# On Markov intertwinings 

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(1) Pitman's theorem [1975, Pitman and Rogers 1981]
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Let $B:=\left(B_{t}\right)_{t \geq 0}$ be a Brownian motion. Consider the Pitman's transformation $R:=\left(R_{t}\right)_{t \geq 0}$ given by

$$
R_{t}:=-B_{t}+2 \max _{s \in[0, t]} B_{s}
$$

Let $\Lambda$ the Markov kernel from $\mathbb{R}_{+}$to $\mathbb{R}$ given by

$$
\forall r \geq 0, \quad \Lambda(r, d x):=\mathcal{U}_{[-r, r]}(d x)
$$

The process $R$ is a $\Lambda$-dual process for $B$ : for all time $t \geq 0$,

$$
\left\{\begin{align*}
\mathcal{L}\left(B_{t} \mid R_{[0, t]}\right) & =\Lambda\left(R_{t}, \cdot\right)  \tag{1}\\
\mathcal{L}\left(R_{[0, t]} \mid B\right) & =\mathcal{L}\left(R_{[0, t]} \mid B_{[0, t]}\right)
\end{align*}\right.
$$

Furthermore, the process $R$ is a Bessel- 3 process, i.e. it has the law of the norm of a Brownian motion in dimension 3.

## Pitman＇s theorem in picture



Figure：Trajectories：Brownian motion $B_{[0, t]}, R_{[0, t]},-R_{[0, t]}$ ，and the segment－valued dual：$\left[-R_{t}, R_{t}\right]$

## Generators

The first duality relation of (1) can be deduced from $\delta_{0} \Lambda=\delta_{0}$ and from the following intertwining relation for the generators $G$ and $\mathcal{G}$ of $B$ and $R$ :

$$
\begin{equation*}
\mathcal{G} \Lambda=\Lambda G \tag{2}
\end{equation*}
$$

Here $G=\frac{1}{2} \partial_{x}^{2}\left(\right.$ on $\left.\mathcal{C}^{2}(\mathbb{R})\right)$ and $\mathcal{G}=\frac{1}{2} \partial_{r}^{2}+\frac{1}{r} \partial_{r}\left(\right.$ on $\mathcal{C}_{\mathrm{N}}^{2}\left(\mathbb{R}_{+}\right)$, with Neumann condition at 0 ).

The generator $G$ of a Markov process $X:=\left(X_{t}\right)_{t \geq 0}$ is understood in the sense of martingale problems: for any function $f$ from the domain of $G$ (=nice observable on the underlying state space), the process $M^{f}:=\left(M_{t}^{f}\right)_{t \geq 0}$ is a (local) martingale, where

$$
M_{t}^{f}:=f\left(X_{t}\right)-f\left(X_{0}\right)-\int_{0}^{t} G[f]\left(X_{s}\right) d s
$$

## Strong stationary times

A strong stationary time $\tau$ associated to a positive recurrent Markov process $X$ is a finite stopping time such that

$$
\tau \Perp X_{\tau} \quad \text { and } \quad X_{\tau} \sim \mu
$$

where $\mu$ is the invariant probability of $X$. They provide exact simulations of $\mu$ and estimates on the speed of convergence:

$$
\forall t \geq 0, \quad\left\|\mathcal{L}\left(X_{t}\right)-\mu\right\|_{\mathrm{tv}} \leq \mathfrak{s}\left(\mathcal{L}\left(X_{t}\right), \mu\right) \leq \mathbb{P}[\tau>t]
$$

in total variation and separation discrepancy: for any probability measures $m$ and $\mu$ on the same state space:

$$
\mathfrak{s}(m, \mu):=\operatorname{esssup}_{\mu} 1-\frac{d m}{d \mu} \geq \frac{1}{2}\left\|\frac{d m}{d \mu}-1\right\|_{\mathbb{L}^{1}(\mu)}=:\|m-\mu\|_{\mathrm{tv}}
$$

Strong stationary times were introduced by Aldous and Diaconis [1986] to investigate the quantitative convergence to equilibrium of the top-to-random card shuffle.

As a consequence of (1), we get estimates on the convergence of the Brownian motion $W:=\left(W_{t}\right)_{t \geq 0}:=\left(B_{t}[2 \pi]\right)_{t \geq 0}$ on the circle $\mathbb{T}:=\mathbb{R} /(2 \pi \mathbb{Z})$ : let $\tau$ be the hitting time of $\pi$ by $R$ :

$$
\tau:=\inf \left\{t \geq 0: R_{t}=\pi\right\}
$$

It is a strong stationary time for $W$. The hitting times of Bessel processes are well-studied, we have:

$$
\forall \lambda>0, \quad \mathbb{E}[\exp (-\lambda \tau)]=\frac{\sqrt{\pi} \sqrt[4]{\lambda}}{\sqrt[4]{2} \Gamma(3 / 2)} \frac{1}{l_{1 / 2}(\pi \sqrt{2 \lambda})}
$$

where $I_{1 / 2}$ is the modified Bessel function of index $1 / 2$.
A Tauberien theorem enables to deduce the behavior for large $t \geq 0$ of $\mathbb{P}[\tau>t]$ and thus of $\mathfrak{s}\left(\mathcal{L}\left(W_{t}\right), \mathcal{U}_{\mathbb{T}}\right)$ and $\left\|\mathcal{L}\left(W_{t}\right)-\mathcal{U}_{\mathbb{T}}\right\|_{\text {tv }}$, where $\mathcal{U}_{\mathbb{T}}$ is the uniform probability on $\mathbb{T}$.
(1) Pitman's theorem [1975, Pitman and Rogers 1981]
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On $\mathbb{R}$, consider a diffusion $X$ whose generator is $G:=a \partial^{2}+b \partial$, where $a, b$ are smooth functions and $a$ is positive. Consider the speed measure $\mu(d x):=\mu(x) d x$ with density

$$
\mu(x):=\frac{\exp (c(x))}{a(x)} \text { with } \quad c(x):=\int_{0}^{x} \frac{b(y)}{a(y)} d y
$$

It is invariant for $X$ and even reversible: $G$ can be extended into a self-adjoint operator on $\mathbb{L}^{2}(\mu)$.
Consider

$$
\mathcal{D}:=\{[y, z]: y, z \in(-\infty,+\infty), y \leq z\}
$$

Define the conditioning Markov kernel $\Lambda$ from $\mathcal{D}$ to $\mathbb{R}$ via
$\forall[y, z] \in \mathcal{D}, \Lambda([y, z], d x):= \begin{cases}\delta_{y}(d x) & \text {, if } y=z \\ \frac{\mu(x)}{\mu(y, z])} \mathbb{1}_{[y, z]}(x) d x & , \text { otherwise }\end{cases}$

On $\mathcal{D}$ define the degenerate generator

$$
\begin{aligned}
\mathcal{G}:= & \left(\sqrt{a(z)} \partial_{z}-\sqrt{a(y)} \partial_{y}\right)^{2} \\
& +\left(a^{\prime}(y) / 2-b(y)\right) \partial_{y}+\left(a^{\prime}(z) / 2-b(z)\right) \partial_{z} \\
& +2 \frac{\sqrt{a(y)} \underline{\mu}(y)+\sqrt{a(z)} \underline{\mu}(z)}{\mu([y, z])}\left(\sqrt{a(z)} \partial_{z}-\sqrt{a(y)} \partial_{y}\right),
\end{aligned}
$$

Its interest is in the intertwining relation $\mathcal{G} \Lambda=\Lambda G$.

## Proposition

There exists a unique process $[Y, Z]$ whose generator is $\mathcal{G}$, up to its explosion time $\zeta$. The diagonal is an entrance boundary for this process.

The proof is based on the fact that $\left(\mu\left(\left[Y_{t}, Z_{t}\right]\right)\right)_{t \in[0, \zeta]}$ is a (stopped) Bessel-3 process, up to the time-change $\left(\theta_{t}\right)_{t \in[0, \varsigma]}$ given by

$$
2 \int_{0}^{\theta_{t}}\left(\sqrt{a\left(Y_{s}\right)} \underline{\mu}\left(Y_{s}\right)+\sqrt{a\left(Z_{s}\right)} \underline{\mu}\left(Z_{s}\right)\right)^{2} d s=t
$$

## Strong stationary times in 1-dimension

Application to the existence of strong stationary times:

## Theorem

Assume that $X$ is positive recurrent. There exists a strong stationary time for $X$, whatever its initial distribution, if and only if $-\infty$ and $+\infty$ are entrance boundaries.

Positive recurrence or ergodicity: $\mu$ is finite and in large time $X_{t}$ converges to the renormalization of $\mu$, analytically this is characterized by

$$
\int_{-\infty}^{0} \exp (-c(y)) d y=+\infty \quad \text { and } \quad \int_{0}^{\infty} \exp (-c(y)) d y=+\infty
$$

Entrance boundary: e.g. for $+\infty$, it means that $X$ can be started from $+\infty$ (comes down from infinity), it amounts to:

$$
\int_{0}^{+\infty}\left(\int_{0}^{x} \exp (-c(y)) d y\right) \mu(d x)<+\infty
$$

Consider again the generator $G:=a \partial^{2}+b \partial$, on $\mathbb{R}$ or $\mathbb{T}$, where $a$ is allowed to vanish on a finite number of points, but such that $\sqrt{a}$ remains smooth. On the vanishing points, assume $b$ does not vanish: 1-dimensional hypoellipticity in the sense of Hörmander [1967].
Up to some adjustments (modification of the measure used in $\Lambda$, dual processes which can disconnect, ...), the above approach is valid and enables to recover the density theorem (i.e. the law of $X_{t}$ admits a density for all $t>0$ ) and to obtain estimates on the speed of convergence to equilibrium (even when the invariant measure does not charge the whole state space). The Bessel-3 process is still there, the hypoellipticity is only felt at the level of the time change.

Is it possible to recover the generality of Hörmander's theorem in this probabilistic way?
（1）Pitman＇s theorem［1975，Pitman and Rogers 1981］
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Consider a (complete) Riemannian manifold $V$ of dimension $n \geq 2$. The Laplacian $G$ associated to $V$ is the generator of the Brownian motion $X:=\left(X_{t}\right)_{t \geq 0}$ on $V$ (up to speeding of time by 2 ). The generator $G$ is reversible with respect to the Riemannian measure $\mu$. When $V$ is compact, $\mu$ can be normalized into a probability measure.
Let $\mathcal{D}$ be the set of compact subdomains of $V$ with a smooth boundary. Consider $\Lambda$ the Markov kernel from $\mathcal{D}$ to $V$, corresponding to the conditioning of $\mu$.

Goal:

- To find a Markov generator $\mathcal{G}$ on $\mathcal{D}$ intertwined with $G$ through $\Lambda$ :

$$
\begin{equation*}
\mathcal{G} \Lambda=\Lambda G \tag{3}
\end{equation*}
$$

- To associate to $\mathcal{G}$ a Markov evolution of domains $\left(\mathfrak{D}_{t}\right)_{t \geq 0}$, starting from a singleton: $\mathfrak{D}_{0}=\left\{x_{0}\right\}$.
- To couple the evolutions $X$ and $\left(\mathfrak{D}_{t}\right)_{t \geq 0}$ so that

$$
\left\{\begin{aligned}
\mathcal{L}\left(X_{t} \mid \mathfrak{D}_{[0, t]}\right) & =\Lambda\left(\mathfrak{D}_{t}, \cdot\right) \\
\mathcal{L}\left(\mathfrak{D}_{[0, t]} \mid X\right) & =\mathcal{L}\left(\mathfrak{D}_{[0, t]} \mid X_{[0, t]}\right)
\end{aligned}\right.
$$

- To recover density theorems and to construct strong stationary times when $V$ is compact.


## Mean curvature flow

Let be given $D_{0} \in \mathcal{D}$. To any $y \in \partial D_{0}$, associate the exit unitary normal vector $\nu_{y}$ and the mean curvature $\rho(y)$.
At least for small time $t \geq 0$, it is possible to make the domain evolve according to

$$
\forall y_{t} \in \partial D_{t}, \quad \dot{y}_{t}=-\rho\left(y_{t}\right) \nu_{y_{t}}
$$

The domains $D_{t}$ have a tendency to round up and to shrink to a point in finite time.

## Stochastic modification of the mean curvature flow

Modify the previous deterministic evolution into an infinite-dimensional stochastic differential equation on $\left(\mathfrak{D}_{t}\right)_{t \in[0, \zeta]}$ :

$$
\begin{equation*}
\forall Y_{t} \in \partial \mathfrak{D}_{t}, d Y_{t}=\left(\sqrt{2} d B_{t}+\left(2 \frac{\mu\left(\partial \mathfrak{D}_{t}\right)}{\mu\left(\mathfrak{D}_{t}\right)}-\rho\left(Y_{t}\right)\right) d t\right) \nu_{Y_{t}} \tag{4}
\end{equation*}
$$

where $\left(B_{t}\right)_{t \geq 0}$ is a one-dimensional Brownian motion and $\underline{\mu}$ is ( $n-1$ )-dimensional Hausdorff measure. The global isoperimetric ratio $\mu\left(\partial \mathfrak{D}_{t}\right) / \mu\left(\mathfrak{D}_{t}\right)$ counters the effect of the mean curvature and prevents the evolution to collapse to a singleton.

## Theorem

Starting from a non-singleton element of $\mathcal{D}$, it is possible to define $\left(\mathfrak{D}_{t}\right)_{t \in[0, \zeta]}$, where $\zeta$ is a positive random time, solving (4) and whose generator $\mathcal{G}$ satisfies the intertwining relation (3).

The proof is based on an extension of the Doss-Sussman method to the infinite dimensional setting of $\mathcal{D}$.

When $V$ has constant curvature, (4) can be solved for all times, starting from a singleton $\left\{x_{0}\right\}$. In this situation $D_{t}$ is a ball centered at $x_{0}$ and of radius $R_{t}$ solving the following stochastic differential equations:

- Euclidean space $\mathbb{R}^{n}$ (null curvature):

$$
d R_{t}=\sqrt{2} d B_{t}+\frac{n+1}{R_{t}} d t
$$

(Bessel process of dimension $n+2$, up to scaling time by $1 / 2$ ).
Furthermore when $n=2$, it can be proved that starting from any $D \in \mathcal{D}$, the normalized domain $\mathfrak{D}_{t} / \sqrt{\mu\left(\mathfrak{D}_{t}\right)}$ converges to the disk of diameter $1 / \sqrt{\pi}$ for large times (under the restriction that (4) can be solved for any time $t \geq 0$ ).

## Constant curvature

- Spherical space $\mathbb{S}^{n}$ (positive curvature $=1$ ):

$$
d R_{t}=\sqrt{2} d B_{t}+\left(\frac{2 \sin ^{n-1}\left(R_{t}\right)}{\int_{0}^{R_{t}} \sin ^{n-1}(z) d z}-(n-1) \cot \left(R_{t}\right)\right) d t
$$

Enable to construct strong stationary times and to recover the cut-off phenomenon with respect to the dimension [Saloff-Coste 1994, Méliot 2014].

- Poincaré's model of hyperbolic space $\mathbb{H}^{n}$ (negative curvature $=-1$ )

$$
d R_{t}=\sqrt{2} d B_{t}+\left(\frac{2 \sinh ^{n-1}\left(R_{t}\right)}{\int_{0}^{R_{t}} \sinh ^{n-1}(z) d z}-(n-1) \operatorname{coth}\left(R_{t}\right)\right) d t
$$

## Observables

To define rigorously the generator $\mathcal{G}$, we must have at our disposal "nice observables".

- Elementary observables:

$$
F_{f}: \mathcal{D} \ni D \quad \mapsto \quad F_{f}(D):=\int_{D} f d \mu
$$

associated to the functions $f \in \mathcal{C}^{\infty}(V)$, the space of smooth mappings on $V$.

- Composite observables: the functionals of the form $\mathfrak{F}:=\mathfrak{f}\left(F_{f_{1}}, \ldots, F_{f_{n}}\right)$, where $n \in \mathbb{Z}_{+}, f_{1}, \ldots, f_{n} \in \mathcal{C}^{\infty}(V)$ and $\mathfrak{f}: \mathcal{R} \rightarrow \mathbb{R}$ is a $\mathcal{C}^{\infty}$ mapping, with $\mathcal{R}$ an open subset of $\mathbb{R}^{n}$ containing the image of $\mathcal{D}$ by $\left(F_{f_{1}}, \ldots, F_{f_{n}}\right)$.

On elementary observables:

$$
\forall D \in \mathcal{D}, \quad \mathcal{G}\left[F_{f}\right](D):=\int_{D} G[f] d \mu+2 \frac{\underline{\mu}(\partial D)}{\mu(D)} \int_{\partial D} f d \underline{\mu}
$$

For the extension to composite observables, the carré du champs is also required:

$$
\forall D \in \mathcal{D}, \quad \Gamma_{\mathcal{G}}\left[F_{f}, F_{g}\right](D)=\left(\int_{\partial D} f d \mu\right)\left(\int_{\partial D} g d \underline{\mu}\right)
$$

Then on composite observables $\mathfrak{F}$ as above:
$\mathcal{G}[\mathfrak{F}]=\sum_{j \in \llbracket 1, n \rrbracket} \partial_{j} \mathfrak{f}\left(F_{f_{1}}, \ldots, F_{f_{n}}\right) \mathcal{G}\left[F_{f_{j}}\right]+\sum_{k, l \in \llbracket 1, n \rrbracket} \partial_{k, l} \mathfrak{f}\left(F_{f_{1}}, \ldots, F_{f_{n}}\right) \Gamma_{\mathcal{G}}\left[F_{f_{k}}, F_{f_{l}}\right]$
(consequence of the continuity of the trajectories of $\mathfrak{D}$ ).

The above constructions can be extended to any elliptic second order differential generator $G$ on a manifold $V$ admitting an invariant measure $\mu$. The definition of the generator $\mathcal{G}$ is exactly the same, but there is a difference in the description of the infinitesimal evolution of the boundaries.
The operator $G$ induces on $V$ a Riemannian structure so that $G=\triangle+b$, where $b$ is a vector field. Write $\exp (U)$ the density of $\mu$ with respect to the Riemannian measure. Then $b$ admits a (weighted Hodge) decomposition $\nabla U+\beta$. The s.d.e. (4) must be replaced by
$d Y_{t}=$

$$
\left(\sqrt{2} d B_{t}+\left(2 \frac{\mu\left(\partial \mathfrak{D}_{t}\right)}{\mu\left(\mathfrak{D}_{t}\right)}+\langle\beta-\nabla U, \nu\rangle\left(Y_{t}\right)-\rho\left(Y_{t}\right)\right) d t\right) \nu\left(Y_{t}\right)
$$

## Pitman property

Up to the stopping time until which everything is well-defined, we always have:

## Theorem

The volume process $\left(\mu\left(\mathfrak{D}_{\theta_{t}}\right)\right)_{t \geq 0}$ is a Bessel process of dimension 3, where the time change is given by

$$
2 \int_{0}^{\theta_{t}}\left(\underline{\mu}\left(\partial \mathfrak{D}_{s}\right)\right)^{2} d s=t
$$

The ubiquity of the Bessel-3 process suggests that hypoellipticity in general could be investigated in a similar probabilistic way.

# （1）Pitman＇s theorem［1975，Pitman and Rogers 1981］ 

（2）One－dimensional diffusions
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## Markov chains

Given an intertwining relation between finite generators or transition matrices, Diaconis and Fill [1990] constructed a coupling between the corresponding Markov processes or chains.
Problem: it is difficult to manipulate, since it is not so explicit.
Finite Markov chain setting: $V$ a finite state space and $P$ an irreducible transition matrix, denote $\mu$ the invariant probability. Consider the conditioning Markov kernel $\Lambda$,

$$
\forall S \in \mathcal{D}, \forall x \in X, \quad \Lambda(S, x):=\frac{\mu(x)}{\mu(S)}
$$

where $\mathcal{D}:=\{S \subset V: S \neq \emptyset\}$. Let $X:=\left(X_{n}\right)_{n \in \mathbb{Z}_{+}}$be a Markov chain associated to $P$, we want to find a Markov chain $\mathfrak{X}:=\left(\mathfrak{X}_{n}\right)_{n \in \mathbb{Z}_{+}}$on $\mathcal{D}$ which is a $\Lambda$-dual to $X$ and an explicit coupling $(X, \mathfrak{X})$.

Consider the adjoint transition matrix (in $\mathbb{L}^{2}(\mu)$ ):

$$
\forall x, y \in V, \quad P^{*}(x, y):=\frac{\mu(y)}{\mu(x)} P(y, x)
$$

A random mapping $\psi: \Omega \times V \rightarrow V$ is said to be associated to $P^{*}$ when

$$
\forall x, x^{\prime} \in V, \quad \mathbb{P}\left[\psi(x)=x^{\prime}\right]=P^{*}\left(x, x^{\prime}\right)
$$

For any $S \in \overline{\mathcal{D}}:=\mathcal{D} \sqcup\{\emptyset\}$, let be given a random mapping $\psi_{S}$. It defines a random mapping $\Psi$ from $\overline{\mathcal{D}}$ to $\overline{\mathcal{D}}$ via

$$
\forall S \in \overline{\mathcal{D}}, \quad \Psi(S):=\quad\left\{y \in V: \psi_{S}(y) \in S\right\}
$$

Let be given a trajectory $\left(x_{n}\right)_{n \in \mathbb{Z}_{+}}$of $X$. A adapted finite trajectory $\left(S_{n}\right)_{n \in \mathbb{Z}_{+}}$of $\mathfrak{X}$ is constructed iteratively via

- $S_{0}:=\left\{x_{0}\right\}$.
- When $S_{n}$ has been constructed, consider a random mapping $\psi_{S_{n}}$ associated to $P^{*}$ (independent of everything already done, except for the index $S_{n}$ ) and condition by the event $\left\{\psi_{S_{n}}\left(x_{n+1}\right)=x_{n}\right\}$, to get a random mapping $\phi_{n}$ (no longer associated to $P^{*}$ ). Construct

$$
S_{n+1}:=\left\{y \in V: \phi_{n}(y) \in S_{n}\right\}
$$

A stochastic chain $\mathfrak{X}$ is obtained by integrating with respect to $\left(x_{n}\right)_{n \in \mathbb{Z}_{+}}$.

## Theorem

The stochastic chain $\mathfrak{X}$ is Markovian and is a $\Lambda$-dual of $X$.

The spirit of this construction is related to the coupling-from-the-past sampling algorithm of Propp and Wilson [1996] and particular instances of random mappings enable to recover the Doob-transform of the evolving sets of Morris and Peres [2005].

Consider the usual random walk $X:=\left(X_{n}\right)_{n \in \mathbb{Z}_{+}}$on $\mathbb{Z}$, starting from 0 and whose transitions are given by

$$
\forall x, y \in \mathbb{Z}, \quad P(x, y):= \begin{cases}1 / 2 & , \text { if }|y-x|=1 \\ 0 & , \text { otherwise }\end{cases}
$$

The counting measure is invariant and even reversible: $P^{*}=P$. The conditioning Markov kernel $\Lambda$ is defined as before, by restricting $\mathcal{D}$ to contain only finite subsets. Introduce the process $R:=\left(R_{n}\right)_{n \in \mathbb{Z}_{+}}$via

$$
\forall n \in \mathbb{Z}_{+}, \quad R_{n}:=2 \max \left\{X_{m}: m \in \llbracket 0, n \rrbracket\right\}-X_{n}
$$

Then the discrete equivalent of (1) holds: the stochastic chain $\mathfrak{X}:=\left(\mathfrak{X}_{n}\right)_{n \in \mathbb{Z}_{+}}$, given by

$$
\forall n \in \mathbb{Z}_{+}, \quad \mathfrak{X}_{n}:=\left\{R_{n}-2 m: m \in \llbracket 0, R_{n} \rrbracket\right\}
$$

is Markovian and a $\Lambda$-dual of $X$.

## Schematic proof by random mappings

Consider the random mapping $\psi_{s}$ given by

$$
\forall x \in \mathbb{Z}, \quad \psi_{S}(x):= \begin{cases}x+B & , \text { if } x>\max (S) \\ x-B & , \text { if } x \leq \max (S)\end{cases}
$$

where $B$ is a Rademacher variable, and the picture:


In the discrete Pitman theorem, a backward construction of the dual is possible. This property extends to restless birth and death chains on $\mathbb{Z}$, i.e. satisfying $\forall x \in \mathbb{Z}, P(x, x-1)+P(x, x+1)=1$. Extra randomness is in general necessary. But first computations suggest that this extra randomness disappears for one-dimensional diffusions, through approximation by restless birth and death chains on $\mathbb{Z}$.

The hope is that random mappings can be replaced by random flows in diffusion frameworks, leading to direct constructions of $\Lambda$-duals in the context of hypoelliptic processes.
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