On the forward algorithm for stopping problems on continuous-time Markov chains

Laurent Miclo[†*] and Stéphane Villeneuve[‡*]

[†] Toulouse School of Economics,
Institut de Mathématiques de Toulouse,
Université de Toulouse and CNRS, France

[‡] Toulouse School of Economics,
Université de Toulouse Capitole, France

Abstract

In this paper, we revisit the forward algorithm, developed by Irle, to characterize both the value function and the stopping set for a large class of optimal stopping problems on continuous-time Markov Chains. Our objective is to renew the interest of this constructive method by showing its usefulness to solve some constrained optimal stopping problems that have emerged recently.

Keywords: Forward algorithm, Continuous-time Markov chains, constrained Optimal Stopping, American option pricing

1 Introduction

Optimal stopping problems have received a lot of attention in the literature on stochastic control since the seminal work of Wald [23] about sequential analysis while the most recent application of optimal stopping problems have emerged from mathematical finance with both the valuation of American options and the theory of real options, see e.g. [19] and [5]. The first general result of optimal stopping theory for stochastic processes was obtained in discrete time by Snell [21] who characterized the value function of an optimal stopping problem as the least excessive function that is a majorant of the reward[1]. From there on, the theoretical and numerical aspects of the valuation of optimal stopping problems on Markov processes have been the subject of numerous articles in many different models including discrete-time Markov chains (see e.g. [3],[15]), time-homogenous diffusions (see e.g. [4]) and Lévy processes (see e.g.

*LM and SV acknowledge funding from the French National Research Agency (ANR) under the Investments for the Future (Investissements d'Avenir) program, grant ANR-17-EURE-0010.

[1]For a survey of optimal stopping theory for Markov processes, see the book by Shiryaev [20].

[18]) with the appropriate extension of the Snell characterization. This paper is concerned with optimal stopping problems in the setting of a general continuous-time Markov chain. This class of processes, which contains the classic birth-death process, have recently been introduced in finance to model the state of the order book, see [1] or to model growth stocks, see [14]. The paper follows in the footsteps of Eriksson and Pistorius [8] who have shown that the value of an optimal stopping problem for a continuous-time Markov chain can be characterized as the unique solution to a system of variational inequalities when assuming a uniform integrability condition for the payoff function. Furthermore, when the state space of the underlying Markov chain is a subset of $\mathbb{R}$ and when the stopping region is assumed to be an interval, their paper also provides an algorithm to compute the value function.

Eager not to make any a priori assumptions about the shape of the stopping region, we use a different approach that relies on the forward algorithm for optimal stopping problems on Markov chains, introduced by Irle, in the two papers [11] and [12]. Inspired by Howard's policy improvement, Irle proposed a very nice monotone algorithm to compute the value of an optimal stopping problem, which unfortunately did not receive the attention it deserves.[2]Relying on the Snell characterization as the smallest excessive majorant of the payoff function, it consists in a monotone recursive construction of both the value function and the stopping region along a sequence of almost excessive functions build with the hitting times of explicit sets. The main advantage of the monotone approach developed here, is that it converges to the value with minimal assumptions about the continuous-time Markov chain and the payoff function. In particular, we abandon the uniform integrability condition while, unlike [8], the state space is not necessary a subset of the set of real numbers. Such an approach gives a generic constructive method of finding the value function and seems to be designed for computational methods. It is fair to notice however, that this procedure may only give the exact value of the value function after infinite number of steps. A practical exception is given when considering the case of Markov chains with finite number of states where the resulting algorithm resembles the elimination algorithm proposed in [11], [12] and [22] and thus converges in a finite number of steps. For completeness, other constructive iterative methods based on the Snell characterization have to be mentioned. The two contemporaneous papers [10] and [2] in the one-dimensional diffusion case where upper bounds of the value function are build using linear programming while [13] produces an increasing sequence of approximations of the optimal stopping time of a Bermudean option. As applications, we revisit two constrained optimal stopping problems. The first one proposed by Dupuis and Wang in [6] considers the case where the decision-maker is only allowed to stop a Geometric Brownian motion at the jump times of an independent Poisson process. The second problem is an example stemming from the class of optimal stopping problems with stochastic stopping time constraints expressed in terms of the states of a Markov process, see e.g. [7].

2 Formulation of the problem

On a countable state space V endowed with the discrete topology, we consider a Markov generator $\mathcal{L} \coloneqq (L(x,y))_{x,y\in V}$, that is an infinite matrix whose entries are real numbers satisfying

$$\begin{aligned} \forall x \neq y \in V, \quad & L(x,y) \geq 0 \\ \forall x \in V, \quad & L(x,x) = -\sum_{y \neq x} L(x,y) \end{aligned}$$

[2]To be perfectly honest, we didn't know Irle's papers when we started working on the same type of algorithm in the first version of this paper. We are indebted to Sören Christensen for introducing us the Irle's groundbreaking works.

We define $L(x) = -L(x,x)$ and assume that $L(x) < +\infty$ for every $x \in V$.
For any probability measure m on V, let us associate to L a Markov process $X \coloneqq (X_t)_{t\geq 0}$ defined on some probability space $(\Omega, \mathcal{G}, \mathbb{P})$ whose initial distribution is m. First we set $\sigma_0 \coloneqq 0$ and X_0 is sampled according to m. Then we consider an exponential random variable σ_1 of parameter $L(X_0) \coloneqq -L(X_0, X_0)$. If $L(X_0) = 0$, we have a.s. $\sigma_1 = +\infty$ and we take $X_t \coloneqq X_0$ for all $t > 0$, as well as $\sigma_n \coloneqq +\infty$ for all $n \in \mathbb{N}, n \geq 2$. If $L(X_0) > 0$, we take $X_t \coloneqq X_0$ for all $t \in (0, \sigma_1)$ and we sample X_{σ_1} on $V \setminus \{X_0\}$ according to the probability distribution $L(X_0, .)/L(X_0)$. Next, still in the case where $\sigma_1 < +\infty$, we sample an inter-time $\sigma_2 - \sigma_1$ as an exponential distribution of parameter $L(X_{\sigma_1})$. If $L(X_{\sigma_1}) = 0$, we have a.s. $\sigma_2 = +\infty$ and we take $X_t \coloneqq X_{\sigma_1}$ for all $t \in [\sigma_1, +\infty)$, as well as $\sigma_n \coloneqq +\infty$ for all $n \in \mathbb{N}, n \geq 3$. If $L(X_{\sigma_1}) > 0$, we take $X_t \coloneqq X_{\sigma_1}$ for all $t \in [\sigma_1, \sigma_2)$ and we sample X_{σ_2} on $V \setminus \{X_{\sigma_1}\}$ according to the probability distribution $(L(X_{\sigma_1}, .)/L(X_{\sigma_1}))_{x\in V\setminus\{X_{\sigma_1}\}}$. We keep on following the same procedure, where all the ingredients are independent, except for the explicitly mentioned dependences.
In particular, we get a non-decreasing family $(\sigma_n)_{n\in\mathbb{Z}_+}$ of jump times taking values in $\bar{\mathbb{R}}_+ \coloneqq \mathbb{R}_+ \sqcup \{+\infty\}$. Denote the corresponding exploding time

$$\sigma_\infty \;\coloneqq\; \lim_{n\to\infty} \sigma_n \;\in\; \bar{\mathbb{R}}_+$$

When $\sigma_\infty < +\infty$, we must still define X_t for $t \geq \sigma_\infty$. So introduce $\triangle$ a cemetery point not belonging to V and denote $\bar{V} \coloneqq V \sqcup \{\triangle\}$. $\bar{V}$ is seen as the Alexandrov compactification of V. We take $X_t \coloneqq \triangle$ for all $t \geq \sigma_\infty$ to get a $\bar{V}$-valued Markov process X. Let $(\mathcal{G}_t)_{t\geq 0}$ be the completed right-continuous filtration generated by $X \coloneqq (X_t)_{t\geq 0}$ and let $\mathcal{F}$ (resp.$\bar{\mathcal{F}}_+$) be the set of functions defined on V taking values in $\mathbb{R}_+$ (resp. $\bar{\mathbb{R}}_+ \coloneqq \mathbb{R}_+ \sqcup \{+\infty\}$). The generator L acts on $\mathcal{F}$ via

$$\begin{aligned}\forall\, f \in \mathcal{F}, \forall\, x \in V, \qquad \mathcal{L}[f](x) &\coloneqq \sum_{y\in V} L(x,y) f(y) \\ &= \sum_{y\in V\setminus\{x\}} L(x,y)(f(y) - f(x)).\end{aligned}$$

We would like to extend this action on $\bar{\mathcal{F}}_+$, but since its elements are allowed to take the value $+\infty$, it leads to artificial conventions such as $(+\infty)-(+\infty) = 0$. The only reasonable convention is $0\times(+\infty) = 0$, so let us introduce $\mathcal{K}$, the infinite matrix whose diagonal entries are zero and which is coinciding with $\mathcal{L}$ outside the diagonal. Its interest is that $\mathcal{K}$ acts obviously on $\bar{\mathcal{F}}_+$ through

$$\forall\, f \in \bar{\mathcal{F}}_+, \forall\, x \in V, \qquad \mathcal{K}[f](x) \;\coloneqq\; \sum_{y\in V\setminus\{x\}} L(x,y) f(y) \;\in\; \bar{\mathbb{R}}_+. \tag{2.1}$$

In this paper, we will consider an optimal stopping problem with payoff $e^{-rt}\phi(X_t)$, where $\phi \in \bar{\mathcal{F}}_+$ and $r > 0$, given by

$$u(x) \coloneqq \sup_{\tau\in\mathcal{T}} \mathbb{E}_x[e^{-r\tau}\phi(X_\tau)], \tag{2.2}$$

where $\mathcal{T}$ is a set of $\mathcal{G}_t$-adapted stopping times and where the x in index of the expectation indicates that X starts from $x \in V$. A stopping time τ^* is said to be optimal for u if

$$u(x) \;=\; \mathbb{E}_x(e^{-r\tau^*}\phi(X_{\tau^*})).$$

Observe that with our convention, we have $e^{-r\tau}\phi(X_\tau) = 0$ on the set $\{\tau = +\infty\}$.

There are two questions to be solved in connection with Definition (2.2). The first question is to value the function u while the second is to find an optimal stopping time τ^*. Note that optimal stopping times may not exist (see [20] Example 5 p.61) . According to the general optimal stopping theory, an optimal stopping time, if it exists, is related to the set

$$D = \{x \in V : \, u(x) = \phi(x)\} \tag{2.3}$$

called the stopping region. In particular, when ϕ satisfies the uniform integrability condition

$$\mathbb{E}_x \left[\sup_{t \geq 0} e^{-rt} \phi(X_t) \right] \; < \; \infty,$$

the stopping time $\tau_D = \inf\{t \geq 0, \, X_t \in D\}$ is optimal if for all $x \in V, \mathbb{P}_x(\tau_D < \infty) = 1$ (see Shiryaev [20] Theorem 4 p.52).

The main objective of this paper is to provide a recursive construction of both the value function u and the stopping region D when the payoff function is measurable and positive and does not satisfy the uniform integrability condition. However, to present the idea of the Irle's forward approach developed in this paper, Section 3 first consider the case of a finite state space V for which the uniform integrability condition is obviously satisfied. Section 4 is devoted to the general case. Section 5 contains the applications.

3 Finite state space

On a finite set V, the payoff function is bounded and thus the value function u defined by (2.2) is well-defined for every $x \in V$. Moreover, it is well-known (see [20], Theorem 3) that the value function u is the minimal r-excessive function which dominates ϕ. Recall that a function f is r-excessive if $0 \geq \mathcal{L}[f] - rf$. Moreover, on the set $\{u > \phi\}$, u satisfies $\mathcal{L}[u](x) - ru(x) = 0$. Because of the finiteness of V, the process

$$e^{-rt} f(X_t) - f(x) - \int_0^t e^{-rs} \left(\mathcal{L}[f] - rf \right) (X_s) \, ds$$

is a $\mathcal{G}_t$-martingale under $\mathbb{P}_x$ for every function f defined on V and every $x \in V$ which yields by taking expectations, the so-called Dynkin's formula.

We first establish some properties of the stopping region $\mathcal{D}$. Let us introduce the set

$$\mathcal{D}_1 \;\; \coloneqq \;\; \{x \in V, \, \mathcal{L}[\phi](x) - r\phi(x) \leq 0\}$$

and assume that $\phi(x_0) > 0$ for some $x_0 \in V$. We recall that a Markov process X is said to be irreducible if for all $x, y \in V \times V$, $\mathbb{P}_x(T_y < +\infty) > 0$ where

$$T_y \;\; \coloneqq \;\; \inf\{t \geq 0, \, X_t = y\}.$$

Lemma 1. *We have the inclusion $\mathcal{D} \subset \mathcal{D}_1$ and when we assume furthermore that X is irreducible, we have $\mathcal{D} \subset \{x \in V, \phi(x) > 0\}$.*

Proof. Because u is r-excessive, we have for all $x \in \mathcal{D}$,

$$\begin{aligned} 0 &\geq \mathcal{L}[u](x) - ru(x) \\ &= \sum_{y \neq x} L(x,y)u(y) - (r + L(x))\phi(x) \quad \text{because } x \in \mathcal{D} \\ &\geq \sum_{y \neq x} L(x,y)\phi(y) - (r + L(x))\phi(x) \quad \text{because } u \geq \phi \\ &= \mathcal{L}[\phi](x) - r\phi(x). \end{aligned}$$

Therefore, $x \in \mathcal{D}_1$.

For the second inclusion, let T_{x_0} be the first time X hits x_0. We have for all $x \in V$, and every $t \geq 0$,

$$\begin{aligned} u(x) &\geq \mathbb{E}_x[e^{-r(T_{x_0} \wedge t)} \phi(X_{T_{x_0} \wedge t})] \\ &= \phi(x_0)\mathbb{E}_x[e^{-rT_{x_0}} \mathbb{1}_{T_{x_0} \leq t}] + \mathbb{E}_x[e^{-rt}\phi(X_t)\mathbb{1}_{T_{x_0} \geq t}] \end{aligned}$$

Letting t tend to $+\infty$, we obtain because ϕ is bounded on the finite sate space V

$$u(x) \geq \phi(x_0)\mathbb{E}_x[e^{-rT_{x_0}} \mathbb{1}_{T_{x_0} < +\infty}] > 0$$

where the last strict inequality follows from the fact that X is irreducible.

□

Now, we introduce u_1 as the value associated to the stopping strategy *Stop the first time X enters in $\mathcal{D}_1$*. Formally, let us define

$$\tau_1 \coloneqq \inf\{t \geq 0 : X_t \in \mathcal{D}_1\}$$

and

$$u_1(x) \coloneqq \mathbb{E}_x[e^{-r\tau_1}\phi(X_{\tau_1})\mathbb{1}_{\tau_1 < +\infty}]$$

Clearly $u \geq u_1$ by Definition (2.2). Moreover, we have $u_1 = \phi$ on $\mathcal{D}_1$.

Lemma 2. *We have*

- $\forall x \notin \mathcal{D}_1$, $u_1(x) > \phi(x)$ *and* $\mathcal{L}[u_1](x) - ru_1(x) = 0$.
- $\forall x \in \mathcal{D}$, $\mathcal{L}[u_1](x) - ru_1(x) \leq 0$.

Proof. Let $x \notin \mathcal{D}_1$. Applying the Optional Sampling theorem to the bounded martingale

$$M_t = e^{-rt}\phi(X_t) - \phi(x) - \int_0^t e^{-rs}\left(\mathcal{L}[\phi] - r\phi\right)(X_s)\, ds,$$

we have,

$$\begin{aligned} u_1(x) &= \mathbb{E}_x[e^{-r\tau_1}\phi(X_{\tau_1})] \\ &= \phi(x) + \mathbb{E}_x\left[\int_0^{\tau_1} e^{-rs}(L[\phi](X_s) - r\phi(X_s))\, ds\right] \\ &> \phi(x), \end{aligned}$$

because $L[\phi](y) - r\phi(y) > 0$ for $y \notin \mathcal{D}_1$. Moreover, for $x \notin \mathcal{D}_1$, $\tau_1 \geq \sigma_1$ almost surely. Thus, the Strong Markov property yields

$$\begin{aligned} u_1(x) &= \mathbb{E}_x[e^{-r\tau_1}\phi(X_{\tau_1})] \\ &= \mathbb{E}_x[e^{-r\sigma_1}u_1(X_{\sigma_1})] \\ &= \frac{\mathcal{L}[u_1](x) + L(x)u_1(x)}{r + L(x)}, \end{aligned}$$

from which we deduce $\mathcal{L}[u_1](x) - ru_1(x) = 0$.

Because u is r-excessive, we have for all $x \in \mathcal{D}$,

$$\begin{aligned} 0 &\geq \mathcal{L}[u](x) - ru(x) \\ &= \sum_{y \in V} L(x,y)u(y) - r\phi(x) \quad \text{because } x \in \mathcal{D} \\ &\geq \sum_{y \neq x} L(x,y)u_1(y) - (r + L(x))\phi(x) \\ &= \mathcal{L}[u_1](x) - ru_1(x) \quad \text{because } \mathcal{D} \subset \mathcal{D}_1. \end{aligned}$$

□

To start the recursive construction, we introduce the set

$$\mathcal{D}_2 \quad \coloneqq \quad \{x \in \mathcal{D}_1,\, \mathcal{L}[u_1](x) - ru_1(x) \leq 0\}$$

and the function

$$u_2(x) \quad \coloneqq \quad \mathbb{E}_x[e^{-r\tau_2}\phi(X_{\tau_2})\mathbb{1}_{\tau_2 < +\infty}]$$

where

$$\tau_2 \quad \coloneqq \quad \inf\{t \geq 0 :\, X_t \in \mathcal{D}_2\}$$

Observe that if $\mathcal{D}_2 = \mathcal{D}_1$, u_1 is a r-excessive majorant of ϕ and therefore $u_1 \geq u$. Because the reverse inequality holds by definition, the procedure stops.
By induction, we shall define a sequence $(u_n, \mathcal{D}_n)$ for $n \in \mathbb{Z}_+$ starting from $(u_1, \mathcal{D}_1)$ by

$$\mathcal{D}_{n+1} \quad \coloneqq \quad \{x \in \mathcal{D}_n,\, \mathcal{L}[u_n](x) - ru_n(x) \leq 0\}$$

and

$$u_{n+1}(x) \quad \coloneqq \quad \mathbb{E}_x[e^{-r\tau_{n+1}}\phi(X_{\tau_{n+1}})\mathbb{1}_{\tau_{n+1} < +\infty}]$$

where

$$\tau_{n+1} \quad \coloneqq \quad \inf\{t \geq 0 :\, X_t \in \mathcal{D}_{n+1}\}.$$

Next lemma proves a key monotonicity result.

Lemma 3. *We have $u_{n+1} \geq u_n$ and $\forall x \notin \mathcal{D}_{n+1}$, $u_{n+1}(x) > \phi(x)$.*

Proof. To start the induction, we assume using Lemma 2 that u_n satisfies

$$\forall x \in V \setminus D_n, \mathcal{L}[u_n](x) - ru_n(x) = 0 \text{ and } u_n(x) > \phi(x) \tag{3.1}$$
$$\forall x \in D_n, \quad u_n(x) = \phi(x). \tag{3.2}$$

For $x \in \mathcal{D}_{n+1} \subset \mathcal{D}_n$, we have $u_{n+1}(x) = \phi(x) = u_n(x)$. On the other hand, for $x \notin \mathcal{D}_{n+1}$, we have

$$\begin{aligned} u_{n+1}(x) &= \mathbb{E}_x[e^{-r\tau_{n+1}}\phi(X_{\tau_{n+1}})\mathbb{1}_{\tau_{n+1}<+\infty}] \\ &= \mathbb{E}_x[e^{-r\tau_{n+1}}u_n(X_{\tau_n})\mathbb{1}_{\tau_{n+1}<+\infty}] \quad \text{because } \mathcal{D}_{n+1} \subset \mathcal{D}_n \\ &= u_n(x) + \mathbb{E}_x\left[\int_0^{\tau_{n+1}} e^{-rs}(\mathcal{L}[u_n](X_s) - ru_n(X_s))\, ds\right] \\ &\geq u_n(x), \end{aligned}$$

because $\mathcal{L}[u_n] - u_n \geq 0$ outside $\mathcal{D}_{n+1}$.
Let $x \notin \mathcal{D}_{n+1}$. If $x \notin \mathcal{D}_n$, we have $u_n(x) > \phi(x)$ and thus $u_{n+1}(x) > \phi(x)$. Now, let $x \in \mathcal{D}_n \cap \mathcal{D}_{n+1}^c$ and let us define

$$\hat{\tau} \;\; := \;\; \inf\{t \geq 0,\, X_t \notin \mathcal{D}_n \cap \mathcal{D}_{n+1}^c\}.$$

Clearly, $\hat{\tau} \leq \tau_{n+1}$. Therefore by the Strong Markov property,

$$\begin{aligned} u_{n+1}(x) &= \mathbb{E}_x[e^{-r\hat{\tau}}u_{n+1}(X_{\hat{\tau}})\mathbb{1}_{\hat{\tau}<+\infty}] \\ &\geq \mathbb{E}_x[e^{-r\hat{\tau}}u_n(X_{\hat{\tau}})\mathbb{1}_{\hat{\tau}<+\infty}] \\ &> u_n(x), \end{aligned}$$

because $\mathcal{L}[u_n] - u_n > 0$ on the set $\mathcal{D}_n \cap \mathcal{D}_{n+1}^c$. □

According to Lemma 3, the sequence $(u_n)_n$ is increasing and satisfies $u_n \geq \phi$ with strict inequality outside $\mathcal{D}_n$, while by construction, the sequence $(\mathcal{D}_n)_n$ is decreasing. It follows that we can define a function u_∞ on V by

$$u_\infty(x) \;\; = \;\; \lim_{n\to\infty} u_n(x)$$

and a set by

$$\mathcal{D}_\infty \;\; := \;\; \bigcap_{n\in\mathbb{Z}_+} \mathcal{D}_n.$$

We are in a position to state our first result.

Theorem 4. *We have $u_\infty = u$ and $\mathcal{D}_\infty = D$.*

Proof. By definition, $u \geq u_n$ for every $n \in \mathbb{Z}_+$ and thus passing to the limit, we have $u \geq u_\infty$. To show the reverse inequality, we first notice that for every $n \in \mathbb{Z}_+$, we have $u_n \geq \phi$ and thus $u_\infty \geq \phi$.
If $x \in \mathcal{D}_\infty$ then $x \in \mathcal{D}_{n+1}$ for every $n \in \mathbb{Z}_+$ and thus $\mathcal{L}[u_n](x) - ru_n(x) \leq 0$ for every $n \in \mathbb{Z}_+$. Passing to the limit, we obtain

$$\mathcal{L}[u_\infty](x) - ru_\infty(x) \;\; \leq \;\; 0 \qquad \forall x \in \mathcal{D}_\infty.$$

If $x \notin \mathcal{D}_\infty$ then there is some n_0 such that $x \notin \mathcal{D}_n$ for $n \geq n_0$. Thus, for such a $n \geq n_0$, we have

$$\mathcal{L}[u_n](x) - ru_n(x) \quad = \quad 0.$$

Passing to the limit, we obtain

$$\mathcal{L}[u_\infty](x) - ru_\infty(x) \quad = \quad 0 \qquad \forall x \notin \mathcal{D}_\infty.$$

To conclude, we observe that because for every $x \in V$, we have $\mathcal{L}[u_\infty](x) - ru_\infty(x) \leq 0$, we have for every stopping time τ

$$u_\infty(x) \quad \geq \quad \mathbb{E}[e^{-r\tau} u_\infty(X_\tau)],$$

from which we deduce that

$$u_\infty(x) \geq \mathbb{E}[e^{-r\tau}\phi(X_\tau)], \tag{3.3}$$

because $u_\infty \geq \phi$. Taking the supremum over τ at the right-hand side of (3.3), we obtain $u_\infty \geq u$. Equality $u = u_\infty$ implies that $\mathcal{D}_\infty \subset \mathcal{D}$. To show the reverse inclusion, let $x \notin \mathcal{D}_\infty$ which means that $x \notin \mathcal{D}_n$ for n larger than some n_0. Lemma 3 yields that $u_n(x) > \phi(x)$ for $n \geq n_0$ and because u_n is increasing, we deduce that $u_\infty(x) > \phi(x)$ for $x \notin \mathcal{D}_\infty$ which concludes the proof. □

Remark 5. *Because V is finite, the sequence $(u_n)_n$ is constant after some $n_0 \leq card(V)$ and therefore the procedure stops after at most $card(V)$ steps.*

Example 6. *Let $(X_t)_{t\geq 0}$ be a birth-death process on the set of integers $V_N = \{-N, \ldots, N\}$ stopped the first time it hits $-N$ or N. We define for $x \in V_N \setminus \{-N, N\}$,*

$$\begin{cases} L(x, x+1) &= \lambda \geq 0, \\ L(x, x-1) &= \mu \geq 0, \\ \quad L(x) &= \lambda + \mu, \end{cases}$$

and $L(-N) = L(N) = 0$. We define $\phi(x) = \max(x, 0)$ as the reward function.
Clearly, $u(-N) = 0 = \phi(-N)$ and $u(N) = N = \phi(N)$ thus the stopping region contains the extreme points $\{-N, N\}$. We define

$$\mathcal{D}_1 \quad \coloneqq \quad \{x \in V_N, \, \mathcal{L}[\phi](x) - r\phi(x) \leq 0\}.$$

A direct computation shows that $\mathcal{L}[\phi](x) - r\phi(x) = 0$ for $-N+1 \leq x \leq -1$, $\mathcal{L}[\phi](0) - r\phi(0) = \lambda$ and $\mathcal{L}[\phi](x) - r\phi(x) = \lambda - \mu - rx$ for $1 \leq x \leq N-1$. Therefore,

$$\mathcal{D}_1 \quad = \quad \{-N, -N+1, \ldots, -1\} \cup \{x_1, \ldots, N\},$$

with $x_1 = \lceil \frac{\lambda-\mu}{r} \rceil$, where $\lceil x \rceil$ is the least integer greater than or equal to x. In particular, when $\lambda \leq \mu$, we have $x_1 = 0$ and thus $\mathcal{D}_1 = V_N = \mathcal{D}$. Assume now that $\lambda > \mu$. To start the induction, we define

$$\tau_1 \quad \coloneqq \quad \inf\{t \geq 0 : \, X_t \in \mathcal{D}_1\}$$

and

$$u_1(x) \quad \coloneqq \quad \mathbb{E}_x[e^{-r\tau_1}\phi(X_{\tau_1})\mathbb{1}_{\tau_1 < +\infty}]$$

and we construct u_1 *by solving for* $0 \leq x \leq x_1 - 1$, *the linear equation*

$$\lambda u_1(x+1) + \mu u_1(x-1) - (r+\lambda+\mu)u_1(x) \;=\; 0 \text{ with } u_1(-1) \;=\; 0 \text{ and } u_1(x_1) \;=\; x_1.$$

The function u_1 *is thus explicit and denoting*

$$\Delta \;:=\; (r+\lambda+\mu)^2 - 4\lambda\mu > 0,$$

$$\theta_1 \;=\; \frac{r+\lambda+\mu-\sqrt{\Delta}}{2\lambda} \text{ and } \theta_2 \;=\; \frac{r+\lambda+\mu+\sqrt{\Delta}}{2\lambda},$$

we have

$$u_1(x) \;=\; x_1 \frac{\theta_2^{x+1} - \theta_1^{x+1}}{\theta_2^{x_1+1} - \theta_1^{x_1+1}}.$$

Observe that $\theta_2 + \theta_1 > 0$ *and thus* $\mathcal{L}[u_1](-1) - ru_1(-1) = \lambda u_1(0) > 0$. *As a consequence,* -1 *does not belong to the set*

$$\mathcal{D}_2 \;=\; \{x \in \mathcal{D}_1,\, \mathcal{L}[u_1](x) - ru_1(x) \leq 0\}.$$

Therefore, if $\mathcal{L}[u_1](x_1) - ru_1(x_1) = \mu u_1(x_1 - 1) + \lambda - (r+\mu)x_1 \leq 0$, *we have*

$$\mathcal{D}_2 \;=\; \{-N, -N+1, \ldots, -2\} \cup \{x_1, \ldots, N\},$$

or, if $\mathcal{L}[u_1](x_1) - ru_1(x_1) = \mu u_1(x_1 - 1) + \lambda - (r+\mu)x_1 > 0$

$$\mathcal{D}_2 \;=\; \{-N, -N+1, \ldots, -2\} \cup \{x_1+1, \ldots, N\}.$$

Following our recursive procedure, after N *steps, we shall have eliminated the negative integers and thus obtain*

$$\mathcal{D}_N \;=\; \{-N\} \cup \{x_N, \ldots, N\}$$

for some $x_1 \leq x_N \leq N$. *Note that for* $-N \leq x \leq x_N$, *we have*

$$u_N(x) \;=\; x_N \frac{\theta_2^{x+N} - \theta_1^{x+N}}{\theta_2^{x_N+N} - \theta_1^{x_N+N}}.$$

If $\lambda u_N(x_n+1) + \mu u_N(x_N-1) - (r+\lambda+\mu)u_N(x_N) = \lambda - (r+\mu)x_N + \mu x_N \frac{\theta_2^{x_N-1+N} - \theta_1^{x_N-1+N}}{\theta_2^{x_N+N} - \theta_1^{x_N+N}} \leq 0$, *the stopping region coincides with* $\mathcal{D}_N$, *else we define*

$$\mathcal{D}_{N+1} \;=\; \{-N\} \cup \{x_{N+1}, \ldots, N\}, \text{ with } x_{N+1} \;=\; x_N + 1$$

and

$$u_{N+1}(x) \;=\; x_{N+1} \frac{\theta_2^{x+N} - \theta_1^{x+N}}{\theta_2^{x_{N+1}+N} - \theta_1^{x_{N+1}+N}}$$

and we repeat the procedure.

4 General state space

4.1 Countable State Space

When considering countable finite state space, Dynkin's formula that has been used in the proofs of Lemma 2 and 3 is not directly available, because nothing prevents the payoff to take arbitrarily large values. Nevertheless, we will adapt the strategy used in the case of a finite state space to build a monotone dynamic approach to the value function in the case of a countable finite state space.
Hereafter, we set some payoff function $\phi \in \bar{\mathcal{F}}_+ \setminus \{0\}$ and $r > 0$. We will construct a subset $D_\infty \subset V$ and a function $u_\infty \in \bar{\mathcal{F}}_+$ by the following recursive algorithm.
We begin by taking $D_0 \coloneqq V$ and $u_0 \coloneqq \phi$. Next, let us assume that $D_n \subset V$ and $u_n \in \bar{\mathcal{F}}_+$ have been built for some $n \in \mathbb{Z}_+$ such that

$$\forall x \in V \setminus D_n, \quad (r + L(x))u_n(x) = \mathcal{K}[u_n](x) \tag{4.1}$$

$$\forall x \in D_n, \quad u_n(x) = \phi(x). \tag{4.2}$$

Observe that it is trivially true for $n = 0$. Then, we define the subset D_{n+1} as follows

$$D_{n+1} \quad \coloneqq \quad \{x \in D_n \,:\, \mathcal{K}[u_n](x) \leq (r + L(x))u_n(x)\} \tag{4.3}$$

where the inequality is understood in $\bar{\mathbb{R}}_+$.
Next, we consider the stopping time

$$\tau_{n+1} \quad \coloneqq \quad \inf\{t \geq 0 \,:\, X_t \in D_{n+1}\}$$

with the usual convention that $\inf \emptyset = +\infty$. For $m \in \mathbb{Z}_+$, define furthermore the stopping time

$$\tau_{n+1}^{(m)} \quad \coloneqq \quad \sigma_m \wedge \tau_{n+1}$$

and the function $u_{n+1}^{(m)} \in \bar{\mathcal{F}}_+$ given by

$$\forall\, x \in V, \qquad u_{n+1}^{(m)}(x) \quad \coloneqq \quad \mathbb{E}_x[\exp(-r\tau_{n+1}^{(m)})u_n(X_{\tau_{n+1}^{(m)}})]. \tag{4.4}$$

Remark 7. *The non-negative random variable* $\exp(-r\tau_{n+1}^{(m)})u_n(X_{\tau_{n+1}^{(m)}})$ *is well-defined, even if* $\tau_{n+1}^{(m)} = +\infty$, *since the convention* $0 \times (+\infty) = 0$ *imposes that* $\exp(-r\tau_{n+1}^{(m)})u_n(X_{\tau_{n+1}^{(m)}}) = 0$ *whatever would be* $X_{\tau_{n+1}^{(m)}}$, *which is not defined in this case. The occurrence of* $\tau_{n+1}^{(m)} = +\infty$ *should be quite exceptional: we have*

$$\{\tau_{n+1}^{(m)} = +\infty\} \quad = \quad \{\tau_{n+1} = +\infty \text{ and } L(X_{\tau_{n+1}^{(m)}}) = 0\}$$

in particular it never happens if $L(x) > 0$ *for all* $x \in V$, *i.e. when* $\triangle$ *is the only possible absorbing point for* X.

Our first result shows that the sequence $(u_{n+1}^{(m)})_{m \in \mathbb{Z}_+}$ is non-decreasing.

Lemma 8. *We have*

$$\forall\, m \in \mathbb{Z}_+, \forall\, x \in V, \qquad u_{n+1}^{(m)}(x) \quad \leq \quad u_{n+1}^{(m+1)}(x)$$

Proof. We first compute

$$\begin{aligned} u_{n+1}^{(m+1)}(x) &\coloneqq \mathbb{E}_x[\exp(-r\tau_{n+1}^{(m+1)})u_n(X_{\tau_{n+1}^{(m+1)}})] \\ &= \mathbb{E}_x[\mathbb{1}_{\tau_{n+1}\leq\sigma_m}\exp(-r\tau_{n+1}^{(m+1)})u_n(X_{\tau_{n+1}^{(m+1)}})] + \mathbb{E}_x[\mathbb{1}_{\tau_{n+1}>\sigma_m}\exp(-r\tau_{n+1}^{(m+1)})u_n(X_{\tau_{n+1}^{(m+1)}})] \end{aligned}$$

Note that on the event $\{\tau_{n+1} \leq \sigma_m\}$, we have that $\tau_{n+1}^{(m+1)} = \tau_{n+1} = \tau_{n+1}^{(m)}$, so the first term in the above r.h.s. is equal to

$$\mathbb{E}_x[\mathbb{1}_{\tau_{n+1}\leq\sigma_m}\exp(-r\tau_{n+1}^{(m+1)})u_n(X_{\tau_{n+1}^{(m+1)}})] = \mathbb{E}_x[\mathbb{1}_{\tau_{n+1}\leq\sigma_m}\exp(-r\tau_{n+1}^{(m)})u_n(X_{\tau_{n+1}^{(m)}})] \tag{4.5}$$

On the event $\{\tau_{n+1} > \sigma_m\}$, we have that $\tau_{n+1}^{(m+1)} = \tau_{n+1}^{(m)} + \sigma_1 \circ \theta_{\tau_{n+1}^{(m)}}$, where θ_t, for $t \geq 0$, is the shift operator by time $t \geq 0$ on the underlying canonical probability space $\mathbb{D}(\mathbb{R}_+, \bar{V})$ of RCLL trajectories. Using the Strong Markov property of X, we get that

$$\mathbb{E}_x[\mathbb{1}_{\tau_{n+1}>\sigma_m}\exp(-r\tau_{n+1}^{(m+1)})u_n(X_{\tau_{n+1}^{(m+1)}})] = \mathbb{E}_x\left[\mathbb{1}_{\tau_{n+1}>\sigma_m}\exp(-r\tau_{n+1}^{(m)})\mathbb{E}_{X_{\tau_{n+1}^{(m)}}}[\exp(-r\sigma_1)u_n(X_{\sigma_1})]\right] \tag{4.6}$$

For $y \in V$, consider two situations:

- if $L(y) = 0$, we have a.s. $\sigma_1 = +\infty$ and as in Remark 7, we get

$$\mathbb{E}_y[\exp(-r\sigma_1)u_n(X_{\sigma_1})] = 0$$

- if $L(y) > 0$, we compute, in $\bar{\mathbb{R}}_+$,

$$\begin{aligned} \mathbb{E}_y[\exp(-r\sigma_1)u_n(X_{\sigma_1})] &= \int_0^{+\infty} \exp(-rs)L(y)\exp(-L(y)s) \sum_{z\in V\setminus\{y\}} \frac{L(y,z)}{L(y)} u_n(z) \\ &= \int_0^{+\infty} \exp(-rs)\exp(-L(y)s) \sum_{z\in V\setminus\{y\}} L(y,z)u_n(z) \\ &= \frac{1}{r+L(y)}\mathcal{K}[u_n](y) \end{aligned}$$

By our conventions, the equality

$$\mathbb{E}_y[\exp(-r\sigma_1)u_n(X_{\sigma_1})] = \frac{1}{r+L(y)}\mathcal{K}[u_n](y) \tag{4.7}$$

is then true for all $y \in V$.
For $y \in D_n$, due to (4.1), the r.h.s. is equal to $u_n(y)$. For $y \in D_n \setminus D_{n+1}$, by definition of D_{n+1} in (4.3), the r.h.s. of (4.7) is bounded below by $u_n(y)$. It follows that for any $y \notin D_{n+1}$,

$$\mathbb{E}_y[\exp(-r\sigma_1)u_n(X_{\sigma_1})] \geq u_n(y)$$

On the event $\{\tau_{n+1} > \sigma_m\}$, we have $X_{\tau_{n+1}^{(m)}} \notin D_{n+1}$ and thus

$$\mathbb{E}_{X_{\tau_{n+1}^{(m)}}}[\exp(-r\sigma_1)u_n(X_{\sigma_1})] \geq u_n(X_{\tau_{n+1}^{(m)}})$$

Coming back to (4.6), we deduce that

$$\mathbb{E}_x[\mathbb{1}_{\tau_{n+1}>\sigma_m}\exp(-r\tau_{n+1}^{(m+1)})u_n(X_{\tau_{n+1}^{(m+1)}})] \quad \geq \quad \mathbb{E}_x[\mathbb{1}_{\tau_{n+1}>\sigma_m}\exp(-r\tau_{n+1}^{(m)})u_n(X_{\tau_{n+1}^{(m)}})]$$

and taking into account (4.5), we conclude that

$$\begin{aligned} u_{n+1}^{(m+1)}(x) &\geq \mathbb{E}_x[\mathbb{1}_{\tau_{n+1}\leq\sigma_m}\exp(-r\tau_{n+1}^{(m)})u_n(X_{\tau_{n+1}^{(m)}})]+\mathbb{E}_x[\mathbb{1}_{\tau_{n+1}>\sigma_m}\exp(-r\tau_{n+1}^{(m)})u_n(X_{\tau_{n+1}^{(m)}})] \\ &= \mathbb{E}_x[\exp(-r\tau_{n+1}^{(m)})u_n(X_{\tau_{n+1}^{(m)}})] \\ &= u_{n+1}^{(m)}(x) \end{aligned}$$

□

The monotonicity property of Lemma 8 enables us to define the function $u_{n+1}\in\bar{\mathcal{F}}_+$ via

$$\forall\, x\in V, \qquad u_{n+1}(x) \quad \coloneqq \quad \lim_{m\to\infty} u_{n+1}^{(m)}(x)$$

ending the iterative construction of the pair (D_{n+1}, u_{n+1}) from (D_n, u_n). It remains to check that:

Lemma 9. *The assertion* (4.1) *is satisfied with n replaced by $n+1$.*

Proof. Consider $x\in V\setminus D_{n+1}$ for which $\tau_{n+1}^{(m+1)}\geq\sigma_1$, $\mathbb{P}_x$ a. s.. For the Markov process X starting from x, we have for any $m\in\mathbb{Z}_+$,

$$\tau_{n+1}^{(m+1)} \quad = \quad \sigma_1+\tau_{n+1}^{(m)}\circ\sigma_1$$

The Strong Markov property of X then implies that

$$\begin{aligned} u_{n+1}^{(m+1)}(x) &= \mathbb{E}_x\left[\exp(-r\sigma_1)\mathbb{E}_{X_{\sigma_1}}[\exp(-r\tau_{n+1}^{(m)})u_n(X_{\tau_{n+1}^{(m)}})]\right] \\ &= \mathbb{E}_x\left[\exp(-r\sigma_1)u_{n+1}^{(m)}(X_{\sigma_1})\right] \\ &= \frac{1}{r+L(x)}\mathcal{K}[u_{n+1}^{(m)}](x) \end{aligned}$$

by resorting again to the computations of the proof of Lemma 8. Monotone convergence insures that

$$\lim_{m\to\infty}\mathcal{K}[u_{n+1}^{(m)}](x) \quad = \quad \mathcal{K}[u_{n+1}](x)$$

so we get that for $x\in V\setminus D_{n+1}$,

$$(r+L(x))u_{n+1}(x) \quad = \quad \mathcal{K}[u_{n+1}](x)$$

as wanted. □

The sequence $(D_n)_{n\in\mathbb{Z}_+}$ is non-increasing by definition, as a consequence we can define

$$D_\infty \quad \coloneqq \quad \bigcap_{n\in\mathbb{Z}_+} D_n$$

From Lemma 8, we deduce that for any $n \in \mathbb{Z}_+$,

$$\forall\, x \in V, \qquad \begin{array}{rcl} u_{n+1}(x) & \geq & u_{n+1}^{(0)}(x) \\ & = & u_n(x) \end{array}$$

It follows that we can define the function $u_\infty \in \bar{\mathcal{F}}_+$ as the non-decreasing limit

$$\forall\, x \in V, \qquad u_\infty(x) \;=\; \lim_{n\to\infty} u_n(x) \;\in\; \bar{\mathbb{R}}_+$$

The next two propositions establish noticeable properties of the pair (D_∞, u_∞):

Proposition 10. *We have:*

$$\forall\, x \in D_\infty, \qquad \left\{ \begin{array}{rcl} u_\infty(x) & = & \phi(x) \\ \mathcal{K}[u_\infty](x) & \leq & (r+L(x))u_\infty(x) \end{array} \right.$$

$$\forall\, x \in V \setminus D_\infty, \qquad \left\{ \begin{array}{rcl} u_\infty(x) & \geq & \phi(x) \\ \mathcal{K}[u_\infty](x) & = & (r+L(x))u_\infty(x) \end{array} \right.$$

Proof. Since $u_0 = \phi$, the fact that $(u_n)_{n\in\mathbb{Z}_+}$ is a non-decreasing sequence implies that $u_\infty \geq \phi$. To show there is an equality on D_∞, it is sufficient to show that

$$\forall\, n \in \mathbb{Z}_+, \forall\, x \in D_n, \qquad u_n(x) \;=\; \phi(x)$$

This is proven by an iterative argument on $n \in \mathbb{Z}_+$. For $n = 0$, it corresponds to the equality $u_0 = \phi$. Assume that $u_n = \phi$ on D_n, for some $n \in \mathbb{Z}_+$. For $x \in D_{n+1}$, we have $\tau_{n+1} = 0$ and thus for any $m \in \mathbb{Z}_+$, we get $\tau_{n+1}^{(m)} = 0$. From (4.4), we deduce that

$$\forall\, x \in D_{n+1}, \qquad u_{n+1}^{(m)} \;=\; u_n(x) \;=\; \phi(x)$$

Letting m go to infinity, it yields that $u_{n+1} = \phi$ on D_{n+1}.
Consider $x \in V \setminus D_\infty$. There exists $N(x) \in \mathbb{Z}_+$ such that for any $n \geq N(x)$, we have $x \in V \setminus D_n$. Then passing at the limit for large n in (4.1), we get, via another use of monotone convergence, that

$$\forall\, x \in V \setminus D_\infty, \qquad (r+L(x))u_\infty(x) = \mathcal{K}[u_\infty](x)$$

For $x \in D_\infty$, we have $x \in D_{n+1}$ for any $n \in \mathbb{Z}_+$ and thus from (4.3), we have $\mathcal{K}[u_n](x) \leq (r+L(x))u_n(x)$. Letting n go to infinity, we deduce that

$$\forall\, x \in D_\infty, \qquad \mathcal{K}[u_\infty](x) \leq (r+L(x))u_\infty(x)$$

□

In fact, u_∞ is a strict majorant of ϕ on $V \setminus D_\infty$ as proved in the following

Proposition 11. *We have*

$$\forall\, x \in V \setminus D_\infty, \qquad u_\infty(x) \;>\; \phi(x)$$

It follows that

$$D_\infty \;=\; \{x \in V \,:\, u_\infty(x) = \phi(x)\}$$

Proof. Consider $x \in V \setminus D_\infty$, there exists a first integer $n \in \mathbb{Z}_+$ such that $x \in D_n$ and $x \notin D_{n+1}$. From (4.1) and $x \in V \setminus D_{n+1}$, we deduce that

$$\mathcal{K}[u_{n+1}](x) = (r + L(x))u_{n+1}(x)$$

From (4.3) and $x \in V \setminus D_{n+1}$, we get

$$\mathcal{K}[u_n](x) > (r + L(x))u_n(x)$$

Putting together these two inequalities and the fact that $\mathcal{K}[u_{n+1}] \geq \mathcal{K}[u_n]$, we end up with

$$(r + L(x))u_{n+1}(x) > (r + L(x))u_n(x)$$

which implies that

$$\begin{aligned} \phi(x) &\leq u_n(x) \\ &< u_{n+1}(x) \\ &\leq u_\infty(x) \end{aligned}$$

namely $\phi(x) < u_\infty(x)$.
This argument shows that

$$\{x \in V \,:\, u_\infty(x) = \phi(x)\} \subset D_\infty$$

The reverse inclusion is deduced from Proposition 10. □

Another formulation of the functions u_n, for $n \in \mathbb{N}$, will be very useful for the characterization of their limit u_∞. For $n, m \in \mathbb{Z}_+$, let us modify Definition (4.4) to define a function $\widetilde{u}_{n+1}^{(m)}$ as

$$\forall\, x \in V, \qquad \widetilde{u}_{n+1}^{(m)}(x) \coloneqq \mathbb{E}_x\left[\exp(-r\tau_{n+1}^{(m)})\phi(X_{\tau_{n+1}^{(m)}})\right] \tag{4.8}$$

A priori there is no monotonicity with respect to m, so we define

$$\forall\, x \in V, \qquad \widetilde{u}_{n+1}(x) \coloneqq \liminf_{m\to\infty} \widetilde{u}_{n+1}^{(m)}(x)$$

A key observation is:

Lemma 12. *For any $n \in \mathbb{N}$, we have $\widetilde{u}_n = u_n$.*

Proof. Since for any $n \in \mathbb{Z}_+$, we have $u_n \geq \phi$, we get from a direct comparison between (4.4) and (4.8) that for any $m \in \mathbb{Z}_+$, $\widetilde{u}_{n+1}^{(m)} \leq u_{n+1}^{(m)}$, so letting m go to infinity, we deduce that

$$\widetilde{u}_{n+1} \leq u_{n+1} \tag{4.9}$$

The reverse inequality is proven by an iteration over n.
More precisely, since $u_0 = \phi$, we get by definition that $\widetilde{u}_1 = u_1$.

Assume that the equality $\widetilde{u}_n = u_n$ is true for some $n \in \mathbb{N}$, and let us show that $\widetilde{u}_{n+1} = u_{n+1}$. For any $m \in \mathbb{Z}_+$, we have

$$\begin{aligned}
\forall\, x \in V, \qquad u_{n+1}^{(m)}(x) &= \mathbb{E}_x\left[\exp(-r\tau_{n+1}^{(m)})u_n(X_{\tau_{n+1}^{(m)}})\right] \\
&= \mathbb{E}_x\left[\exp(-r\tau_{n+1}^{(m)})\widetilde{u}_n(X_{\tau_{n+1}^{(m)}})\right] \\
&= \mathbb{E}_x\left[\exp(-r\tau_{n+1}^{(m)})\liminf_{l\to\infty}\widetilde{u}_n^{(l)}(X_{\tau_{n+1}^{(m)}})\right] \\
&\leq \liminf_{l\to\infty}\mathbb{E}_x\left[\exp(-r\tau_{n+1}^{(m)})\widetilde{u}_n^{(l)}(X_{\tau_{n+1}^{(m)}})\right]
\end{aligned}$$

where we used Fatou's lemma. From (4.8) and the Strong Markov property, we deduce that

$$\begin{aligned}
\mathbb{E}_x\left[\exp(-r\tau_{n+1}^{(m)})\widetilde{u}_n^{(l)}(X_{\tau_{n+1}^{(m)}})\right] &= \mathbb{E}_x\left[\exp(-r\tau_{n+1}^{(m)})\mathbb{E}_{X_{\tau_{n+1}^{(m)}}}\left[\exp(-r\tau_{n+1}^{(l)})\phi(X_{\tau_{n+1}^{(l)}})\right]\right] \\
&= \mathbb{E}_x\left[\exp(-r\tau_{n+1}^{(m+l)})\phi(X_{\tau_{n+1}^{(m+l)}})\right]
\end{aligned}$$

It follows that

$$\begin{aligned}
u_{n+1}^{(m)}(x) &\leq \liminf_{l\to\infty}\mathbb{E}_x\left[\exp(-r\tau_{n+1}^{(m+l)})\phi(X_{\tau_{n+1}^{(m+l)}})\right] \\
&= \widetilde{u}_{n+1}(x)
\end{aligned}$$

It remains to let m go to infinity to get $u_{n+1} \leq \widetilde{u}_{n+1}$ and $u_{n+1} = \widetilde{u}_{n+1}$, taking into account (4.9). □

Let $\mathcal{T}$ be the set of $\bar{\mathbb{R}}_+$-valued stopping times with respect to the filtration generated by X. For $\tau \in \mathcal{T}$ and $m \in \mathbb{Z}_+$, we define

$$\tau^{(m)} \coloneqq \sigma_m \wedge \tau$$

Extending the observation of Remark 7, it appears that for any $m \in \mathbb{Z}_+$, the quantity

$$\forall\, x \in V, \qquad u^{(m)}(x) \coloneqq \sup_{\tau\in\mathcal{T}}\mathbb{E}_x\left[\exp(-r\tau^{(m)})\phi(X_{\tau^{(m)}})\right] \tag{4.10}$$

is well-defined in $\bar{\mathbb{R}}_+$. It is non-decreasing with respect to $m \in \mathbb{Z}_+$, since for any $\tau \in \mathcal{T}$ and any $m \in \mathbb{Z}_+$, $\tau^{(m)}$ can be written as $\widetilde{\tau}^{(m+1)}$, with $\widetilde{\tau} \coloneqq \tau^{(m)} \in \mathcal{T}$. Thus we can define a function $\hat{u}$ by

$$\forall\, x \in V, \qquad \hat{u}(x) \coloneqq \lim_{m\to\infty} u^{(m)}(x)$$

By definition of the value function u given by (2.2), we have $u^{(m)}(x) \leq u(x)$ for every $m \in \mathbb{Z}_+$ and thus $\hat{u}(x) \leq u(x)$ for every $x \in V$. To show the reverse inequality, consider any stopping time $\tau \in \mathcal{T}$ and apply Fatou Lemma to get

$$\begin{aligned}
\mathbb{E}_x\left[\exp(-r\tau)\phi(X_\tau)\right] &= \mathbb{E}_x\left[\liminf_{m\to\infty}\exp(-r\tau^{(m)})\phi(X_{\tau^{(m)}})\right] \\
&\leq \liminf_{m\to\infty}\mathbb{E}_x\left[\exp(-r\tau^{(m)})\phi(X_{\tau^{(m)}})\right] \\
&\leq \liminf_{m\to\infty} u^{(m)}(x) \\
&\leq \hat{u}(x).
\end{aligned}$$

Therefore, the value function u coincides with the limit of the sequence $(u^{(m)})_{m\in\mathbb{Z}_+}$. At this stage, we recall the definition of the stopping region

$$D \;\coloneqq\; \{x \in V \,:\, u(x) = \phi(x)\} \tag{4.11}$$

We are in a position to state our main result

Theorem 13. *We have*

$$\begin{aligned} u_\infty &= u \\ D_\infty &= D. \end{aligned}$$

Proof. It is sufficient to show that $u_\infty = u$, since $D_\infty = D$ will then follow from Proposition 11 and (4.11). We begin by proving the inequality $u_\infty \le u$. Fix some $x \in V$. By considering in (4.10) the stopping time $\tau \coloneqq \tau_{n+1}$ defined in (4.4), we get for any given $m \in \mathbb{Z}_+$,

$$\begin{aligned} u^{(m)}(x) &\ge \mathbb{E}_x\left[\exp(-r\tau_{n+1}^{(m)})\phi(X_{\tau_{n+1}^{(m)}})\right] \\ &= \widetilde{u}_{n+1}^{(m)}(x) \end{aligned}$$

considered in (4.8). Taking Lemma 12 into account, we deduce that

$$u^{(m)}(x) \;\ge\; u_{n+1}^{(m)}(x)$$

and letting m go to infinity, we get $u(x) \ge u_{n+1}(x)$. It remains to let n go to infinity to show that $u(x) \ge u_\infty(x)$.

To prove the reverse inequality $u_\infty \ge u$, we will show by induction that for every $x \in V$, every $m \in \mathbb{Z}_+$ and every $\tau \in \mathcal{T}$, we have

$$\mathbb{E}_x\left[\exp(-r(\sigma_m \wedge \tau)u_\infty(X_{\sigma_m\wedge\tau})\right] \le u_\infty(x). \tag{4.12}$$

For $m = 1$, we have, because $u_\infty(X_\tau) = u_\infty(x)$ on the set $\{\tau < \sigma_1\}$,

$$\begin{aligned} \mathbb{E}_x\left[\exp(-r(\sigma_1 \wedge \tau)u_\infty(X_{\sigma_1\wedge\tau})\right] &= \mathbb{E}_x\left[\exp(-r\sigma_1)u_\infty(X_{\sigma_1})\mathbb{1}_{\sigma_1\le\tau}\right] + \mathbb{E}_x\left[\exp(-r\tau)u_\infty(X_\tau)\mathbb{1}_{\tau<\sigma_1}\right] \\ &= \mathbb{E}_x\left[\exp(-r\sigma_1)u_\infty(X_{\sigma_1})\mathbb{1}_{\sigma_1\le\tau}\right] + u_\infty(x)\mathbb{E}_x\left[\exp(-r\tau)\mathbb{1}_{\tau<\sigma_1}\right] \\ &= \mathbb{E}_x\left[\exp(-r\sigma_1)u_\infty(X_{\sigma_1})\right] + u_\infty(x)\mathbb{E}\left[\exp(-r\tau)\mathbb{1}_{\sigma_1>\tau}\right] \\ &- \mathbb{E}\left[\exp(-r\sigma_1)u_\infty(X_{\sigma_1})\mathbb{1}_{\sigma_1>\tau}\right]. \end{aligned}$$

Focusing on the third term, we observe, that on the set $\{\tau < \sigma_1\}$, we have $\sigma_1 = \tau + \hat{\sigma}_1 \circ \theta_\tau$ where $\hat{\sigma}_1$ is an exponential random variable with parameter $L(x)$ independent of τ. Therefore, the Strong Markov property yields

$$\begin{aligned} \mathbb{E}\left[\exp(-r\sigma_1)u_\infty(X_{\sigma_1})\mathbb{1}_{\sigma_1>\tau}\right] &= \mathbb{E}_x\left[\exp(-r\tau)\mathbb{E}_x\left[\exp(-r\hat{\sigma}_1)u_\infty(X_{\hat{\sigma}_1})\right]\mathbb{1}_{\sigma_1>\tau}\right] \\ &= \frac{\mathcal{K}[u_\infty](x)}{r + L(x)}\mathbb{E}_x\left[\exp(-r\tau)\mathbb{1}_{\sigma_1>\tau}\right]. \end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{E}_x\left[\exp(-r(\sigma_1\wedge\tau)u_\infty(X_{\sigma_1\wedge\tau})\right] &= \frac{\mathcal{K}[u_\infty](x)}{r+L(x)}\left(1-\mathbb{E}_x\left[\exp(-r\tau)\mathbb{1}_{\sigma_1>\tau}\right]\right)+u_\infty(x)\mathbb{E}_x\left[\exp(-r\tau)\mathbb{1}_{\sigma_1>\tau}\right]\\ &\leq u_\infty(x)\end{aligned}$$

where the last inequality follows from Proposition 10. This proves the assertion for $m=1$. Assume now that for every $x\in V$ and every $\tau\in\mathcal{T}$, we have

$$\mathbb{E}_x\left[\exp(-r(\sigma_m\wedge\tau)u_\infty(X_{\sigma_m\wedge\tau})\right] \quad\leq\quad u_\infty(x).$$

Observing that $\sigma_{m+1}\wedge\tau=\sigma_m\wedge\tau+(\sigma_1\wedge\tau)\circ\theta_{\sigma_m\wedge\tau}$, we get

$$\begin{aligned}\mathbb{E}_x\left[\exp(-r(\sigma_{m+1}\wedge\tau)u_\infty(X_{\sigma_{m+1}\wedge\tau})\right] &= \mathbb{E}_x\left[\exp(-r(\sigma_m\wedge\tau)\mathbb{E}_{X_{\sigma_m\wedge\tau}}\left(\exp(-r(\sigma_1\wedge\tau)u_\infty(X_{\sigma_1\wedge\tau})\right)\right]\\ &\leq \mathbb{E}_x\left[\exp(-r(\sigma_m\wedge\tau)u_\infty(X_{\sigma_m\wedge\tau})\right]\\ &\leq u_\infty(x),\end{aligned}$$

which ends the argument by induction. To conclude, we take the limit at the right-hand side of inequality (4.12) to obtain $u(x)\leq u_\infty(x)$ for every $x\in V$. □

Remark 14. *Because we have financial applications in mind, we choose to work directly with payoffs of the form $e^{-rt}\phi(X_t)$. Observe, however, that our methodology applies when $r=0$ pending the assumption $\phi(X_\tau)=0$ on the set $\{\tau=+\infty\}$.*

We close this section by giving a very simple example on the countable state space $\mathbb{Z}$ with a bounded reward function ϕ such that the recursive algorithm does not stop in finite time because it only eliminates one point at each step.

Example 15. *Let $(X_t)_{t\geq 0}$ be a birth-death process with the generator on $\mathbb{Z}$*

$$\begin{cases} L(x,x+1) &= \lambda\geq 0,\\ L(x,x-1) &= \mu\geq 0,\\ L(x) &= \lambda+\mu,\end{cases}$$

We define the reward function as

$$\phi(x)=\begin{cases}0 \text{ for } x\leq 0\\ 1 \text{ for } x=1\\ 2 \text{ for } x\geq 2.\end{cases}$$

We assume $r=0$ and $\lambda\geq\mu$. Therefore, $\mathcal{D}_1=\mathbb{Z}\setminus\{1\}$. It is easy to show that

$$u_1(1)\quad=\quad\frac{2\lambda}{\lambda+\mu},\ \text{and thus } \mathcal{L}[u_1](0)\quad=\quad\lambda u_1(1)>0.$$

Therefore, $\mathcal{D}_2=\mathbb{Z}\setminus\{1,0\}$. At each step $n\in\mathbb{Z}_+$, because $u_n(1-n)>0$, the algorithm will only remove the integer $1-n$ in the set $\mathcal{D}_n$. Therefore, it will not reach the stopping region $\mathcal{D}=\{2,3,\ldots\}$ in finite time.

4.2 Measurable state space

Up to now, we have only considered continuous-time Markov chains with a discrete state space. But, it is not difficult to see that the results of the previous section can be extended to the case where the state space of the Markov chain is a measurable space. More formally, we consider on a measurable state space $(V, \mathcal{V})$, a non-negative finite kernel K. It is a mapping

$$K : V \times \mathcal{V} \ni (x, S) \quad \mapsto \quad K(x, S) \in \mathbb{R}_+$$

such that

- for any $x \in V$, $K(x, \cdot)$ is a non-negative finite measure on $(V, \mathcal{V})$ (because $K(x, V) \in \mathbb{R}_+$),
- for any $S \in \mathcal{V}$, $K(\cdot, S)$ is a non-negative measurable function on $(V, \mathcal{V})$.

For any probability measure m on V, let us associate to K a continuous-time Markov process $X := (X_t)_{t\geq 0}$ whose initial distribution is m. First we set $\sigma_0 := 0$ and X_0 is sampled according to m. Then we consider an exponential random variable σ_1 of parameter $K(X_0) := K(X_0, V)$. If $K(X_0) = 0$, we have a.s. $\sigma_1 = +\infty$ and we take $X_t := X_0$ for all $t > 0$, as well as $\sigma_n := +\infty$ for all $n \in \mathbb{N}, n \geq 2$. If $K(X_0) > 0$, we take $X_t := X_0$ for all $t \in (0, \sigma_1)$ and we sample X_{σ_1} on $V \setminus \{X_0\}$ according to the probability distribution $K(X_0, \cdot)/K(X_0)$. Next, still in the case where $\sigma_1 < +\infty$, we sample an inter-time $\sigma_2 - \sigma_1$ as an exponential distribution of parameter $K(X_{\sigma_1})$. If $K(X_{\sigma_1}) = 0$, we have a.s. $\sigma_2 = +\infty$ and we take $X_t := X_{\sigma_1}$ for all $t \in (\sigma_1, +\infty)$, as well as $\sigma_n := +\infty$ for all $n \in \mathbb{N}, n \geq 3$. If $K(X_{\sigma_1}) > 0$, we take $X_t := X_0$ for all $t \in (\sigma_1, \sigma_2)$ and we sample X_{σ_2} on $V \setminus \{X_{\sigma_1}\}$ according to the probability distribution $(K(X_{\sigma_1}, x)/K(X_{\sigma_1}))_{x \in V \setminus \{X_{\sigma_1}\}}$. We keep on following the same procedure, where all the ingredients are independent, except for the explicitly mentioned dependences.
In particular, we get a non-decreasing family $(\sigma_n)_{n\in\mathbb{Z}_+}$ of jump times taking values in $\bar{\mathbb{R}}_+ := \mathbb{R}_+ \sqcup \{+\infty\}$. Denote the corresponding exploding time

$$\sigma_\infty \quad := \quad \lim_{n\to\infty} \sigma_n \quad \in \quad \bar{\mathbb{R}}_+$$

When $\sigma_\infty < +\infty$, we must still define X_t for $t \geq \sigma_\infty$. So introduce $\triangle$ a cemetery point not belonging to V and denote $\bar{V} := V \sqcup \{\triangle\}$. We take $X_t := \triangle$ for all $t \geq \sigma_\infty$ to get a $\bar{V}$-valued process X.
Let $\mathcal{B}$ be the space of bounded and measurable functions from V to $\mathbb{R}$. For $f \in \mathcal{B}$, the infinitesimal generator of $X = (X_t)_{t\geq 0}$ is given by

$$\mathcal{L}[f](x) \quad = \quad \int_V f(y)K(x, dy) - K(x)f(x) \;:=\; \mathcal{K}[f](x) - K(x)f(x).$$

As in Section 4.1, we set some payoff function $\phi \in \bar{\mathcal{F}}_+ \setminus \{0\}$ and $r > 0$. We will construct a subset $D_\infty \subset V$ and a function $u_\infty \in \bar{\mathcal{F}}_+$ by our recursive algorithm as follows:
We begin by taking $D_0 := V$ and $u_0 := \phi$. Next, let us assume that $D_n \subset V$ and $u_n \in \bar{\mathcal{F}}_+$ have been built for some $n \in \mathbb{Z}_+$ such that

$$\begin{aligned} \forall x \in V \setminus D_n, \quad (r + L(x))u_n(x) &= \mathcal{K}[u_n](x) \\ \forall x \in D_n, \quad u_n(x) &= \phi(x). \end{aligned}$$

Observe that it is trivially true for $n = 0$. Then, we define the subset D_{n+1} as follows

$$D_{n+1} \quad := \quad \{x \in D_n \,:\, \mathcal{K}[u_n](x) \leq (r + K(x))u_n(x)\}$$

where the inequality is understood in $\bar{\mathbb{R}}_+$.
Next, we consider the stopping time

$$\tau_{n+1} \;\coloneqq\; \inf\{t \geq 0 \,:\, X_t \in D_{n+1}\}$$

with the usual convention that $\inf \emptyset = +\infty$. It is easy to check that the proofs of Section 4.1. are directly deduced.

5 Applications

The objective of this section is to show on the study of two examples, how the forward algorithm can be used in practice.

5.1 Optimal stopping with random intervention times

We revisit the paper by Dupuis and Wang [6] where they consider a class of optimal stopping problems that can only be stopped at Poisson jump times[3]. Consider a probability space $(\Omega, \mathcal{F} \coloneqq (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. For $x > 0$, let $(S_t^x)_{t\geq 0}$ be a geometric Brownian motion solving the stochastic differential equation

$$\frac{dS_t^x}{S_t^x} \;=\; b\,dt + \sigma\,dW_t, \qquad S_0^x \;=\; x,$$

where $W = (W_t)_{t\geq 0}$ is a standard $\mathcal{F}$-Brownian motion and b and $\sigma > 0$ are constants. When $x = 0$, we take $S_t^0 = 0$ for all times $t \geq 0$. The probability space is rich enough to carry a $\mathcal{F}$-Poisson process $N = (N_t)_{t\geq 0}$ with intensity $\lambda > 0$ that is assumed to be independent from W. The jump times of the Poisson process are denoted by T_n with $T_0 = 0$
In [6], the following optimal stopping problem is considered

$$u(0,x) \;=\; \sup_{\tau \in \mathcal{S}_0} \mathbb{E}\left[e^{-r\tau}(S_\tau^x - K)_+\right],$$

where $r > b$ and $\mathcal{S}_0$ is the set of $\mathcal{F}$-adapted stopping time τ for which $\tau(\omega) = T_n(\omega)$ for some $n \in \mathbb{Z}_+$. Similarly to [6], let us define $\mathcal{G}_n = \mathcal{F}_{T_n}$ and the $\mathcal{G}_n$-Markov chain $Z_n = (T_n, S_{T_n}^x)$ to have

$$u(0,x) \;=\; \sup_{N \in \mathcal{N}_0} \mathbb{E}\left[\psi(Z_N) | Z_0 = (0,x)\right], \text{ where } \psi(t,x) \;=\; e^{-rt}(x-K)_+$$

and $\mathcal{N}_0$ is the set of $\mathcal{G}$-stopping time with values in $\mathbb{Z}_+$. To enter the continuous-time framework of the previous sections, we use the following Remark with an independent Poisson process $\widetilde{N} = (\widetilde{N}_t)_t$ with intensity 1.

Remark 16. *Our methodology also applies for discrete Markov chains according to the Poissonization technique that we recall briefly. Consider a Poisson process $N = (N_t)_t$ of intensity λ and a discrete*

[3]Let us mention two recent generalizations of the problem by Lempa [16] and Menaldi and Robin [17] for which the forward algorithm is applicable

Markov chain $(X_n)_{n\in\mathbb{Z}_+}$ with transition matrix or kernel P. Assume that $(X_n)_{n\in\mathbb{Z}_+}$ and $N=(N_t)_t$ are independent. Then, the process

$$X_t = \sum_{n=0}^{N_t} X_n$$

is a continuous-time Markov chain with generator $\mathcal{L} \coloneqq \lambda(P - \mathrm{Id})$.

To start our recursive approach, we need to compute the infinitesimal generator $\widetilde{\mathcal{L}}$ of the continuous Markov chain $(\widetilde{Z}_t = \sum_{i=0}^{\widetilde{N}_t} Z_n)_{t\geq 0}$ with state space $V = \mathbb{R}_+ \times \mathbb{R}_+$ in order to define

$$\widetilde{\mathcal{D}}_1 \coloneqq \{(t,x) \in V;\ \widetilde{\mathcal{L}}[\psi](t,x) \leq 0\}.$$

Let f be a bounded and measurable function on V. According to Remark 16, we have,

$$\widetilde{\mathcal{L}}[f](t,x) \;=\; \lambda \int_0^{+\infty} \mathbb{E}\left[f(t+u, S_u^x)\right] e^{-\lambda u}\, du - f(t,x).$$

Because $\psi(t,x) = e^{-rt}\phi(x)$ with $\phi(x) = (x-K)_+$, we have

$$\widetilde{\mathcal{L}}[\psi](t,x) \;=\; e^{-rt}\left(\lambda R_{r+\lambda}[\phi](x) - \phi(x)\right),$$

where

$$R_{r+\lambda}[\phi](x) \;=\; \int_0^{+\infty} \mathbb{E}[\phi(S_u^x)] e^{-(r+\lambda)u}\, du$$

is the resolvent of the continuous Markov process $S^x = (S_t^x)_{t\geq 0}$. Therefore, we have $\widetilde{\mathcal{D}}_1 = \mathbb{R}_+ \times \mathcal{D}_1$ with

$$\mathcal{D}_1 \;\coloneqq\; \{x \in \mathbb{R}_+,\ \lambda R_{r+\lambda}[\phi](x) - \phi(x) \leq 0\}.$$

First, we observe that $\mathcal{D}_1$ is an interval $[x_1, +\infty[$. Indeed, let us define

$$\eta(x) \;\coloneqq\; \lambda R_{r+\lambda}[\phi](x) - \phi(x).$$

Clearly, $\eta(x) > 0$ for $x \leq K$. Moreover, for $x > K$,

$$\eta'(x) \;=\; \left(\lambda \int_0^{+\infty} \partial_x \mathbb{E}[\phi(S_u^x)] e^{-(r+\lambda)u}\, du\right) - \phi'(x).$$

It is well-known that $\partial_x \mathbb{E}[\phi(S_u^x)] \leq e^{bu}$ for any $x \geq 0$ and thus, because $r > b$,

$$\eta'(x) \leq \frac{\lambda}{r-b+\lambda} - 1 \;<\; 0,$$

which gives that η is a decreasing function on $[K, +\infty)$. It follows that if $\mathcal{D}_1$ is not empty, then it will be an interval of the form $[x_1, +\infty)$. Now,

$$\eta(x) \;=\; x\left(\lambda \int_0^{+\infty} \frac{\mathbb{E}[\phi(S_u^x)]}{x} e^{-(r+\lambda)u}\, du\right) - \phi(x).$$

Because $\frac{\mathbb{E}[\phi(S_u^x)]}{x} \le e^{bu}$, we have $\eta(x) \le \left(\frac{\lambda}{r-b+\lambda} - 1\right) x + K$ and therefore, $\eta(x) \le 0$ for $x \ge \left(1 + \frac{\lambda}{r-b}\right) K$ which proves that $\mathcal{D}_1$ is not empty.

We will now prove by induction that for every $n \in \mathbb{N}$, $\widetilde{\mathcal{D}}_n = \mathbb{R}_+ \times \mathcal{D}_n$ with $\mathcal{D}_n = [x_n, +\infty)$ and $x_n > K$. Assume that it is true for some $n \in \mathbb{N}$. Following our monotone procedure with Remark 14, we define the solution $u_{n+1} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ of the equation

$$\begin{cases} \widetilde{\mathcal{L}}[u_{n+1}] &= 0 \quad , \text{ on } \widetilde{\mathcal{D}}_n^c \\ u_{n+1} &= \psi \quad , \text{ on } \widetilde{\mathcal{D}}_n \end{cases}$$

and define

$$\widetilde{\mathcal{D}}_{n+1} \coloneqq \{(t,x) \in \widetilde{D}_n,\ \widetilde{\mathcal{L}}[u_{n+1}] \le 0\}.$$

Let us check that $\widetilde{\mathcal{D}}_{n+1} = \mathbb{R}_+ \times \mathcal{D}_{n+1}$ with $\mathcal{D}_{n+1} = [x_{n+1}, +\infty)$ and $x_{n+1} \ge x_n$.

To do this, we look for a function of the form

$$\forall\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}_+, \qquad u_{n+1}(t,x) \;=\; \exp(-rt) v_{n+1}(x) \tag{5.1}$$

We end up with the following equation on v_{n+1}:

$$\begin{cases} \lambda R_{r+\lambda}[v_{n+1}] - v_{n+1} &= 0 \quad , \text{ on } \mathcal{D}_n^c \\ v_{n+1} &= \phi \quad , \text{ on } \mathcal{D}_n \end{cases} \tag{5.2}$$

or equivalently, (see [9], Proposition 2.1 page 10)

$$\begin{cases} \mathcal{L}^S[v_{n+1}] - r v_{n+1} &= 0 \quad , \text{ on } \mathcal{D}_n^c \\ v_{n+1} &= \phi \quad , \text{ on } \mathcal{D}_n \end{cases}$$

where $\mathcal{L}^S$ is the infinitesimal generator of $S^x = (S_t^x)_{t \ge 0}$, that is, acting on any $f \in \mathcal{C}^2(\mathbb{R}_+)$ via

$$\mathcal{L}^S[f](x) \;=\; \frac{\sigma^2 x^2}{2} f''(x) + bx f'(x),$$

With this formulation we see that v_{n+1} is given by

$$\forall\ x \in \mathbb{R}_+, \qquad v_{n+1}(x) \;=\; \mathbb{E}_x[\exp(-r\tau_{x_n}^x)\phi(S_{\tau_{x_n}}^x)]$$

where τ_{x_n} is the first hitting time of $\mathcal{D}_n = [x_n, +\infty[$ by our induction hypothesis.

By definition, we have

$$\begin{aligned} \widetilde{\mathcal{D}}_{n+1} &\coloneqq \{(t,x) \in \widetilde{\mathcal{D}}_n \,:\, \widetilde{\mathcal{L}}[u_{n+1}](t,x) \le 0\} \\ &= \mathbb{R}_+ \times \{x \in \mathcal{D}_n \,:\, \lambda R_{r+\lambda}[v_{n+1}](x) - v_{n+1}(x) \le 0\} \\ &= \mathbb{R}_+ \times \{x \in \mathcal{D}_n \,:\, \lambda R_{r+\lambda}[v_{n+1}](x) - \phi(x) \le 0\} \end{aligned}$$

thus $\widetilde{\mathcal{D}}_{n+1} = \mathbb{R}_+ \times \mathcal{D}_{n+1}$ where

$$\mathcal{D}_{n+1} \;\coloneqq\; \{x \in \mathcal{D}_n \,:\, \zeta_{n+1}(x) \le 0\}$$

with

$$\forall\ x \ge 0, \qquad \zeta_{n+1}(x) \;\coloneqq\; \lambda R_{r+\lambda}[v_{n+1}](x) - \phi(x)$$

To prove that $\mathcal{D}_{n+1}$ is of the form $[x_{n+1}, +\infty)$, we begin by showing that

$$\forall\; y \geq x \geq x_n, \qquad \zeta_{n+1}(x) = 0 \;\;\Rightarrow\;\; \zeta_{n+1}(y) \leq 0 \tag{5.3}$$

To do so, introduce the hitting time

$$\tau_x^y \;\; \coloneqq \;\; \inf\{t \geq 0 \,:\, S_t^y = x\}$$

Recall that the solution of (5.1) is given by

$$\forall\; x \in \mathbb{R}+, \forall\; t \geq 0, \qquad S_t^x \;\; = \;\; x \exp\left(\sigma W_t - \frac{\sigma^2}{2} t + bt\right)$$

It follows that

$$\tau_x^y \;\; = \;\; \inf\{t \geq 0 \,:\, W_t - \frac{\sigma^2}{2} t + bt = \ln(x/y)\}$$

In particular τ_x^y takes the value $+\infty$ with positive probability when $b > \sigma^2/2$, but otherwise τ_x^y is a.s. finite. Nevertheless, taking into account that for any $z \geq x_n$, we have $v_{n+1}(z) = \phi(z)$, we can always write for $y \geq x \geq x_n$:

$$\begin{aligned}
& R_{r+\lambda}[v_{n+1}](y) \\
&= \mathbb{E}\left[\int_0^\infty v_{n+1}(S_u^y)\exp(-(r+\lambda)u)\,du\right] \\
&= \mathbb{E}\left[\int_0^{\tau_x^y} v_{n+1}(S_u^y)\exp(-(r+\lambda)u)\,du + \int_{\tau_x^y}^\infty v_{n+1}(S_u^y)\exp(-(r+\lambda)u)\,du\right] \\
&= \mathbb{E}\left[\int_0^{\tau_x^y} \phi(S_u^y)\exp(-(r+\lambda)u)\,du\right] + \mathbb{E}\left[\exp(-(r+\lambda)\tau_x^y)\int_0^\infty v_{n+1}(S_{\tau_x^y+u}^y)\exp(-(r+\lambda)u)\,du\right] \\
&= \mathbb{E}\left[\int_0^{\tau_x^y} \phi(S_u^y)\exp(-(r+\lambda)u)\,du\right] + \mathbb{E}\left[\exp(-(r+\lambda)\tau_x^y)\right] R_{r+\lambda}[v_{n+1}](x)
\end{aligned}$$

where we use the strong Markov property with the stopping time τ_x^y. Reversing the same argument, with v_{n+1} replaced by ϕ, we deduce that

$$\begin{aligned}
R_{r+\lambda}[v_{n+1}](y) \;\; &= \;\; \mathbb{E}\left[\int_0^{\tau_x^y} \phi(S_u^y)\exp(-(r+\lambda)u)\,du\right] + \mathbb{E}\left[\exp(-(r+\lambda)\tau_x^y)\right] R_{r+\lambda}[\phi](x) \\
&\quad + \mathbb{E}\left[\exp(-(r+\lambda)\tau_x^y)\right]\left(R_{r+\lambda}[v_{n+1}](x) - R_{r+\lambda}[\phi](x)\right) \\
&= \;\; R_{r+\lambda}[\phi](y) + \mathbb{E}\left[\exp(-(r+\lambda)\tau_x^y)\right]\left(R_{r+\lambda}[v_{n+1}](x) - R_{r+\lambda}[\phi](x)\right)
\end{aligned}$$

Thus, we have

$$\zeta_{n+1}(y) \;\; = \;\; \lambda R_{r+\lambda}[\phi](y) - \phi(y) + \lambda \mathbb{E}\left[\exp(-(r+\lambda)\tau_x^y)\right]\left(R_{r+\lambda}[v_{n+1}](x) - R_{r+\lambda}[\phi](x)\right)$$

In the first part of the above proof, to get the existence of x_1, we have shown that the mapping $\zeta_1 \coloneqq R_{r+\lambda}[\phi] - \phi$ is non-increasing on $[x_1, +\infty) \supset [x_n, +\infty)$, and in particular

$$\begin{aligned}
\lambda R_{r+\lambda}[\phi](y) - \phi(y) \;\; &\leq \;\; \lambda R_{r+\lambda}[\phi](x) - \phi(x) \\
&= \;\; \zeta_1(x)
\end{aligned}$$

so we get

$$\zeta_{n+1}(y) \leq \zeta_1(x)\mathbb{E}\left[\exp(-(r+\lambda)\tau_x^y)\right]\left(\lambda R_{r+\lambda}[v_{n+1}](x) - \lambda R_{r+\lambda}[\phi](x)\right)$$

Assume now that $\zeta_{n+1}(x) = 0$. It means that

$$\lambda R_{r+\lambda}[v_{n+1}](x) = \phi(x)$$

implying that

$$\begin{aligned} \zeta_{n+1}(y) &\leq \zeta_1(x) + \mathbb{E}\left[\exp(-(r+\lambda)\tau_x^y)\right]\left(\phi(x) - \lambda R_{r+\lambda}[\phi](x)\right) \\ &\leq \left(1 - \mathbb{E}\left[\exp(-(r+\lambda)\tau_x^y)\right]\right)\zeta_1(x) \\ &\leq 0 \end{aligned}$$

since $x \geq x_n \geq x_1$ so that $\zeta_1(x) \leq 0$.
This proves (5.3) and ends the induction argument.
According to Proposition 10 and Theorem 13 (more precisely its extension given in Subsection 4.2), the value function u and the stopping set $\mathcal{D} = \{x \in \mathbb{R}_+,\ u(x) = \phi(x)\}$ satisfy $u(t,x) = e^{-rt}v(x)$, where

$$v(x) = \lambda R_{r+\lambda}[v](x) \text{ on } \mathbb{R}_+ \setminus \mathcal{D},$$

$$v = \phi \text{ on } \mathcal{D}$$

and

$$\mathcal{D} = \bigcap_{n\in\mathbb{N}} [x_n, +\infty[.$$

The stopping set is an interval $[x^*, +\infty[$ that may be empty if x^* is not finite. Using again [9], Proposition 2.1, we obtain

$$\mathcal{L}^S[v](x) - rv(x) = 0 \qquad \forall x \in \mathbb{R}_+ \setminus \mathcal{D}.$$

Therefore, the function w given by $w(x) \coloneqq \lambda R_{r+\lambda}[v](x)$ for any $x \in (0, +\infty)$ also satisfies

$$\mathcal{L}^S[w](x) - rw(x) = 0 \qquad \forall x \in \mathbb{R}_+ \setminus \mathcal{D}.$$

Moreover,

$$(-\mathcal{L}^S + (r+\lambda))[w](x) = \lambda v(x) = \lambda\phi(x) \qquad \forall x \in \mathcal{D},$$

which yields

$$\mathcal{L}^S[w](x) - rw(x) + \lambda(\phi(x) - w(x)) = 0 \qquad \forall x \in \mathcal{D}.$$

This corresponds to the variational inequality (3.4)-(3-9) page 6 solved in [6], establishing that $\mathcal{D}$ is non-empty.

5.2 Stochastic time constraints

Our last example builds on the paper [7]. Let us consider a standard Brownian motion $B \coloneqq (B_t)_{t\geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, and denote by $\mathcal{T}_0$ the set of all $\mathcal{F}_t$-stopping times. We are interested in the following optimal stopping problem with stochastic constraints

$$v(x) = \sup_{\tau\in\mathcal{T}_{\mathbb{Z}}} \mathbb{E}\left[\phi(X^x_{\tau\wedge\tau_0})\right], \tag{5.4}$$

where $x \geq 0$, $X^x_t = x + B_t$ for all $t \geq 0$, ϕ is a bounded function, $\tau_0 = \inf\{t \geq 0,\ X^x_t = 0\}$, and

$$\mathcal{T}_{\mathbb{Z}_+} = \{\tau \in \mathcal{T}_0, \text{ such that } X^x_\tau \in \mathbb{Z}_+\}.$$

In words, the holder of such an American option can only exercise when the Brownian motion has integer values, and this as long as the Brownian remains non-negative. The idea is to embed the constrained optimal stopping problem (5.4) into an unconstrained stopping problem written on a birth-death process killed at 0. To do this, let us define the continuous increasing process

$$L_t \;\coloneqq\; \sum_{n\in\mathbb{Z}_+} L^{(n)}_t(X) \tag{5.5}$$

where $L^{(n)}_t(X)$ is the local time of X at $n \in \mathbb{Z}_+$, and its pseudo-inverse

$$A_t = \inf\{s > 0,\ L_s > t\}.$$

Let us define the process $Y \coloneqq (Y_t)_{t\geq 0} \coloneqq (X^x_{A_t})_{t\geq 0}$ starting from $X^x_{A_0} \in \mathbb{Z}_+$. The process $(Y_t)_{t\geq 0}$ is a continuous-time Markov chain with values in $\mathbb{Z}_+$ whose generator is denoted by $\mathcal{L}$. Observe that when $x \notin \mathbb{Z}_+$ and in some sense, Y has immediately jumped from the entrance state x at 0_- to $\lfloor x \rfloor$ or $\lfloor x \rfloor + 1$, where $\lfloor x \rfloor$ is the integer part of x. Because the Brownian motion has continuous paths, it is clearly a birth-death process. By the strong Markov property and the translation invariance of the Brownian motion, the infinitesimal generator of X is fully determined by the two numbers: $\mathcal{L}(n, n+1) = \mathcal{L}(1, 2)$ and $\mathcal{L}(n, n-1) = \mathcal{L}(1, 0)$ for all $n \geq 1$. Moreover, by symmetry, we have $\mathcal{L}(1, 2) = \mathcal{L}(1, 0)$.

Proposition 17. *Let $T \coloneqq \inf\{t \geq 0,\ |B_t| = 1\}$ be the hitting time of $\{-1, 1\}$ by the Brownian motion. We have*

$$\mathcal{L}(1) = \frac{1}{\mathbb{E}\big(L^{(0)}_T(B)\big)} = 1$$

where we recall that $\mathcal{L}(1) = \mathcal{L}(1, 2) + \mathcal{L}(1, 0)$ is the total jump rate of Y from 1 (and from all $n \geq 1$). We deduce that $\mathcal{L}(1, 2) = \mathcal{L}(1, 0) = \frac{1}{2}$.

Proof. Take any $n \geq 1$ and consider $\tau \coloneqq \inf\{t \geq 0 : |Y_t - n| = 1\}$, when $Y_0 = n = X^n_0$. Note that for any $t \geq 0$, in the r.h.s. of (5.5), there is only a finite number of terms which are non-zero (this statement as the following ones are to be understood a.s.). It follows that the mapping $\mathbb{R}_+ \ni t \mapsto L_t$ is continuous, by continuity of each of the local times $(L^{(n)}_t)_{t\geq 0}$, for $n \in \mathbb{Z}_+$. As a consequence, we get that $L_{A_t} = t$, for any $t \geq 0$. Thus, we have

$$\begin{aligned} \tau &= \inf\{L_{A_t} \geq 0 : |B_{A_t}| = 1\} \\ &= L_{\inf\{A_t : |B_{A_t}|=1\}} \end{aligned}$$

A priori, we only have $\inf\{A_t : |B_{A_t}| = 1\} \geq \inf\{s \geq 0 : |B_s| = 1\}$, because the former infimum is on a smaller set than the latter one. We deduce

$$\tau \;\geq\; L_{\inf\{s\geq 0:|B_s|=1\}} \;=\; L_T$$

Conversely, note that by the strong Markov property applied to T, we have for any $\epsilon > 0$, $L_{T+\epsilon} > L_T$, because any level set of the Brownian motion has no isolated point, so that $A_{L_T} = T$ and $|B_{A_{L_T}}| = |B_T| = 1$. In particular, we get

$$\begin{aligned}\tau &= \inf\{L_{A_t} \geq 0 : |B_{A_t}| = 1\}\\ &\leq L_{A_{L_T}} \;=\; L_T\end{aligned}$$

Putting together these two inequalities, we obtain $\tau = L_T$. It is clear, on one hand that $L_T = L_T^{(0)}(B)$ by definition of T and on the other hand that the total jump rate from n of Y satisfies $\mathcal{L}(n) = 1/\mathbb{E}[\tau]$. These facts lead to the first equality.

Next, observe that the process $(M_t)_{t\geq 0} \coloneqq (|B_t| - L_t^{(0)}(B))_{t\geq 0}$ is a Brownian motion by Tanaka's formula. Because, the stopping time T is integrable, the optional sampling theorem gives

$$\mathbb{E}\big(L_T^{(0)}(B)\big) = \mathbb{E}\big(|B_T|\big) = 1,$$

which ends the proof. □

We are in a position to embed the constrained optimal stopping problem (5.4) in the setting of unconstrained optimal stopping problems on a birth-death process killed at 0. We first build on [7],Theorem 3.3, to characterize the optimal stopping time for (5.4). Let us consider the hitting time

$$H_{\mathbb{Z}_+} \coloneqq \inf\{t \geq 0,\, X_t^x \in \mathbb{Z}_+\}$$

and the function h given by

$$\forall\, x \in \mathbb{R}_+, \qquad h(x) \;\coloneqq\; \mathbb{E}[\phi(X^x_{H_{\mathbb{Z}_+}})]$$

Note that h is harmonic on $\mathbb{R}_+ \setminus \mathbb{Z}_+$. Finally, let us define

$$\forall\, x \in \mathbb{R}_+, \qquad \hat{v}(x) \;\coloneqq\; \sup_{\tau\in\mathcal{T}_0} \mathbb{E}\left[h(X^x_{\tau\wedge\tau_0})\right],$$

and the associated stopping set $S \coloneqq \{x \geq 0,\, \hat{v}(x) = h(x)\}$. Using Strong Markov property, Theorem 3.3 in [7] proves both $v = \hat{v}$ and the entrance time in S is the smallest optimal time for $\hat{v}$, because h is bounded. Next lemma is key to show our result.

Lemma 18. *The boundaries of all connected components of S^c, the complementary set of S in $\mathbb{R}_+$, are in $\mathbb{Z}_+$. Therefore, the smallest optimal stopping time for $\hat{v}$ belongs to $\mathcal{T}_{\mathbb{Z}_+}$.*

Proof. According to optimal stopping theory, the value function $\hat{v}$ is sub-harmonic on $(0,\infty)$ and harmonic on $(0,\infty) \setminus S$. Let $x \in (0,\infty) \setminus S$ and define $a \coloneqq \sup(S \cap [0,x))$ (this supremum is attained, because S is closed and $0 \in S$) and $b \coloneqq \inf(S \cap (x,+\infty)) \leq +\infty$. Thus, on one hand, $\hat{v}$ is linear on (a,b). Denote by $\hat{p}$ the slope of $\hat{v}$ on (a,b). On the other hand, h is linear both on $(\lfloor a\rfloor, \lfloor a\rfloor + 1)$ with slope p_a and on $(\lfloor b\rfloor, \lfloor b\rfloor + 1)$ with slope p_b.

Assume that $a \notin \mathbb{Z}_+$. First, we will show either a is the unique point of $(\lfloor a \rfloor, b)$ in S or all the interval $[\lfloor a \rfloor, a]$ lies in S. Assume there exists some $\lfloor a \rfloor \leq c < a$ (which is possible because $a \notin \mathbb{Z}_+$) that belongs to S. Both functions $\hat{v}$ and h are linear and coincide on c and a, therefore they are equal everywhere on the interval $[c, a]$ and thus $[\lfloor a \rfloor, a] \subset S$. In that case, because $\hat{v}(y) > h(y)$ for $y \in (a, b)$ and $\hat{v}(y) = h(y)$ for $\lfloor a \rfloor \leq y \leq a$, we have $p_a < \hat{p}$ which contradicts the sub-harmonicity of $\hat{v}$.
On the other hand, if a is the unique point of $(\lfloor a \rfloor, b)$ in S, $\hat{v}$ is linear at both sides of a with slopes p^- for $y < a$ and $\hat{p}$ for $y > a$. Since h is below or equal to $\hat{v}$, we must have $p^- \leq p_a \leq \hat{p}$, yielding to the same contradiction of sub-harmonicity, if $p^- < \hat{p}$. If on the contrary $p^- = \hat{p}$, then h and $\hat{v}$ coincide on $[\lfloor a \rfloor, a]$, in contradiction with the assumption that $\{a\} = (\lfloor a \rfloor, b) \cap S$. Therefore, a must be in $\mathbb{Z}_+$. By symmetry, the same reasoning applies to prove that $b \in \mathbb{Z}_+$. □

For $n \in \mathbb{Z}_+$, let us define the following optimal stopping problem on the birth-death process Y,

$$u(n) := \sup_{\sigma \in \mathcal{T}_0^Y} \mathbb{E}\left(\phi(Y_\sigma^n)\right) \tag{5.6}$$

where $\mathcal{T}_0^Y$ is the set of all $\mathcal{F}^Y$ stopping time. We have,

Proposition 19. *For any $n \in \mathbb{Z}_+$, $v(n) = u(n)$ and thus for every $x > 0$,*

$$v(x) = (\lfloor x \rfloor + 1 - x)u(\lfloor x \rfloor) + (x - \lfloor x \rfloor)u(\lfloor x \rfloor + 1).$$

Proof. According to Lemma 18, the stopping time

$$\hat{T} = \inf\{t \geq 0,\, X_{t \wedge \tau_0} \in S\} \in \mathcal{T}_{\mathbb{Z}_+}$$

and is optimal for $v = \hat{v}$. Moreover, proceeding analogously as in the proof of Proposition 17, the stopping time

$$\hat{\sigma} = L_{\hat{T}} = \inf\{t \geq 0, Y_{t \wedge \sigma_0} \in S\} \in \mathcal{T}_0^Y$$

where $\sigma_0 = L_{\tau_0}$. Therefore,

$$\begin{aligned} v(n) &= \mathbb{E}[\phi(X_{\hat{T}}^n)] \\ &= \mathbb{E}[\phi(Y_{\hat{\sigma}}^n)] \\ &\leq u(n). \end{aligned}$$

To prove the converse inequality, let us define $\mathcal{T}_0^Y(M)$ as the set of all Markov stopping times of type $\inf\{t \geq 0,\, Y_t \in A\}$ for A a subset of $\mathbb{Z}_+$. Optimal stopping theory tells us the smallest optimal time associated to problem (5.6) belongs to $\mathcal{T}_0^Y(M)$. Moreover, if $\sigma \in \mathcal{T}_0^Y(M)$ then $A_\sigma \in \mathcal{T}_{\mathbb{Z}_+}$ and thus

$$\begin{aligned} u(n) &= \sup_{\sigma \in \mathcal{T}_0^Y(M)} \mathbb{E}\left(\phi(Y_\sigma^n)\right) \\ &= \sup_{\sigma \in \mathcal{T}_0^Y(M)} \mathbb{E}\left(\phi(X_{A_\sigma}^n)\right) \\ &\leq v(n), \end{aligned}$$

which ends the proof. □

6 Conclusion

The paper brings up to date the forward algorithm that seems to be very effective for Markovian optimal stopping problems. In particular, we want to draw attention to this constructive method which provides an alternative to tackle some class of constrained optimal stopping problems.

References

[1] Abergel, F. and Jedidi, A.: A Mathematical Approach to Order book Modeling, *International Journal of Theoretical and Applied Finance*, Vol 16, 1350025 (2013).

[2] Christensen, S.: A method for pricing American options using semi-infinite linear programming, *Mathematical Finance*, Vol 24, pp. 156-172, (2014).

[3] Cox J.C, Ross S. and Rubinstein M.: Option pricing: A simplified approach, *Journal of Financial economics*, Vol 7, pp.229-262, (1979).

[4] Dayanik S. and Karatzas, I.: On the optimal stopping problem for one-dimensional diffusions, *Stochastic processes and their Applications*, 107,pp. 173-212, (2003).

[5] Dixit, A.K. and Pindyck, R.S. *Investment under Uncertainty*, Princeton University Press, (1994).

[6] Dupuis P. and Wang H.: Optimal stopping with random intervention times, *Advances in Applied Probability*, Vol 34, pp. 141-157,(2002).

[7] Egloff D. and M. Leippold: The Valuation of American options with Stochastic Time Constraints, *Applied Mathematical Finance*, Vol 16, pp. 287-305, (2009).

[8] Eriksson B. and Pistorius M.R.: American Option Valuation under Continuous-Time Markov Chains, *Advances in Applied Probability*, Vol 47, 2, pp. 378-401 (2015).

[9] Ethier, S.N. and Kurtz, T.G.:*Markov Processes: Characterization and Convergence*, Wiley Series in Probability and Statistics, (2005)

[10] Helmes K. and Stockbridge R. : Construction of the value function and optimal stopping rules in optimal stopping of one-dimensional diffusions,*Advances in Applied Probability*, Vol 42, pp. 158-182, (2010).

[11] Irle, A.: A forward algorithm for solving optimal stopping problems. *Journal of Applied Probability*,Vol 43, pp.102–113, (2006).

[12] Irle, A: On forward improvement iteration for stopping problems. *In Proceeding of the International Workshop of Sequential Methodologies*, Troyes, (2009).

[13] Kolodko, A. and J. Schoenmakers: Iterative Construction of the optimal Bermudean stopping time, *Finance and Stochastics*, Vol 10, pp. 27-49, (2006).

[14] Kou, S.C. and S.G. Kou: Modeling Growth Stocks with Birth-death processes. *Advances in Applied Probability*, Vol 35, 641-664 (2003).

[15] Lamberton, D.: Error estimates for the Binomial approximation of American Put options, *Annals of Applied Probability*, Vol 8, pp. 206-233, (1998).

[16] Lempa, J.: Optimal stopping with information constraint,*Appl. Math. Optim.*, Vol 66, pp. 147-173, (2012).

[17] Menaldi, J.L. and Robin, M.: On Some Optimal Stopping Problems with Constraint, *SIAM Journal of Control and Optimization*, Vol 54, pp 2650-2671, (2016).

[18] Mordecki, E.: Optimal Stopping and Perpetual options for Lévy processes, *Finance and Stochastics*, Vol 6, pp. 473-493,(2002).

[19] Myneni, R.: The Pricing of American option, *Annals of Applied Probability*, Vol 2, pp. 1-23 (1992).

[20] Shiryaev A.N. *Optimal Stopping Rules*, Springer, New-York, (1978).

[21] Snell L.: Applications of Martingale System Theorems, *Trans. Am. Math. Soc.*, Vol 73, pp. 293-312, (1953).

[22] Sonin I.: The Elimination algorithm for the problem of optimal stopping, *Math. Methods for Operation research*, Vol 49, pp. 111-123, (1999).

[23] Wald, A.: Sequential tests of statistical hypotheses,*Ann. Math. Statist.*, Vol 16, pp. 117-186,(1945).