}\right) dt + \sqrt{2} \, dW_t,$$

where $W := (W_t)_{t \geq 0}$ is the Brownian motion defined by

$$\forall t \geq 0, \quad W_t := \int_0^t \frac{1}{\sqrt{Y_s^2 + \tilde{Y}_s^2}} (Y_s \, dB_s + \tilde{Y}_s \, d\tilde{B}_s).$$

Comparing with (47), where we replace B with W, it appears that $|Y_t| \leq \widehat{Y}_t \leq Y_t^*$, at least for $t \leq \tau_a^*$. In particular, for any $M \in (0, a]$, the law of τ_M is stochastically dominated by that of τ_M^*. Using the strong Markov property at τ_M^*, to prove the same domination for $M > a$, it is sufficient to deal with the following situation. Assume that Y^* is the solution of (47) starting from a and that Y is solution of (54) with an initial distribution supported on $[0, a]$. Let B be the same in (47) and in (54), then a.s., for all $t \geq 0$, $|Y_t| \leq Y_t^*$. Indeed, using Tanaka's formula (see for instance Chapter 6 of the book of Revuz and Yor [31]), we have

$$\forall t \geq 0, \quad d|Y_t| = |Y_t| \, dt + \sqrt{2} \, dB_t + dl_t,$$

where $(l_t)_{t \geq 0}$ is the local time at 0 of Y. Consider

$$\sigma := \inf\{t \geq 0 : |Y_t| > Y_t^*\}.$$

If $\sigma < +\infty$, then we have $Y_\sigma = Y_\sigma^*$. Recall from Section 2 that necessarily $Y_\sigma^* > 0$ and since l_t is only increasing when $Y_t = 0$, there exists a random interval of the form $[\sigma, \sigma + \varepsilon)$ on which this local time remains constant. But we have

$$\forall t \geq 0, \quad d\big(Y_t^* - |Y_t|\big) = \big(Y_t^* - |Y_t|\big) \, dt + g\big(Y_t^*\big) \, dt - dl_t,$$

which, via the parameter variation method, leads to

$$\forall t \geq 0, \quad Y_{\sigma+t}^* - |Y_{\sigma+t}| = e^t \int_0^t e^{-s} \big(g\big(Y_s^*\big) \, ds - dl_s\big).$$

If $t \in [0, \varepsilon)$, the r.h.s. is non-negative, which in contradiction with the definition of σ. □

In particular, we get that

$$\forall M > 0, \forall t \geq 0, \quad \mathbb{P}\big[\tau_M^* > t\big] \leq \mathbb{P}[\tau_M > t]. \tag{55}$$

The advantage is that the r.h.s. is simpler to evaluate:

Lemma 39. *We have for any $M > 0$ and any $t \geq 0$,*

$$\mathbb{P}[\tau_M > t] \leq \sqrt{\frac{2}{(1 - e^{-2t})\pi}} e^{-t} M.$$

Proof. Using once again the parameter variation method, we get that

$$\forall t \geq 0, \quad Y_t = \sqrt{2} \int_0^t \exp(t-s)\, dB_s,$$

In particular, Y_t is a centered Gaussian random variable of variance $e^{2t} - 1$. Besides, by definition, we have

$$\begin{aligned}
\mathbb{P}[\tau_M > t] &= \mathbb{P}\big[\forall s \in [0,t], |Y_s| \leq M\big] \\
&\leq \mathbb{P}\big[|Y_t| \leq M\big] \\
&= \int_{-M}^{M} \exp\left(-\frac{y^2}{2(e^{2t}-1)}\right) \frac{dy}{\sqrt{2\pi(e^{2t}-1)}} \\
&\leq \frac{2M}{\sqrt{2\pi(e^{2t}-1)}}.
\end{aligned}$$

□

These computations leads to the bound

$$\forall t > 0, \forall M > 0, \quad \mathbb{P}\big[\tau_M^* > t\big] \leq \sqrt{\frac{2}{(1-e^{-2t})\pi}} M e^{-t},$$

which asymptotically for $t > 0$ large, has not the optimal exponential rate (1 instead of 2).

So where is the weak link in the above arguments? It is the stochastic dominance (55), because the exponential rate of $\mathbb{P}[\tau_M > t]$ for large $t > 0$ is almost 1 (for large $M > 0$), as will be shown below. So the strong repulsion of Y^* in 0 is the reason for the exponential rate 2 for τ_M^*. The process Y (or $|Y|$) has more freedom to wander around 0, which is the best place to "stay" to avoid the points $-M$ and M, and this accounts for their exit rate 1.

Since the generator of Y is $\tilde{L} := \exp(-\tilde{V})\partial \exp(\tilde{V})\partial$, where $\tilde{V} : \mathbb{R} \ni y \mapsto y^2/2$, it appears that the measure $\tilde{\nu}$ admitting the density $\exp(\tilde{V})$ with respect to the Lebesgue measure is "reversible": $\tilde{L}$ can be extended into its self-adjoint Friedrichs extension on $\mathbb{L}^2(\tilde{\nu})$. From the general Markovian theory of absorption (see e.g. the book [10] of Collet, Martínez and San Martín), we have

$$\lim_{t\to+\infty} \frac{1}{t} \ln\big(\mathbb{P}[\tau_M > t]\big) = -\lambda_0(M),$$

where

$$\lambda_0(M) := \inf_{f \in \mathcal{C}^\infty([-M,M]) \,:\, f(-M)=f(M)=0} \frac{\int_{-M}^{M} (f')^2\, d\tilde{\nu}}{\int_{-M}^{M} f^2\, d\tilde{\nu}}.$$

Lemma 39 implies that $\lambda_0(M) \geq 1$ for all $M > 0$ and this bound is asymptotically optimal as M goes to infinity:

Lemma 40. *We have*

$$\lim_{M\to+\infty} \lambda_0(M) = 1.$$

Proof. Let f_M be the function defined on $[-M, M]$ by

$$\forall y \in [-M, M], \quad f_M(y) := \exp\big(-y^2/2\big) - \exp\big(-M^2/2\big).$$

Elementary computations show that

$$\begin{aligned}
\liminf_{M\to+\infty} \lambda_0(M) &\geq \lim_{M\to+\infty} \frac{\int_{-M}^{M} (f_M')^2\, d\tilde{\nu}}{\int_{-M}^{M} f_M^2\, d\tilde{\nu}} \\
&= 1.
\end{aligned}$$

□

The functions f_M, for $M>0$, were suggested by the spectral decomposition of $\tilde{L}$ on $\mathbb{L}^2(\gamma)$, which can be obtained by a method somewhat dual to the one presented in the previous section. Consider, on the appropriate domain of $\mathbb{L}^2(\gamma)$, the linear mapping $K: f \mapsto \exp(-V)\partial f \in \mathbb{L}^2(\tilde{\nu})$. Since $\tilde{L} = \exp(-\tilde{V})\partial \exp(\tilde{V})\partial$ and $L = \exp(\tilde{V})\partial \exp(-\tilde{V})\partial$, we get at once the intertwining relation $\tilde{L}K = KL$ (with a non-Markovian link K, but its inverse is a positive kernel quite close to $\Lambda^\dagger$). So a priori the $\tilde{H}_n := K[H_n]$, for $n \in \mathbb{N}$, are good candidates to be the eigenvectors of $\tilde{L}$, associated respectively to the eigenvalues $-n$. Indeed, we compute that

$$\forall n \in \mathbb{N}, \forall y \in \mathbb{R}, \quad \tilde{H}_n(y) = n \exp\bigl(-y^2/2\bigr) H_{n-1}(y),$$

so that $(\tilde{H}_n)_{n\in\mathbb{N}}$ is an orthogonal Hilbertian basis of $\mathbb{L}^2(\tilde{\nu})$, so the spectrum of $\tilde{L}$ is $-\mathbb{N}$. The measure $\tilde{\eta} := \tilde{H}_1 \tilde{\nu} = \exp(-\tilde{V})$ is quasi-stationary for $\tilde{L}$ and its adaptation to the Dirichlet boundary conditions on $[-M, M]$ furnishes f_M, for $M > 0$. One can also deduce the spectral decomposition of the generator of $|Y|$: restrict everything to $\mathbb{R}_+$, but just keep the $\tilde{H}_n$ with n odd. In particular its spectrum is $\{-1, -3, -5, \dots\}$.

Appendix B: Liggett duality for one-dimensional diffusions

The assertion that the generator $\tilde{L}$ defined in (13) is a continuous analogue of the discrete generator of the Morris–Peres evolving sets will be justified here.

We begin by recalling the notion of duality introduced by Liggett in [24]. Let $X := (X_t)_{t\geq 0}$ and $\tilde{Z} := (\tilde{Z}_t)_{t\geq 0}$ be two Markov processes, on respective state spaces E and $\tilde{E}$. Let Ξ be a measurable mapping from $\tilde{E} \times E$ to $\mathbb{R}$. The processes X and $\tilde{Z}$ are said to be dual with respect to the link Ξ if

$$\forall x \in E, \forall \tilde{z} \in \tilde{E}, \forall t \geq 0, \quad \mathbb{E}_x\bigl[\Xi(\tilde{z}, X_t)\bigr] = \mathbb{E}_{\tilde{z}}\bigl[\Xi(\tilde{Z}_t, x)\bigr]. \tag{56}$$

This definition equally holds in the discrete-time setting.

Let us next consider the example of the evolving sets introduced by Morris and Peres in [28]. On a denumerable space E, let P be a transition probability, reversible with respect to a probability π giving a positive weight to all points of E. Denote by L the generator $I - P$ where I is the identity operator and let $X := (X_t)_{t\geq 0}$ be a jump Markov process generated by L. Let $\tilde{E}$ be the set of all non-empty subsets of E and let Ξ be the link defined by

$$\forall \tilde{z} \in \tilde{E}, \forall x \in E, \quad \Xi(\tilde{z}, x) := \mathbb{1}_{\tilde{z}}(x).$$

Morris and Peres [28] constructed a $\tilde{E}$-valued process $\tilde{Z} := (\tilde{Z}_t)_{t\geq 0}$ which is dual to X with respect to Ξ. It is defined in the following way. For any $\tilde{z} \in \tilde{E}$ and $u \in [0, 1]$, denote

$$\tilde{z}^{(u)} := \left\{ y \in E : \sum_{v\in\tilde{z}} P(y, v) \geq u \right\}.$$

The generator $\tilde{L}$ of $\tilde{Z}$ acts on any bounded function F on $\tilde{E}$ via

$$\forall \tilde{z} \in \tilde{E}, \quad \tilde{L}[F](\tilde{z}) := \int_0^1 F\bigl(\tilde{z}^{(u)}\bigr) - F(\tilde{z})\, du$$

(note that $\tilde{Z}$ is absorbed at E).

A straightforward computation enables checking that

$$\forall \tilde{z} \in \tilde{E}, \forall x \in E, \quad \tilde{L}\bigl[\Xi(\cdot, x)\bigr](\tilde{z}) = L\bigl[\Xi(\tilde{z}, \cdot)\bigr](x).$$

So, if $(P_t)_{t\geq 0}$ and $(\tilde{P}_t)_{t\geq 0}$ are the semi-groups generated by L and $\tilde{L}$, differentiating the mapping

$$[0, T] \ni t \mapsto \tilde{P}_t \otimes P_{T-t}[\Xi](\tilde{z}, x),$$

shows it is constant and in particular for any $T \geq 0$,

$$P_T\big[\Xi(\tilde{z},\cdot)\big](x) = \tilde{P}_T\big[\Xi(\cdot,x)\big](\tilde{z}),$$

which is just a rewriting of (56).

We now come back to the positive recurrent one-dimensional diffusion X considered in the introduction. Let $\tilde{E} := \{(x,y) \in \mathbb{R}^2 : x \leq y\}$ on which we consider the generator $\tilde{L}$ given in (13). The diagonal is not an entrance boundary for this generator and we impose Neumann boundary conditions on D^* on the Markov processes $\tilde{Z} := (\tilde{X}_t, \tilde{Y}_t)_{t\geq 0}$ associated to $\tilde{L}$ (or rather on their minimal versions, namely up to the time when $(-\infty,+\infty)$, $(-\infty,-\infty)$ or $(+\infty,+\infty)$ is reached, note that the two latter exit boundary points can now also be attained with positive probabilities, namely the evolving segment $[\tilde{X}_t, \tilde{Y}_t]$ can vanish in finite time). It amounts to seeing $\tilde{L}$ as a generator on $\mathbb{R}^2$ and to identifying (x,y) with (y,x) for any point (x,y) of the plane.

For any smooth and bounded function f given on $\mathbb{R}$, introduce the mapping $\Xi[f]$ defined on $\tilde{E}$ by

$$\forall (x,y) \in \tilde{E}, \quad \Xi[f](x,y) := \int_x^y f(u)\pi(u)\,du.$$

In the proof of Lemma 19, we have seen that

$$\forall (x,y) \in \tilde{E}, \quad \tilde{L}\big[\Xi[f]\big](x,y) = \Xi\big[L[f]\big](x,y).$$

Taking into account that $\Xi[f]$ satisfies the Neumann condition on the boundary, the arguments of Proposition 21 lead to

$$\forall T \geq 0, \forall (x,y) \in \tilde{E}, \quad \tilde{P}_T\big[\Xi[f]\big](x,y) = \Xi\big[P_T[f]\big](x,y),$$

where $(\tilde{P}_T)_{T\geq 0}$ is the semi-group generated by $\tilde{L}$.

With probabilistic notation, this can written in the form

$$\mathbb{E}_{(x,y)}\left[\int_{\tilde{Z}_t} f(u)\pi(u)\,du\right] = \int_x^y \mathbb{E}_u\big[f(X_t)\big]\pi(u)\,du. \tag{57}$$

By Fubini's theorem, the l.h.s. is equal to

$$\int_{\mathbb{R}} \mathbb{E}_{(x,y)}\big[\mathbb{1}_{\tilde{Z}_t}(u)\big] f(u)\pi(u)\,du.$$

The r.h.s. of (57) can be transformed into

$$\begin{aligned}\mathbb{E}_\pi\big[\mathbb{1}_{[x,y]}(X_0)f(X_t)\big] &= \mathbb{E}_\pi\big[\mathbb{1}_{[x,y]}(X_t)f(X_0)\big] \\ &= \int_{\mathbb{R}} \mathbb{E}_u\big[\mathbb{1}_{[x,y]}(X_t)\big] f(u)\pi(du).\end{aligned}$$

Since this is true for all smooth and bounded function f, it follows that for almost every $u \in \mathbb{R}$,

$$\mathbb{E}_{(x,y)}\big[\mathbb{1}_{\tilde{Z}_t}(u)\big] = \mathbb{E}_u\big[\mathbb{1}_{[x,y]}(X_t)\big].$$

This equality is trivial for $t = 0$ and for $t > 0$ the l.h.s. and the r.h.s. are continuous with respect to $u \in \mathbb{R}$, essentially because the event that u belongs to the boundary of $\tilde{Z}_t$ and the event that $X_t \in \{x,y\}$ are negligible.

Thus (56) holds, with formally the same link as the one considered by Morris and Peres [28].

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