Higher order Cheeger inequalities for Steklov eigenvalues

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Plan of the talk

Plan

Consider the **Laplacian** $\triangle$ on a **compact and connected Riemannian manifold** M of dimension n. The **spectrum** of $-\triangle$ is denoted

$$0 = \lambda_1(M) < \lambda_2(M) \leqslant \cdots \leqslant \lambda_k(M) \leqslant \cdots \nearrow \infty$$

Let $\mathcal{A}$ be the family of all nonempty open subsets A of M with a piecewise smooth boundary, and more generally for $k \in \mathbb{N}$, denote $\mathcal{A}_k$ the set of all k-tuples $(A_1, \cdots, A_k)$ of mutually disjoint elements of $\mathcal{A}$.

Define the k-th **Cheeger** (or isoperimetric) **constant** via

$$h_k(M) \quad := \quad \inf_{(A_1,\cdots,A_k)\in\mathcal{A}_k} \max_{l\in[\![k]\!]} \frac{\underline{\mu}(\partial A_l)}{\mu(A_l)}$$

where μ is the **Riemannian measure** and $\underline{\mu}$ is the $(n-1)$-dimensional **Hausdorff measure** on M.

Higher order Cheeger inequalities

Theorem 1

There exists a universal positive constant c_0 such that

$$\forall\ k \in \mathbb{N}, \qquad \lambda_k(M) \ \geqslant\ \frac{c_0}{k^6} h_k^2(M)$$

The Cheeger's inequality was introduced by [Cheeger, 1970] for $k = 2$, it corresponds to cutting M into two parts.
Higher order Cheeger's inequalities were first proven by [Lee, Gharan and Trevisan, 2012] in a combinatoric framework.
Theorem 1 is also true for Riemannian manifolds with a smooth boundary, endowed with the Laplacian with Neumann's condition.
The boundary of a subset $A \in \mathcal{A}$ must be replaced by the **interior boundary**, i.e. the intersection of ∂A with $\mathrm{Int}(M) := M \backslash \partial M$.

Plan

Steklov operators

Let f be a smooth function on the smooth boundary ∂M of a compact and connected Riemannian manifold M of dimension n. Consider F its harmonic extension on M and define on ∂M

$$S[f] \;\; := \;\; \langle \nu, \nabla F \rangle$$

where ν is the unit inward normal vector along ∂M.
S is called the **Steklov operator** associated to M (or the Dirichlet-to-Neumann operator). It is symmetric and essentially self-adjoint in $\mathbb{L}^2(\underline{\mu})$. Denote the sequence of the **Steklov eigenvalues** of $-S$ by

$$0 = \sigma_1(M) < \sigma_2(M) \leqslant \cdots \leqslant \sigma_k(M) \leqslant \cdots \nearrow \infty$$

We want to get lower bounds on them via isoperimetric quantities, in the spirit of Theorem 1.

Isoperimetric quantities (1)

Define the **interior** and **exterior boundaries** of $A \in \mathcal{A}$ via:

$$\partial_{\mathrm{i}} A \;:=\; \partial A \cap \mathrm{Int}\, M \qquad\qquad \partial_{\mathrm{e}} A \;:=\; \bar{A} \cap \partial M$$

Consider the **isoperimetric ratios**

$$\eta(A) \;:=\; \frac{\underline{\mu}(\partial_{\mathrm{i}} A)}{\mu(A)} \qquad\qquad \eta'(A) \;:=\; \frac{\underline{\mu}(\partial_{\mathrm{i}} A)}{\underline{\mu}(\partial_{\mathrm{e}} A)}$$

$$\rho(A) \;:=\; \inf_{\substack{B \in \mathcal{A} \\ B \subset A \\ \bar{B} \cap \partial_i A = \varnothing}} \eta(B)\,, \qquad\qquad \rho'(A) \;:=\; \inf_{\substack{B' \in \mathcal{A} \\ B' \subset A \\ \bar{B}' \cap \partial_i A = \varnothing}} \eta'(B')$$

Isoperimetric quantities (2)

For $k \in \mathbb{N}$, we define the k-th **Cheeger–Steklov constant** of M by

$$\iota_k(M) := \inf_{(A_1,\cdots,A_k)\in\mathcal{A}_k} \max_{l\in[\![k]\!]} \rho(A_l)\rho'(A_l)$$

It should be compared to the k-th Cheeger constant given above:

$$\begin{aligned} h_k(M) &:= \inf_{(A_1,\cdots,A_k)\in\mathcal{A}_k} \max_{l\in[\![k]\!]} \eta(A_l) \\ &= \inf_{(A_1,\cdots,A_k)\in\mathcal{A}_k} \max_{l\in[\![k]\!]} \rho(A_l) \end{aligned}$$

Higher order Cheeger inequalities for Steklov eigenvalues

Theorem 2

There exists a universal positive constant c_1 such that

$$\forall\ k \in \mathbb{N}, \qquad \sigma_k(M) \ \geqslant\ \frac{c_1}{k^6}\iota_k(M)$$

The Steklov operator was introduced in [Steklov,1902] for bounded domains of the plane. Theorem 2 is an extension of works by [Escobar, 1997, 1999] and [Jammes, 2015] for the Steklov-Cheeger's inequality corresponding to $k = 2$.

Plan

Finite Markov generators

A matrix $L := L(x, y)_{x,y\in M}$ indexed by the finite set M is said to be a **Markov generator** when

$$\forall\ x \neq y \in M, \quad L(x, y) \geqslant 0, \qquad \text{and} \qquad \sum_{y\in M} L(x, y) = 0$$

Assume it is **irreducible**: for every $x, y \in M$ there exists a sequence $x = x_0, x_1, \ldots, x_l = y$ of elements of M such that $L(x_j, x_{j+1}) > 0$ for any $j \in [\![0, l-1]\!]$. Denote by $\mu := (\mu(x))_{x\in M}$ its unique **invariant probability**, characterized by

$$\forall\ y \in M, \qquad \sum_{x\in M} \mu(x) L(x, y) \quad = \quad 0$$

Cheeger's constants

Let $\mathcal{A}$ be the set of non-empty subsets of M. The **boundary** of $A \in \mathcal{A}$ is a set of edges:

$$\partial A \quad := \quad \{(x,y)\ :\ x \in A \text{ and } y \notin A\}$$

The set of edges is endowed with the measure $\underline{\mu}$ given by

$$\forall\ x \neq y \in M, \qquad \underline{\mu}(x,y) \quad := \quad \mu(x)L(x,y)$$

For $k \in [\![|M|]\!]$, define the k-th **Cheeger's constant**:

$$h_k(L) \quad := \quad \min_{(A_1,\cdots,A_k)\in\mathcal{A}_k} \max_{l\in[\![k]\!]} \frac{\underline{\mu}(\partial A_l)}{\mu(A_l)}$$

where $\mathcal{A}_k$ is the set of k-tuples $(A_1, A_2, ..., A_k)$ of disjoints elements from $\mathcal{A}$.
We also denote

$$\|L\| \quad := \quad \max\{|L(x,x)|\ :\ x \in M\}$$

Higher order Cheeger's inequalities

Assume μ is **reversible** for L:

$$\forall\ x, y \in M, \qquad \mu(x)L(x,y) \;=\; \mu(y)L(y,x)$$

Then the eigenvalues of $-L$ are non-negative, write them

$$0 = \lambda_1(L) < \lambda_2(L) \leqslant \cdots \leqslant \lambda_{|M|}(L)$$

Higher order Cheeger inequalities are similar to the bounds deduced in the Riemannian situation:

Theorem 3

There exists a universal positive constant c_2 such that

$$\forall\ k \in \mathbb{N}, \qquad \lambda_k(L) \;\geqslant\; \frac{c_2}{k^8\|L\|}h_k^2(L)$$

Finite Steklov operator

Consider $V \subset M$ a proper subset that will be seen as a *boundary*. A corresponding **Steklov operator** S on $\mathbb{R}^V$ is defined as follows. For $f \in \mathbb{R}^V$, consider F the harmonic extension on M, namely the unique $F \in \mathbb{R}^M$ satisfying

$$\begin{cases} L[F](x) = 0\,, & \text{if } x \in M \backslash V \\ F(x) = f(x)\,, & \text{if } x \in V \end{cases}$$

Then we define

$$\forall\, x \in V, \qquad S[f](x) \;:=\; L[F](x)$$

We have

Proposition 4

The operator S is an irreducible Markov generator on V whose invariant measure is ν, the normalized restriction of μ to V. Furthermore when L is reversible, S is reversible.

Probabilist point of view

To the generator L and to any initial distribution m_0 on M, we can associate **Markov processes** $(X_t)_{t\geqslant 0}$: sample X_0 according to m_0, wait a time τ_1 distributed as an exponential variable of parameter $|L(X_0,X_0)|$ and choose a new position X_{τ_1} with the probability $L(X_0,\cdot)/|L(X_0,X_0)|$. Wait an inter-time $\tau_2-\tau_1$ distributed as an exponential variable of parameter $|L(X_{\tau_1},X_{\tau_1})|$ and choose a new position X_{τ_2} with the probability $L(X_{\tau_1},\cdot)/|L(X_{\tau_1},X_{\tau_1})|$, etc. All the ingredients are independent, except for the mentioned dependences. Assume that the support of m_0 is included into V. A Markov process $(Y_t)_{t\geqslant 0}$ associated to the generator S and whose initial distribution is m_0 can be obtained from $(X_t)_{t\geqslant 0}$ by erasing its passages in $M\backslash V$.

The irreducibility of S follows and we get that the corresponding invariant measure ν is proportional to the restriction of μ to V via the ergodic theorem:

$$\frac{\nu(y)}{\nu(z)} = \lim_{t\to+\infty}\frac{\int_0^t \mathbb{1}_{\{y\}}(Y_s)\,ds}{\int_0^t \mathbb{1}_{\{z\}}(Y_s)\,ds} = \lim_{t\to+\infty}\frac{\int_0^t \mathbb{1}_{\{y\}}(X_s)\,ds}{\int_0^t \mathbb{1}_{\{z\}}(X_s)\,ds} = \frac{\mu(y)}{\mu(z)}$$

Analytic point of view

The **Dirichlet form** associated to L (and μ) is the bilinear form $\mathcal{E}_L$ given by

$$\forall\ F, G \in \mathbb{R}^M, \qquad \mathcal{E}_L(F, G) \ := \ -\int FL[G]\, d\mu$$

By the theory of (non-symmetrical) Dirichlet forms, the knowledge of $\mathcal{E}$ and μ is (essentially) equivalent to the knowledge of the semi-group of $(X_t)_{t \geqslant 0}$.
$\mathcal{E}$ is symmetrical if and only if μ is reversible with respect to L.
For any $f, g \in \mathcal{F}(V)$, let F and G be their harmonic extensions. It appears that

$$\mathcal{E}_S(f, g) \ = \ \frac{\mathcal{E}_L(F, G)}{\mu(V)}$$

The assertion about reversibility of Proposition 4 follows immediately.

Isoperimetric quantities

Assume L reversible and consider the eigenvalues of S

$$0 = \sigma_1(S) < \sigma_2(S) \leqslant \sigma_3(S) \leqslant \cdots \leqslant \sigma_{|V|}(S)$$

Furthermore, we define

$$\eta(A) := \frac{\underline{\mu}(\partial A)}{\mu(A)} \qquad \eta'(A) := \frac{\underline{\mu}(\partial A)}{\mu(A \cap V)}$$

where ∂A and $A \cap V$ can be seen respectively as the **interior** and **exterior boundaries**. Consider

$$\rho(A) := \min_{\substack{B \in \mathcal{A} \\ B \subseteq A}} \eta(B) \qquad \rho'(A) := \min_{\substack{B' \in \mathcal{A} \\ B' \subseteq A}} \eta'(B')$$

For any $k \in [\![v]\!]$, introduce the k-th **Cheeger-Steklov constant** of V via

$$\iota_k(S) \quad := \quad \min_{(A_1,\ldots,A_k)\in\mathcal{A}_k} \max_{l\in[\![k]\!]} \rho(A_l)\rho'(A_l)$$

We have the discrete analogue of Theorem 2:

Theorem 5

Assume that L is reversible. There exists a universal positive constant c_3 such that

$$\forall\ k \in [\![v]\!], \qquad \sigma_k(S) \quad \geqslant \quad \frac{c_3}{k^6}\frac{\iota_k(S)}{\|L\|}$$

Approximation of Steklov operators (1)

Idea of the proof: approximation by a Markov process accelerated on $M \backslash V$. For any $r > 0$, consider the irreducible Markov generator defined by

$$\forall\ x \neq y \in M, \qquad L^{(r)}(x,y) \;:=\; \begin{cases} rL(x,y)\,, & \text{if } x \in M\backslash V \\ L(x,y)\,, & \text{if } x \in V \end{cases}$$

Its invariant probability measure $\mu^{(r)}$ is given by

$$\forall\ x \in M, \qquad \mu^{(r)}(x) \;=\; \begin{cases} \frac{\mu(x)}{\mu(V)+(1-\mu(V))/r}\,, & \text{if } x \in V \\ \frac{\mu(x)}{r\mu(V)+1-\mu(V)}\,, & \text{if } x \in M\backslash V \end{cases}$$

Furthermore, if μ is reversible for L, then $\mu^{(r)}$ is reversible for $L^{(r)}$.

Approximation of Steklov operators (2)

Assume μ is reversible for L and denote

$$0 = \lambda_1^{(r)} < \lambda_2^{(r)} \leqslant \lambda_3^{(r)} \leqslant \cdots \leqslant \lambda_m^{(r)}$$

the eigenvalues of $-L^{(r)}$

Proposition 6

For any $k \in [\![|V|]\!]$, we have

$$\lim_{r\to+\infty} \lambda_k^{(r)} = \sigma_k(S)$$

and for any $k \in [\![|M|]\!] \setminus [\![|V|]\!]$,

$$\lim_{r\to+\infty} \lambda_k^{(r)} = +\infty$$

Dirichlet-Steklov operator

To any $A \in \mathcal{A}$ associate a **Dirichlet-Steklov operator** S_A: given $f \in \mathbb{R}^{A \cap V}$, consider the solution $F \in \mathbb{R}^M$ of

$$\begin{cases} L[F](x) = 0\,, & \text{if } x \in A \backslash V \\ F(x) = 0\,, & \text{if } x \in M \backslash A \\ F(x) = f(x)\,, & \text{if } x \in A \cap V \end{cases} \tag{1}$$

and define

$$\forall\, x \in A \cap V, \qquad S_A[f](x) \;:=\; L[F](x)$$

When $A \cap V \neq \emptyset$, S_A is a **subMarkovian generator** (i.e. $S_A(x,y) \geqslant 0$, for any $x \neq y$, and $\sum_{y \in V} S_A(x,y) \leqslant 0$). It may not be irreducible, but Perron-Frobenius' theorem enables to consider the smallest eigenvalue $\sigma_0(A) \geqslant 0$ of $-S_A$.

Dirichlet-Steklov connectivity spectrum

The **Dirichlet-Steklov connectivity spectrum** $(\kappa_1(S), \kappa_2(S), ..., \kappa_{|V|}(S))$ of S via

$$\forall\ k \in [\![|V|]\!], \qquad \kappa_k(S) \quad := \quad \min_{(A_1,...,A_k)\in\mathcal{A}_k} \max_{l\in[\![k]\!]} \sigma_0(A_l)$$

It shares some similarity with the **Steklov connectivity spectrum** $(\iota_1(S), \iota_2(S), ..., \iota_{|V|}(S))$, but it is more stable, due to its spectral nature.

Theorem 7

Assume that L is reversible. There exists a universal constant $c_4 > 0$ such that

$$\forall\ k \in [\![|V|]\!], \qquad \frac{c_4}{k^6}\kappa_k(S) \leqslant \sigma_k(S) \leqslant \kappa_k(S)$$

The proof is based on accelerated approximation, using the higher order Dirichlet-Cheeger inequalities of [Lee, Gharan and Trevisan, 2012].

End of the proof

Theorem 2 follows, if it can be shown that

$$\forall\ A \in \mathcal{A}, \qquad \sigma_0(A) \;\geqslant\; \frac{\rho(A)\rho'(A)}{8\,\|L\|}$$

Indeed, when F is associated to $f \in \mathbb{R}^{A\cap V}$ as in (1), we have

$$\sigma_0(A) \;\geqslant\; \frac{1}{8\,\|L\|}\,\frac{\mu[|dF^2|]}{\mu[F^2]}\,\frac{\mu[|dF^2|]}{\mu[f^2\mathbb{1}_{A\cap V}]}$$

where for any function $G \in \mathbb{R}^M$ and any edge $(x, y) \in M$, $|dG|(x, y) \coloneqq |G(y) - G(x)|$. It remains to use the discrete co-area formula, in the same spirit as [Jammes, 2015], to get the wanted result.

Plan

The assumption on the state space can be relaxed by considering a probability measure space $(M, \mathcal{M}, \mu)$, endowed with a Markov kernel P leaving μ invariant. The boundary set $V \in \mathcal{M}$ must be such that $0 < \mu(V) < 1$ and we take $\mathcal{A}$ the set of $A \in \mathcal{M}$ such that $0 < \mu(A) \leqslant 1$.
Let $Z := (Z(n))_{n \in \mathbb{Z}_+}$ be a Markov chain whose transition kernel is P. For any $A \in \mathcal{A}$, define the **hitting time of** A by Z via

$$\tau_A \quad := \quad \inf\{n \in \mathbb{Z}_+ \ : \ Z(n) \in A\}$$

We assume that P is **weakly mixing**, in the sense that τ_A is a.s. finite, whatever the initial distribution of $Z(0)$ and for any $A \in \mathcal{A}$.

Steklov operators

Consider f a bounded measurable function on V, we associate a function F on M via

$$\forall\ x \in M, \qquad F_f(x) \quad := \quad \mathbb{E}_x[f(Z(\tau_V))]$$

It can be seen as an harmonic extension: F coincides with f on V and satisfies $(P-I)[F] = 0$ on $M \backslash V$. The Steklov operator S is then defined by

$$\forall\ x \in V, \qquad S[f](x) \quad := \quad P[F](x) - f(x)$$

The normalized restriction of μ on V is invariant for S and reversible when μ is reversible. Then we can consider the eigenvalues of $-S$: $0 = \sigma_1(S) \leqslant \sigma_2(S) \leqslant \sigma_3(S) \leqslant \cdots$.

The other definitions considered in the finite case can be extended to the present setting and in the end we get:

Theorem 8

There exists a universal positive constant c_5 such that

$$\forall\ k \in [\![v]\!], \qquad \sigma_k(S) \quad \geqslant \quad \frac{c_5 \iota_k(S)}{k^6}$$

Nevertheless, there is a technical difficulty: for the approximation by acceleration of the quantities $\sigma_0(A)$ to be uniform in $A \in \mathcal{A}$ (after endowing $[0, +\infty]$ with a finite metric), we need that it is easy to get out of $M \setminus V$, in the sense that the **Dirichlet gap** of $M \setminus V$ is positive, namely

$$\inf_{F \in \mathbb{L}^2(\mu) \setminus \{0\}\,:\, F=0 \text{ on } V} \frac{\mu[F(I-P)[F]]}{\mu[F^2]} \quad > \quad 0$$

Non-homogeneous acceleration

To prove Theorem 8 without the above condition, the underlying Markov process must be accelerated in $M\backslash V$ in a way depending on its current position: the further away from V, the faster it must evolve. It amounts to consider an accelerated generator of the form

$$L^{(r)}(x,dy) \quad := \quad \begin{cases} r\varphi(x)(P(x,dy)-\delta_x(dy))\,, & \text{if } x\in M\backslash V \\ \varphi(x)(P(x,dy)-\delta_x(dy))\,, & \text{if } x\in V \end{cases}$$

where $\varphi\ :\ M\rightarrow[1,+\infty)$ is an appropriate acceleration function. Indeed, choosing

$$\forall\ x\in M, \qquad \varphi(x) \quad := \quad \frac{1}{\mathbb{E}_x[\exp(-2\tau_V/\ln(2))]}$$

one can prove that the corresponding Dirichlet gap is larger to 1/2 and it leads to Theorem 8.

Plan

Mass concentration deformations

The previous approach can be adapted to the Riemannian framework to estimate the eigenvalues σ of the Steklov problem:

$$\begin{cases} \Delta f = 0\,, & \text{in } \; M \\ \frac{\partial f}{\partial \nu} = -\sigma f\,, & \text{on } \; \partial M \end{cases} \tag{2}$$

An acceleration procedure was already used by [Lamberti and Provenzano, 2015]: they considered the eigenproblem

$$\begin{cases} \Delta f + \lambda \rho_\epsilon f = 0 & , \quad \text{in } \; M \\ \frac{\partial f}{\partial \nu} = 0 & , \quad \text{on } \; \partial M \end{cases} \tag{3}$$

where

$$\rho_\epsilon(x) \quad := \quad \begin{cases} \epsilon & , \quad \text{if } d(x, \partial M) \geqslant \epsilon \\ \epsilon^{-1} & , \quad \text{otherwise} \end{cases}$$

and proved the convergence as ϵ goes to zero 0_+ of the eigenvalues of (3) toward those of (2).

Mixed Dirichlet-Steklov eigenvalue problem

The same approximation holds for $\sigma_0(A)$, the first eigenvalue of

$$\begin{cases} \Delta f = 0 & , in \;\; A \\ \frac{\partial f}{\partial \nu} = \sigma f & , on \;\; \partial_{\mathrm{e}} A \\ f = 0 & , on \;\; \partial_{\mathrm{i}} A \end{cases}$$

by the first eigenvalue of the **mixed Dirichlet-Neumann problem**

$$\begin{cases} \Delta f + \lambda \rho_\epsilon f = 0 & , in \;\; A \\ \frac{\partial f}{\partial \nu} = 0 & , on \;\; \partial_{\mathrm{e}} A \\ f = 0 & , on \;\; \partial_{\mathrm{i}} A \end{cases}$$

The convergence is even uniform over $A \in \mathcal{A}$.
Theorem 2 follows by the same arguments as above, taking into account higher order Cheeger inequalities for elliptic reversible diffusions on compact manifolds.
Classical examples can be revisited to show the necessity of terms of the same nature as ρ and ρ' in the definition of the Cheeger-Steklov constants.

Steklov-Buser's inequality?

A reciprocal bound to Cheeger's inequality was proven by [Buser, 1982] under the assumption that the Ricci curvature is bounded below by $-(n-1)^2\delta$:

$$\lambda_2(M) \leqslant 2\delta(n-1)h_2(M) + 10h_2^2(M)$$

It is natural to wonder if a similar relation would hold for Steklov operators, especially in view of some recent results on higher order Buser inequalities, see e.g. [Liu and Peyerimhoff, 2018], in a combinatoric setting.

Plan

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Exemple 4 of [Jammes, 2015] can be used to show the necessity of quantities such as $\eta(B)$ and $\eta'(B)$ in the definition of $\iota_k(M)$ for all $k \geqslant 2$.
Consider $M_l := N \times (-l, l)$, where $l > 0$ and N is a closed manifold. The Steklov spectrum of M_l was calculated explicitly by [Colbois, El Soufi and Girouard, 2011]. The set of eigenvalues is

$$\left\{ 0\, l^{-1}, \sqrt{\lambda_k(N)}\tanh(\sqrt{\lambda_k(N)}l), \sqrt{\lambda_k(N)}\coth(\sqrt{\lambda_k(N)}l) \,:\, k \in \mathbb{N} \right\}$$

where $\lambda_k(N)$ are the Laplace eigenvalues of N. For every $k \in \mathbb{N}$, $\sigma_k(M_l) = \mathcal{O}(l)$ as $l \to 0_+$, while for $k \geqslant 2$, $h_k(M_l) \geqslant h_2(M_l) \geqslant c$ for some positive constant c independent of l. It shows the necessity of a quantity such as $\eta'(B)$ in the definition of $\iota_k(M_l)$ for all $k \geqslant 2$.

Example 2

Let $\mathbb{S}^1$ be the unit circle and $\mathbb{S}^1_m$ denote a circle of radius $m > 0$ with their standard metric. Consider the sequence $(M_m := \mathbb{S}^1_m \times (-m^{3/2}, m^{3/2}))_{m\in\mathbb{N}}$ with the product metric. The set of Steklov eigenvalues $\sigma_k(M_m)$ is given as in the previous example. Since $\lambda_k(\mathbb{S}^1_m) = \frac{1}{m^2}\lambda_k(\mathbb{S}^1)$, we have for any fixed $k \geqslant 2$

$$\sigma_k(M_m) \sim m^{3/2}\lambda_k(\mathbb{S}^1_m) = \frac{1}{\sqrt{m}}\lambda_k(\mathbb{S}^1) \quad \text{as } m \to \infty$$

Therefore

$$\forall\ k \in \mathbb{N}, \qquad \lim_{m\to\infty} \sigma_k(M_m) = 0$$

It is easy to check that for every $k \in \mathbb{N}$, $\lim_{m\to\infty} h_k(M_m) = 0$. Furthermore, there exists a constant $C > 0$ independent of m such that $h'_k(M_m) \geqslant C$ for any $k \geqslant 2$. This example shows the necessity of a quantity such as $\eta(B)$ in the definition of $\iota_k(M_m)$ for all $k \geqslant 2$.

Example 3

[Girouard and Polterovich, 2010] studied a family of Cheeger dumbbells $(M_\epsilon)_{\epsilon>0}$ and showed that $\lim_{\epsilon\to 0_+} \sigma_k(M_\epsilon) = 0$ for every $k \in \mathbb{N}$. In their example, M_ϵ is a domain in $\mathbb{R}^2$ consisting of the union of two disjoint Euclidean unit disks connected with a thin rectangular neck of length ϵ and width ϵ^3. It can be checked that $h_2(M_\epsilon) \to 0$ as $\epsilon \to 0$ and that for $k \geqslant 3$, $h_k(M_\epsilon) \geqslant c > 0$, where c is a constant independent of ϵ. This example, as Example 1, shows the necessity of $\eta'(B)$ in $\iota_k(M_\epsilon)$, at least for $k \geqslant 3$. However, in Example 1 the volume of the family of manifolds tends to zero, while in this example the area and the boundary length of M_ϵ are uniformly controlled.

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