On Markov intertwinings

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Plan of the talk

1. Top-to-random shuffle [Aldous and Diaconis, 1986]
2. Pitman's theorem [1975, Pitman and Rogers 1981]
3. One-dimensional diffusions
4. Multi-dimensional diffusions
5. References

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Decks of cards

N cards labeled $1, 2, ..., N$. A deck of these cards is represented by an element σ of the symmetric group $\mathcal{S}_N$:

positions	labels
1	$\sigma(1)$
2	$\sigma(2)$
3	$\sigma(3)$
$\vdots$	
$N-1$	$\sigma(N-1)$
N	$\sigma(N)$

Figure: The deck of cards σ

Top-to-random shuffle

Starting from the ordered deck $(1, 2, ..., N)$, shuffle it as follows, where the positions I_1, I_2, I_3, ... are independent and uniformly distributed on $\{1, 2, ..., N\}$:

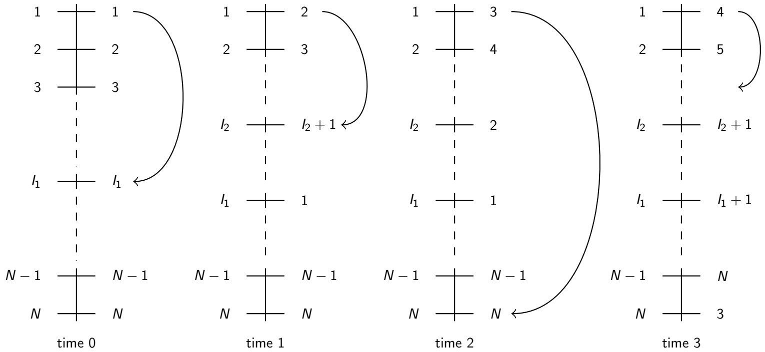

Figure: An example of top-to-random shuffle, here $I_3 = N$

Markov chains

For any $n \in \mathbb{Z}_+$, let $X_n \in \mathcal{S}_N$ be the deck at time n. The random sequence $X := (X_n)_{n\in\mathbb{Z}_+}$ is a Markov chain: to construct X_{n+1}, only the knowledge of X_n (and of I_{n+1}, which is independent from the past) is needed, not the way we ended up with X_n.

The uniform distribution $\mathcal{U}_{\mathcal{S}_N}$ on $\mathcal{S}_N$ is invariant for X: if $X_0 \sim \mathcal{U}_{\mathcal{S}_N}$, then for any time $n \in \mathbb{Z}_+$, $X_n \sim \mathcal{U}_{\mathcal{S}_N}$. Furthermore, by irreducibility and aperiodicity, we know that the law $\mathcal{L}(X_n)$ converges to $\mathcal{U}_{\mathcal{S}_N}$ for large time n.

But from a practical point of view, we want to know for how long we have to shuffle the deck to be close to $\mathcal{U}_{\mathcal{S}_N}$. Quantitative convergence provides such estimates.

Quantitative convergence

Different ways to measure closeness of probability measures m and μ on the same state space:
Total variation:

$$\|m-\mu\|_{\mathrm{tv}} := \sup_{A \text{ event}} |m(A)-\mu(A)| = \frac{1}{2}\left\|\frac{dm}{d\mu}-1\right\|_{\mathbb{L}^1(\mu)}$$

Separation:

$$\mathfrak{s}(m,\mu) := \mathrm{esssup}_\mu 1-\frac{dm}{d\mu} \geq \|m-\mu\|_{\mathrm{tv}}$$

For top-to-random shuffle: for any $c>0$,

$$\left\|\mathcal{L}(X_{\lfloor N\ln(N)+cN\rfloor})-\mathcal{U}_{\mathcal{S}_N}\right\|_{\mathrm{tv}} \leq \exp(-c)$$

i.e. order $N\ln(N)$ shuffles are needed to be almost at equilibrium. Furthermore, it can be shown to be a cut-off time.

Strong stationary times

The proof of Aldous and Diaconis [1986] used strong stationary times associated to a Markov chain X: they are finite stopping times (only the past of X and independent randomness are needed to decide to stop) τ such that

$$\tau \perp\!\!\!\perp X_\tau \quad \text{and} \quad X_\tau \sim \mu$$

where μ is the invariant probability of X.
They provide exact simulations of μ and we have

$$\forall\ n \in \mathbb{Z}_+, \qquad \mathfrak{s}(\mathcal{L}(X_n), \mu) \quad \leq \quad \mathbb{P}[\tau > n]$$

In the top-to-random shuffle, a strong stationary time is given by the first time the last card reaches the top and is inserted in the deck. It is distributed as a sum of independent geometric random variables of parameters $1/N$, $2/N$, ..., $(N-1)/N$ and 1. It leads to the previous bound (coupon collector problem).

Set-valued dual

How to construct a strong stationary time?
Consider $\mathcal{D} := \{A \subset \mathcal{S}_N\} \setminus \{\emptyset\}$ and for any $\sigma \in \mathcal{S}_N$ and $k \in \{1, 2, ..., N\}$

$$A_{\sigma,k} := \{\sigma' \in \mathcal{S}_N \,:\, \sigma'(1) = \sigma(1), \sigma'(2) = \sigma(2), ..., \sigma'(k) = \sigma(k)\}$$

Define a set-valued process $D := (D_n)_{n\in\mathbb{Z}_+}$ via

$$D_n \quad := \quad A_{X_n,\text{position of the initial last card}}$$

as long as the last card has not been reinserted, otherwise take $D_n := \mathcal{S}_N$.
Then D is a $\mathcal{D}$-valued Markov chain absorbed at $\mathcal{S}_N$.
Define the Markov kernel Λ from $\mathcal{D}$ to $\mathcal{S}_N$ via

$$\forall\ A \in \mathcal{D}, \forall\ \sigma \in \mathcal{S}_N, \qquad \Lambda(A, \sigma) := \frac{\mathcal{U}_{\mathcal{S}_N}(\sigma)}{\mathcal{U}_{\mathcal{S}_N}(A)} \mathbb{1}_{\sigma\in A}$$

(=conditional expectation on subsets under the invariant probability).

Intertwining relations

Let P be the transition kernel associated to the time-homogeneous Markov chain X:

$$\forall\ \sigma, \sigma' \in \mathcal{S}_N, \quad P(\sigma, \sigma') \ := \ \mathbb{P}[X_{n+1} = \sigma' | X_n = \sigma]$$

The r.h.s. is $1/N$ if $X_{n+1} \circ X_n^{-1}$ is a cycle of the form $(1 \to i \to i-1 \to i-2 \to \cdots \to 2 \to 1)$ and 0 otherwise.
Let Q be the transition kernel associated to the time-homogeneous Markov chain D. We have the intertwining relation

$$Q\Lambda \ = \ \Lambda P \tag{1}$$

which makes sense, since Q is a $\mathcal{D} \times \mathcal{D}$ matrix, Λ is a $\mathcal{D} \times \mathcal{S}_N$ matrix and P is a $\mathcal{S}_N \times \mathcal{S}_N$ matrix.

Fill and Diaconis [1990] have shown that (1) is a key for the construction of strong stationary times for finite Markov processes, as soon as the transition kernel P is ergodic and Q is absorbing.

Assume (1) is satisfied for general such P and Q. Let $X := (X_n)_{n\in\mathbb{Z}_+}$ and $D := (D_n)_{n\in\mathbb{Z}_+}$ be corresponding Markov chains, with $\mathcal{L}(D_0)\Lambda = \mathcal{L}(X_0)$. Then there exists a coupling of X and D such that for all times $n \in \mathbb{Z}_+$, we have for the conditional expectations:

$$\left\{\begin{array}{rcl} \mathcal{L}(X_n|D_{[\![0,n]\!]}) & = & \Lambda(D_n,\cdot) \\ \mathcal{L}(D_{[\![0,n]\!]}|X) & = & \mathcal{L}(D_{[\![0,n]\!]}|X_{[\![0,n]\!]}) \end{array}\right. \tag{2}$$

It follows that the absorption time of D is a strong stationary time for X. Our goal is to extend such properties to diffusion processes X.

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Brownian motion

The Brownian motion is approximated (in law) by letting $\epsilon \to 0_+$ in the following picture, where each step $\pm\sqrt{\epsilon}$ is chosen with probability $1/2$:

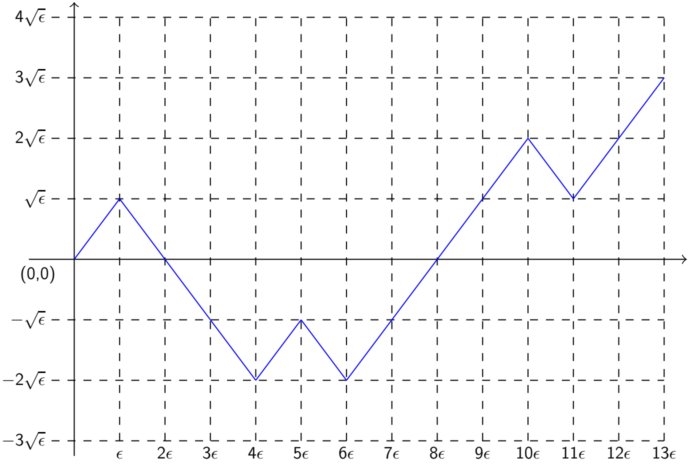


Figure: Approximation of a Brownian path

Pitman's transformation

Let $(B_t)_{t\geq 0}$ be a Brownian motion. Consider the Pitman's transformation $R := (R_t)_{t\geq 0}$ given by

$$R_t \quad := \quad -B_t + 2 \max_{s\in[0,t]} B_s$$

Let Λ the Markov kernel from $\mathbb{R}_+$ to $\mathbb{R}$ given by

$$\forall\ r \geq 0, \qquad \Lambda(r, dx) \quad := \quad \mathcal{U}_{[-r,r]}(dx)$$

Relation (2) can be extended to this continuous setting: for all time $t \geq 0$,

$$\left\{ \begin{array}{rcl} \mathcal{L}(B_t | R_{[0,t]}) & = & \Lambda(R_t, \cdot) \\ \mathcal{L}(R_{[0,t]} | B) & = & \mathcal{L}(R_{[0,t]} | B_{[0,t]}) \end{array} \right. \tag{3}$$

Furthermore, the process R is a Bessel-3 process, i.e. it has the law of the norm of a Brownian motion in dimension 3.

Pitman's theorem in picture

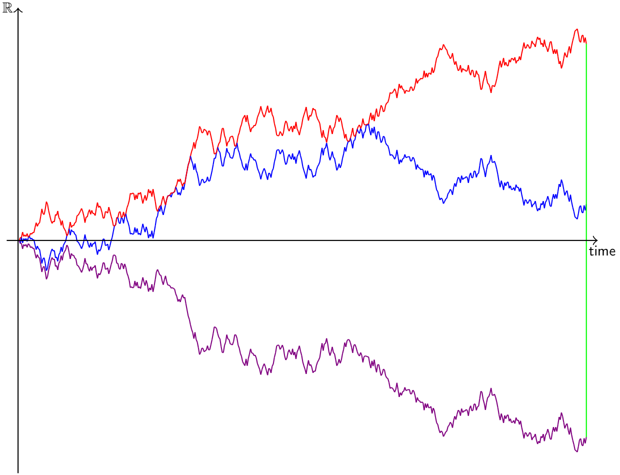


Figure: Trajectories: Brownian motion $B_{[0,t]}$, $R_{[0,t]}$, $-R_{[0,t]}$, and the segment-valued dual: $[-R_t, R_t]$

The intertwining relation (1) holds for the generators G and $\mathcal{G}$ of B and R:

$$\mathcal{G}\Lambda \quad = \quad \Lambda G$$

Here $G = \frac{1}{2}\partial_x^2$ (on $\mathcal{C}^2(\mathbb{R})$) and $\mathcal{G} = \frac{1}{2}\partial_r^2 + \frac{1}{r}\partial_r$ (on $\mathcal{C}^2(\mathbb{R}_+)$).

The generator G of a Markov process $(X_t)_{t\geq 0}$ is understood in the sense of martingale problems: for any function f from the domain of G (=nice observable on the underlying state space), the process $M^f := (M_t^f)_{t\geq 0}$ is a (local) martingale, where

$$M_t^f \quad := \quad f(X_t) - f(X_0) - \int_0^t G[f](X_s)\, ds$$

Martingales are among the main tools of probability theory, they satisfy

$$\forall\ t, s \geq 0, \quad \mathbb{E}[M_{t+s}^f | X_{[0,t]}] \quad = \quad M_t^f$$

Consequently of (3), we get estimates on the convergence of the Brownian motion $W := (W_t)_{t\geq 0}$ on the circle $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$: let τ be the hitting time of π by R:

$$\tau \quad := \quad \inf\{t \geq 0\ :\ R_t = \pi\}$$

It is a strong stationary time for W. The hitting times of Bessel processes are well-studied, we have:

$$\forall\ \lambda > 0, \qquad \mathbb{E}[\exp(-\lambda\tau)] \quad = \quad \frac{\sqrt{\pi}\sqrt[4]{\lambda}}{\sqrt[4]{2}\Gamma(3/2)}\frac{1}{I_{1/2}(\pi\sqrt{2\lambda})}$$

where $I_{1/2}$ is the modified Bessel function of index $1/2$.
A Tauberien theorem enables to deduce the behavior for large $t \geq 0$ of $\mathbb{P}[\tau > t]$ and thus of $\mathfrak{s}(\mathcal{L}(W_t), \mathcal{U}_{\mathbb{T}})$ and $\|\mathcal{L}(W_t) - \mathcal{U}_{\mathbb{T}}\|_{\mathrm{tv}}$.

Plan of the talk

Elliptic real diffusions

On $\mathbb{R}$, consider a diffusion X whose generator is $G := a\partial^2 + b\partial$, where a, b are smooth functions and a is positive. Consider the speed measure $\mu(dx) := \underline{\mu}(x)\,dx$ with density

$$\underline{\mu}(x) := \frac{\exp\left(c(x)\right)}{a(x)} \quad \text{with} \quad c(x) := \int_0^x \frac{b(y)}{a(y)}\,dy$$

It is invariant for X and even reversible: G can be extended into a self-adjoint operator on $\mathbb{L}^2(\mu)$.
Consider

$$\mathcal{D} \;:=\; \{[y,z]\,:\, y,z \in (-\infty,+\infty),\; y \leq z\}$$

As usual, define the associated Markov kernel Λ from $\mathcal{D}$ to $\mathbb{R}$ via

$$\forall\ [y,z] \in \mathcal{D},\ \Lambda([y,z],dx) := \begin{cases} \delta_y(dx) & \text{, if } y = z \\ \frac{\underline{\mu}(x)}{\mu([y,z])}\mathbb{1}_{[y,z]}(x)\,dx & \text{, otherwise} \end{cases}$$

Dual process

On $\mathcal{D}$ define the degenerate generator

$$\begin{aligned}\mathcal{G} \quad := \quad & (\sqrt{a(z)}\partial_z - \sqrt{a(y)}\partial_y)^2 \\ & +(a'(y)/2 - b(y))\partial_y + (a'(z)/2 - b(z))\partial_z \\ & +2\frac{\sqrt{a(y)}\underline{\mu}(y) + \sqrt{a(z)}\underline{\mu}(z)}{\mu([y,z])}(\sqrt{a(z)}\partial_z - \sqrt{a(y)}\partial_y),\end{aligned}$$

Its interest is in the intertwining relation $\mathcal{G}\Lambda = \Lambda G$.

Proposition

There exists a unique process $[Y, Z]$ whose generator is $\mathcal{G}$, up to its explosion time ζ. The diagonal is an entrance boundary for this process.

The proof is based on the fact that $(\mu([Y_t, Z_t]))_{t\in[0,\zeta]}$ is a (stopped) Bessel-3 process, up to the time-change $(\theta_t)_{t\in[0,\varsigma]}$ given by

$$2\int_0^{\theta_t}(\sqrt{a(Y_s)}\underline{\mu}(Y_s) + \sqrt{a(Z_s)}\underline{\mu}(Z_s))^2\, ds \quad = \quad t$$

Application to the existence of strong stationary times:

Theorem

Assume that X is positive recurrent. There exists a strong stationary time for X, whatever its initial distribution, if and only if $-\infty$ and $+\infty$ are entrance boundaries.

Positive recurrence or ergodicity: μ is finite and in large time X_t converges to the renormalization of μ, analytically this is characterized by

$$\int_{-\infty}^{0} \exp(-c(y))\,dy \;=\; +\infty \quad \text{and} \quad \int_{0}^{\infty} \exp(-c(y))\,dy \;=\; +\infty$$

Entrance boundary: e.g. for $+\infty$, it means that X can be started from $+\infty$ (comes down from infinity), it amounts to:

$$\int_{0}^{+\infty} \left(\int_{0}^{x} \exp\left(-c(y)\right)\, dy \right) \mu(dx) \;<\; +\infty$$

1-dimension hypoelliptic diffusions

Consider again the generator $G := a\partial^2 + b\partial$, on $\mathbb{R}$ or $\mathbb{T}$, where a is allowed to vanish on a finite number of points, but such that $\sqrt{a}$ remains smooth. On the vanishing points, assume b does not vanish: 1-dimensional hypoellipticity in the sense of Hörmander [1967].

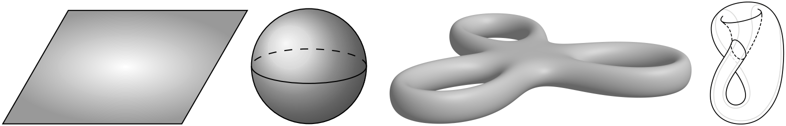
Up to some adjustments (modification of the measure used in Λ, dual processes which can disconnect, ...), the above approach is valid and enables to recover the density theorem (i.e. the law of X_t admits a density for all $t > 0$) and to obtain estimates on the speed of convergence to equilibrium (even when the invariant measure does not charge the whole state space). The Bessel-3 process is still there, the hypoellipticity is only felt at the level of the time change.

Is it possible to recover the generality of Hörmander's theorem in this probabilistic way?

Plan of the talk

Riemannian manifolds

Consider a (complete) Riemannian manifold V of dimension $n \geq 2$.
Examples in dimension $n = 2$:



The Laplacian G associated to V is the generator of the Brownian motion $X := (X_t)_{t \geq 0}$ on V (up to speeding of time by 2). The process X can be approximated as in $\mathbb{R}$ by using independent and uniform increments on spheres of radius $\sqrt{\epsilon}$, as in the real case. The generator G is reversible with respect to the Riemannian measure μ. When V is compact, μ can be normalized into a probability measure.

Evolving domains

Let $\mathcal{D}$ be the set of compact subdomains of V with a smooth boundary. Consider Λ the Markov kernel from $\mathcal{D}$ to V, corresponding to the conditioning of μ. We want:

- to find a Markov generator $\mathcal{G}$ intertwined with G through Λ:

$$\mathcal{G}\Lambda = \Lambda G \tag{4}$$

- To associate to $\mathcal{G}$ a Markov evolution of domains $(\mathfrak{D}_t)_{t\geq 0}$, starting from a singleton: $\mathfrak{D}_0 = \{x_0\}$.
- To couple the evolutions X and $(\mathfrak{D}_t)_{t\geq 0}$ so that

$$\left\{\begin{array}{rcl} \mathcal{L}(X_t|\mathfrak{D}_{[0,t]}) & = & \Lambda(\mathfrak{D}_t, \cdot) \\ \mathcal{L}(\mathfrak{D}_{[0,t]}|X) & = & \mathcal{L}(\mathfrak{D}_{[0,t]}|X_{[0,t]}) \end{array}\right.$$

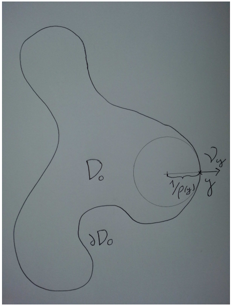
- To recover density theorems and to construct strong stationary times when V is compact.

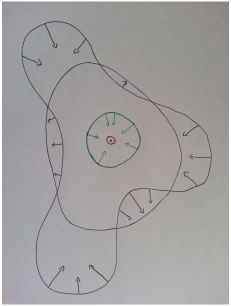
Mean curvature flow

Let be given $D_0 \in \mathcal{D}$. To any $y \in \partial D_0$, associate the exit unitary normal vector ν_y and the mean curvature $\rho(y)$.
At least for small time $t \geq 0$, it is possible to make the domain evolve according to

$$\forall\ y_t \in \partial D_t, \qquad \dot{y}_t \quad = \quad -\rho(y_t)\nu_{y_t}$$

The domains D_t have a tendency to round up and to shrink to a point in finite time.





Stochastic modification of the mean curvature flow

Modify the previous deterministic evolution into an infinite-dimensional stochastic differential equation on $(\mathfrak{D}_t)_{t\in[0,\zeta]}$:

$$\forall\ Y_t \in \partial\mathfrak{D}_t,\ dY_t = \left(\sqrt{2}dB_t + \left(2\frac{\underline{\mu}(\partial\mathfrak{D}_t)}{\mu(\mathfrak{D}_t)} - \rho(Y_t)\right) dt\right) \nu_{Y_t} \quad (5)$$

where $(B_t)_{t\geq 0}$ is a one-dimensional Brownian motion and $\underline{\mu}$ is $(n-1)$-dimensional Hausdorff measure. The global isoperimetric ratio $\underline{\mu}(\partial\mathfrak{D}_t)/\mu(\mathfrak{D}_t)$ counters the effect of the mean curvature and prevents the evolution to collapse to a singleton.

Theorem

Starting from a non-singleton element of $\mathcal{D}$, it is possible to define $(\mathfrak{D}_t)_{t\in[0,\zeta]}$, where ζ is a positive random time, solving (5) and whose generator $\mathcal{G}$ satisfies the intertwining relation (4).

The proof is based on an extension of the Doss-Sussman method to the infinite dimensional setting of $\mathcal{D}$.

When V has constant curvature, (5) can be solved for all times, starting from a singleton $\{x_0\}$. In this situation D_t is a ball centered at x_0 and of radius R_t solving the following stochastic differential equations:

- Euclidean space $\mathbb{R}^n$ (null curvature):

$$dR_t \quad = \quad \sqrt{2}dB_t + \frac{n+1}{R_t}dt$$

(Bessel process of dimension $n+2$, up to scaling time by $1/2$).

Furthermore when $n = 2$, it can be proved that starting from any $D \in \mathcal{D}$, the normalized domain $\mathfrak{D}_t/\sqrt{\mu(\mathfrak{D}_t)}$ converges to the disk of diameter $1/\sqrt{\pi}$ for large times (under the restriction that (5) can be solved for any time $t \geq 0$).

- Spherical space $\mathbb{S}^n$ (positive curvature=1):

$$dR_t = \sqrt{2}dB_t + \left(\frac{2\sin^{n-1}(R_t)}{\int_0^{R_t} \sin^{n-1}(z)\,dz} - (n-1)\cot(R_t) \right) dt$$

Enable to construct strong stationary times (should lead to a cut-off phenomenon with respect to the dimension).

- Poincaré's model of hyperbolic space $\mathbb{H}^n$ (negative curvature=-1)

$$dR_t = \sqrt{2}dB_t + \left(\frac{2\sinh^{n-1}(R_t)}{\int_0^{R_t} \sinh^{n-1}(z)\,dz} - (n-1)\coth(R_t) \right) dt$$

Observables

To define rigorously the generator $\mathcal{G}$, we must have at our disposal "nice observables".

- **Elementary observables**:

$$F_f \ : \ \mathcal{D} \ni D \quad \mapsto \quad F_f(D) := \int_D f \, d\mu$$

associated to the functions $f \in \mathcal{C}^\infty(V)$, the space of smooth mappings on V.

- **Composite observables**: the functionals of the form $\mathfrak{F} := \mathfrak{f}(F_{f_1}, \ldots, F_{f_n})$, where $n \in \mathbb{Z}_+$, $f_1, \ldots, f_n \in \mathcal{C}^\infty(V)$ and $\mathfrak{f} \ : \ \mathcal{R} \to \mathbb{R}$ is a $\mathcal{C}^\infty$ mapping, with $\mathcal{R}$ an open subset of $\mathbb{R}^n$ containing the image of $\mathcal{D}$ by $(F_{f_1}, \ldots, F_{f_n})$.

Generator $\mathcal{G}$

On elementary observables:

$$\forall\ D \in \mathcal{D}, \quad \mathcal{G}[F_f](D) \ := \ \int_D G[f]\, d\mu + 2\frac{\underline{\mu}(\partial D)}{\mu(D)} \int_{\partial D} f\, d\underline{\mu}$$

For the extension to composite observables, the **carré du champs** is also required:

$$\forall\ D \in \mathcal{D}, \quad \Gamma_{\mathcal{G}}[F_f, F_g](D) \ = \ \left(\int_{\partial D} f\, d\underline{\mu}\right)\left(\int_{\partial D} g\, d\underline{\mu}\right)$$

Then on composite observables $\mathfrak{F}$ as above:

$$\mathcal{G}[\mathfrak{F}] = \sum_{j\in[\![1,n]\!]} \partial_j \mathfrak{f}(F_{f_1}, ..., F_{f_n})\mathcal{G}[F_{f_j}] + \sum_{k,l\in[\![1,n]\!]} \partial_{k,l}\mathfrak{f}(F_{f_1}, ..., F_{f_n})\Gamma_{\mathcal{G}}[F_{f_k}, F_{f_l}]$$

(consequence of the continuity of the trajectories of $\mathfrak{D}$).

The above constructions can be extended to any elliptic second order differential generator G on a manifold V admitting an invariant measure μ. The definition of the generator $\mathcal{G}$ is exactly the same, but there is a difference in the description of the infinitesimal evolution of the boundaries.

The operator $\mathcal{G}$ induces on V a Riemannian structure so that $G = \triangle + b$, where b is a vector field. Write $\exp(U)$ the density of μ with respect to the Riemannian measure. Then b admits a (weighted Hodge) decomposition $\nabla U + \beta$. The s.d.e. (5) must be replaced by

$$dY_t = \left(\sqrt{2}dB_t + \left(2\frac{\mu(\partial\mathfrak{D}_t)}{\mu(\mathfrak{D}_t)} + \langle \beta - \nabla U, \nu\rangle (Y_t) - \rho(Y_t)\right) dt\right) \nu(Y_t)$$

Pitman property

Up to the stopping time until which everything is well-defined, we always have:

Theorem

The volume process $(\mu(\mathfrak{D}_{\theta_t}))_{t\geq 0}$ is a Bessel process of dimension 3, where the time change is given by

$$2\int_0^{\theta_t} (\underline{\mu}(\partial \mathfrak{D}_s))^2\, ds \quad = \quad t$$

The ubiquity of the Bessel-3 process suggests that hypoellipticity in general could be investigated in a similar probabilistic way.

Plan of the talk

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