Strong stationary times for finite Heisenberg walks

Laurent Miclo*

Institut de Mathématiques de Toulouse, UMR 5219
Toulouse School of Economics, UMR 5314
CNRS and University of Toulouse

Abstract

The random mapping construction of strong stationary times is applied here to finite Heisenberg random walks over $\mathbb{Z}_M$, for odd $M \geqslant 3$. When they correspond to 3×3 matrices, the strong stationary times are of order $M^4\ln(M)$, estimate which can be improved to $M^3\ln(M)$ if we are only interested in the convergence to equilibrium of the non-Markovian coordinate in the upper right corner. These results are extended to $N \times N$ matrices, with $N \geqslant 3$. All the obtained bounds are believed to be non-optimal, nevertheless this original approach is promising, as it relates the investigation of the previously elusive strong stationary times of such random walks to new absorbing Markov chains with a statistical physics flavor and whose quantitative study is to be pushed further.

Keywords: Random mappings, strong stationary times, finite Heisenberg random walks, absorbing Markov chains.

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1 Introduction

The investigation of the quantitative convergence to equilibrium of random walks on finite groups has lead to a prodigious literature devoted to various techniques, see for instance the overview of Saloff-Coste [20] or the book of Levin, Peres and Wilmer [13]. One of the most probabilistic approaches is based on the strong stationary times introduced by Aldous and Diaconis [1]. Diaconis and Fill [6] presented a general construction of strong stationary times via intertwining dual processes, in particular set-valued dual processes. It was proposed in [16] to obtain the latter processes through the resort of random mappings, in the spirit of Propp and Wilson [19]. Here we apply this method to deduce strong stationary times for finite Heisenberg random walks. It will illustrate that the random mapping technique can be effective in constructing strong stationary times in situations where they are difficult to find and have lead to numerous mistakes in the past. While there is room for improvement in our estimates, we hope this new approach will help the understanding of the convergence to equilibrium of related random walks, see for instance Hermon and Thomas [10], Breuillard and Varjú [2], Eberhard and Varjú [8] or Chatterjee and Diaconis [5] for very recent progress in this direction.

To avoid notational difficulties, we begin by presenting the case of 3×3 matrices.

For $M \geqslant 3$ and M odd, let $\mathbb{H}_M$ be the **Heisenberg group** of matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

where $x, y, z \in \mathbb{Z}_M$. Such matrices will be identified with $[x, y, z] \in \mathbb{Z}_M^3$, the multiplication corresponding to

$$[x, y, z] \cdot [x', y', z'] \quad = \quad [x + x', y + y', z + z' + xy']$$

for any $[x, y, z], [x', y', z'] \in \mathbb{Z}_M^3$.

Consider the usual system of generators of $\mathbb{H}_M$, $\{[1, 0, 0], [-1, 0, 0], [0, 1, 0], [0, -1, 0]\}$, as well as the random walk $[X, Y, Z] := ([X_n, Y_n, Z_n])_{n \in \mathbb{Z}_+}$, starting from the identity $[0, 0, 0]$ and whose transitions are obtained by multiplying on the left by one of these elements, each chosen with probability $1/6$. With the remaining probability $1/3$, the random walk does not move.

The **uniform distribution** $\mathcal{U}$ on $\mathbb{H}_M$ is invariant and reversible for the random walk $[X, Y, Z]$. A finite stopping time τ with respect to the filtration generated by $[X, Y, Z]$, possibly enriched with some independent randomness, is said to be a **strong stationary time** if

- τ and $[X_\tau, Y_\tau, Z_\tau]$ are independent,
- $[X_\tau, Y_\tau, Z_\tau]$ is distributed as $\mathcal{U}$.

The tail probabilities of a strong stationary time enable to estimate the speed of convergence of the law $\mathcal{L}[X_n, Y_n, Z_n]$ of $[X_n, Y_n, Z_n]$ toward $\mathcal{U}$, in the separation sense, as shown by Diaconis and Fill [6]. More precisely, recall that the separation discrepancy $\mathfrak{s}(m, \mu)$ between two probability measures m and μ defined on the same measurable space is defined by

$$\mathfrak{s}(m, \mu) \quad := \quad \operatorname*{ess\,sup}_{\mu} 1 - \frac{m}{\mu}$$

where m/μ is the Radon-Nikodym density of m with respect to μ.

For any strong stationary time τ associated to $[X, Y, Z]$, we have

$$\forall\ n \in \mathbb{Z}_+, \qquad \mathfrak{s}(\mathcal{L}[X_n, Y_n, Z_n], \mathcal{U}) \quad \leqslant \quad \mathbb{P}[\tau > n]$$

It justifies the interest the following bound:

Theorem 1 *There exists a strong stationary time τ for $[X,Y,Z]$ such that for M large enough,*

$$\forall\, r \geqslant 0, \qquad \mathbb{P}[\tau \geqslant r] \ \leqslant\ 8\exp\left(-\frac{r}{101M^4\ln(M)}\right)$$

Taking into account the invariance of the transition matrix of $[X,Y,Z]$ with respect to the right (or left) group multiplication, the above result can be extended to any initial distribution of $[X_0,Y_0,Z_0]$. Note that (X,Y) is an usual random walk on the finite torus $\mathbb{Z}_M^2$, so it needs a time of order M^2 to reach equilibrium in the strong stationary time sense. This estimate will be made more precise in Lemma 9. The main contribution in Theorem 1 will come from an upper bound of order $M^4\ln(M)$ on the time required by $[X,Y,Z]$ to reach equilibrium, once (X,Y) has reached equilibrium.

Nevertheless, the puzzling feature of the 3×3 Heisenberg model over $\mathbb{Z}_M$ is the fast convergence of Z, mixing more rapidly than (X,Y), at a time that should be of order M, up to logarithmic corrections. In the total variation sense, this is known to be true, see e.g. [3, 4] and the references given there. We believe this also holds in the strong stationary time sense and that the new approach presented here can be refined to go in this direction.

Up to our knowledge, no strong stationary time can be found in the literature for finite Heisenberg models. So the main point of this paper is to show that such a strong stationary time can be constructed via the random mapping method of [16], even if it is sub-optimal. Indeed in Theorem 1 the right order should be M^2, the same as for the usual random walk (X,Y) on $\mathbb{Z}_M^2$, the extra time for Z being expected to be negligible as said above. Nevertheless, we will be led to new interesting models of absorbing Markov chains with a statistical physics flavor whose investigation should be pushed further to get the desired estimate, see Remark 18 in Section 4.

If one is only interested in the convergence to equilibrium of (the non-Markovian) Z, the same approach gives a better result, even if it remains sub-optimal according to the above observations. Note that (Y,Z) is a Markov chain.

Theorem 2 *There exists a strong stationary time $\widetilde{\tau}$ for (Y,Z) such that for M large enough,*

$$\forall\, r \geqslant 0, \qquad \mathbb{P}[\widetilde{\tau} \geqslant r] \ \leqslant\ 5\exp\left(-\frac{r}{101M^3\ln(M)}\right)$$

One could think that once the equilibrium has been reached for (Y,Z), it is sufficient to wait for a supplementary time for X of order M^2 to equilibrate to get a strong stationary time for the whole chain $[X,Y,Z]$. But one has to be more careful with this kind of assertion (despite we made one after the statement of Theorem 1), see Remark 19 in Section 5 for more details.

These considerations can be extended to the $N\times N$ Heisenberg $\mathbb{H}_{N,M}$ group model over $\mathbb{Z}_M$. It consists in the matrices of the form

$$\begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,N-1} & x_{1,N}\\ 0 & 1 & x_{2,3} & \cdots & x_{2,N-1} & x_{2,N}\\ 0 & 0 & 1 & \cdots & x_{3,N-1} & x_{3,N}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & x_{N-1,N}\\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

where $x_{k,l}\in\mathbb{Z}_M$ for $1\leqslant k<l\leqslant N$, the group operation corresponds to the matrix multiplication. Such matrices will be identified with $[x_{k,l}]_{1\leqslant k<l\leqslant N}\in\mathbb{Z}_M^{\triangle_N}$, where $\triangle_N \coloneqq \{(k,l)\,:\,1\leqslant k<l\leqslant N\}$.

Consider the usual system of generators of $\mathbb{H}_{N,M}$, $\{\varepsilon\delta_{(I,I+1)}\,:\, I\in[\![N-1]\!]$ and $\varepsilon\in\{\pm1\}\}$, where $\delta_{(I,I+1)}$ is the element of $\mathbb{Z}_M^{\triangle_N}$ whose entries all vanish, except the one indexed by $(I,I+1)$ which is equal to 1. Let $[X] \coloneqq ([X](n))_{n\in\mathbb{Z}_+} \coloneqq ([X_{k,l}(n)])_{1\leqslant k<l\leqslant N,\, n\in\mathbb{Z}_+}$ be the random walk starting from

the identity $[0]_{1\leqslant k<l\leqslant N}$ and whose transitions are obtained by multiplying on the left by one of the generators, each chosen with probability $1/(3(N-1))$. With the remaining probability $1/3$, the random walk does not move. The invariant measure is the uniform distribution on $\mathbb{H}_{N,M}$. We have a result similar to Theorem 1:

Theorem 3 *There exists a strong stationary time τ for $[X]$ such that for M large enough,*

$$\forall\, r\geqslant 0,\qquad \mathbb{P}[\tau\geqslant r]\ \leqslant\ \frac{11N-17}{2}\exp\left(-\frac{2r}{101(N-1)(N-2)M^{N(N-1)/2+1}\ln(M)}\right)$$

The proof is based on intermediate results exploiting the upper diagonal structure of the model. More precisely, let us introduce for $[x]\in\mathbb{H}_{N,M}$ and $l\in[\![N-1]\!]$, the l-th upper diagonal $d_l[x]\coloneqq(x_{k,k+l})_{k\in[\![N-l]\!]}$, as well as $d_{[\![l]\!]}[x]\coloneqq(d_k[x])_{k\in[\![l]\!]}$. Note that $[x]=d_{[\![N-1]\!]}[x]$. Similarly, for $l\in[\![N-1]\!]$, we can associate the stochastic chains $D_l\coloneqq(d_l[X(n)])_{n\in\mathbb{Z}_+}$ as well as $D_{[\![l]\!]}\coloneqq(d_{[\![l]\!]}[X(n)])_{n\in\mathbb{Z}_+}$ to the Markov chain $[X]$. It is not difficult to see that $D_{[\![l]\!]}$ is a Markov chain itself (but D_l is not). We will see that if for some $l\in[\![N-2]\!]$, $D_{[\![l]\!]}$ is starting at equilibrium, then there exists a strong stationary time τ_{l+1} for $D_{[\![l+1]\!]}$ of order at most $(N-1)(N-l+1)M^{Nl-(l-1)(l+2)/2}\ln(M)$, see Proposition 25 in Section 6 where $\tau_{l+1}=\mathfrak{t}_{l+2}-\mathfrak{t}_{l+1}$. Theorem 3 will be obtained by summing these estimates. The estimate of Theorem 3 does match exactly that of Theorem 1 when $N=3$, this is not by chance but because we carefully look for a faithful generalization to facilitate reading. Again, all these bounds are very rough and we hope they are a preliminary step toward the conjecture that the order of convergence for the (non-Markovian) up-diagonal D_l should be $M^{2/l}$ for fixed N and $l\in[\![N-1]\!]$ (see for instance [3]).

Theorem 2 has equally an extension. Denote $C_N[X]$ the last column of $[X]$ and remark this is a Markov chain.

Theorem 4 *There exists a strong stationary time $\widetilde{\tau}$ for $C_N[X]$ such that for M large enough,*

$$\forall\, r\geqslant 0,\qquad \mathbb{P}[\widetilde{\tau}\geqslant r]\ \leqslant\ \frac{6N-7}{2}\exp\left(-\frac{2r}{101(N-1)M^{N+1}\ln(M)}\right)$$

The plan of the paper is as follows. In the next section, as a warming-up computation and to recall the approach of [16], we construct strong stationary times for quite lazy random walks on the finite circle $\mathbb{Z}_M$ (no longer assuming that $M\geqslant 3$ is odd). This construction is extended in Section 3 to produce a strong stationary times for the Markov chain (X,Y) extracted from $[X,Y,Z]$. This procedure is itself (strongly) distorted in Section 4 to prove Theorem 1. Section 5 presents the modification needed for Theorem 2. The extensions to random walks on higher dimensional Heisenberg groups is the object of Section 6. Finally, Appendix A supplements the investigation of random walks on the finite circle $\mathbb{Z}_M$, when the level of laziness is weak.

2 Strong stationary times for finite circles

Here we construct strong stationary times for certain random walks on discrete circles (the remaining cases will be treated in Appendix A). It will enable us to recall the random mapping approach, as developed in [16].

We start by presenting the general situation of random walks on discrete circles with at least 3 points.

Let $M\in\mathbb{N}\backslash\{1,2\}$ and $a\in(0,1/2]$ be fixed. We consider the Markov kernel P on $\mathbb{Z}_M$ given by

$$\forall\, x,y\in\mathbb{Z}_M,\qquad P(x,y)\ \coloneqq\ \begin{cases} a & \text{, if } y=x+1 \text{ or } y=x-1\\ 1-2a & \text{, if } y=x\\ 0 & \text{, otherwise}\end{cases}$$

We are looking for strong stationary times for the corresponding random walk starting from 0 (or from any other initial point by symmetry). From Diaconis and Fill [6], it is always possible to construct such strong stationary times, except when $a = 1/2$ and M is odd, then the random walk has period 2 and does not converge to its equilibrium (instead, we will recover in this case a dual process related to the discrete Pitman's theorem [18]).

More precisely, since $a > 0$, the unique invariant probability associated to P is π the uniform distribution on $\mathbb{Z}_M$. It is even reversible, so that the adjoint matrix P^* of P in $\mathbb{L}^2(\pi)$ is just $P^* = P$. In the sequel and in Appendix A, we will consider certain sets $\mathfrak{V}$ consisting of non-empty subsets of $\mathbb{Z}_M$ and containing the whole state space $\mathbb{Z}_M$. There will be three instances for $\mathfrak{V}$, depending on the values of a and M (a fourth one will be considered in Section A.3). Here we will deal with the simplest case, when $a \in (0, 1/3]$. We will then take $\mathfrak{V} = \mathfrak{I}$, the set of intervals in $\mathbb{Z}_M$ which are symmetric with respect to 0, namely

$$\mathfrak{I} \quad \coloneqq \quad \{B(0,r) \,:\, r \in [\![0, \lfloor M/2 \rfloor]\!]\}$$

where $\lfloor\cdot\rfloor$ is the usual integer part and $B(0,r) = \{-r, -r+1, ..., 0, ..., r-1, r\}$ is the (closed) ball centered at 0 and of radius r, for the usual graph distance on $\mathbb{Z}_M$. Recall that for $k \leqslant l \in \mathbb{Z}$, $[\![k,l]\!]$ stand for the set of integers between k and l (included). By convention, for $k \geqslant 0$, $[\![k]\!] \coloneqq [\![1,k]\!]$, which is the empty set if $k = 0$. The same notation will also be used for the "projected" intervals on $\mathbb{Z}_M$. For the other definitions of $\mathfrak{V}$, when $a \in (1/3, 1/2]$, we refer to Appendix A. These cases, while instructive, will not be helpful for the next sections. For the sake of the general arguments below, just assume that we have chosen a $\mathfrak{V}$ consisting of "nice" subsets of $\mathbb{Z}_M$.

For any $S \in \mathfrak{V}$, we are looking for a **random mapping** $\psi_S \,:\, \mathbb{Z}_M \to \mathbb{Z}_M$ satisfying two conditions:

- the **weak association** with $P^* = P$, namely

$$\forall\, x \in \mathbb{Z}_M,\ \forall\, y \in S, \qquad \mathbb{P}[\psi_S(x) = y] \quad = \quad \frac{1}{\xi(S)} P(x,y) \tag{1}$$

  where $\xi(S) > 0$ is a positive number.

- the **stability** of $\mathfrak{V}$: the set

$$\Psi(S) \quad \coloneqq \quad \psi_S^{-1}(S) \tag{2}$$

  belongs to $\mathfrak{V}$.

The interest of such random mappings is that they enable to construct a $\mathfrak{V}$-valued intertwining dual process, and a strong stationary time if the latter ends up being absorbed in the whole set $\mathbb{Z}_M$. Indeed, introduce the Markov kernel Λ from $\mathfrak{V}$ to $\mathbb{Z}_M$ given by

$$\forall\, S \in \mathfrak{V},\ \forall\, x \in \mathbb{Z}_M, \qquad \Lambda(S,x) \quad \coloneqq \quad \frac{\pi(x)}{\pi(S)} \mathbb{1}_S(x)$$

where $\mathbb{1}_S$ is the indicator function of S.

Consider next the $\mathfrak{V} \times \mathfrak{V}$-matrix $\mathfrak{P}$ given by

$$\forall\, S, S' \in \mathfrak{V}, \qquad \mathfrak{P}(S,S') \quad = \quad \frac{\xi(S)\pi(S')}{\pi(S)} \mathbb{P}[\Psi(S) = S'] \tag{3}$$

We have shown in [16] that $\mathfrak{P}$ is a Markov kernel and that it is intertwined with P through Λ:

$$\mathfrak{P}\Lambda \quad = \quad \Lambda P$$

Note that $\mathbb{Z}_M$ is absorbing for $\mathfrak{P}$, since we always have $\Psi(\mathbb{Z}_M) = \mathbb{Z}_M$ and $\xi(\mathbb{Z}_M) = 1$ (by summing with respect to $y \in \mathbb{Z}_M$ in (1)). Let $\mathfrak{X} \coloneqq (\mathfrak{X}_n)_{n \in \mathbb{Z}_+}$ be a Markov chain on $\mathfrak{V}$ starting from $\{0\}$ and whose transition kernel is $\mathfrak{P}$. Consider $\mathfrak{t}$ its absorbing time:

$$\mathfrak{t} \quad \coloneqq \quad \inf\{n \in \mathbb{Z}_+ \,:\, \mathfrak{X}_n = \mathbb{Z}_M\} \ \in \ \mathbb{N} \sqcup \{\infty\}$$

Let $X := (X_n)_{n\in\mathbb{Z}_+}$ be a Markov chain on $\mathbb{Z}_M$ starting from 0 and whose transition kernel is P. As in the introduction, a finite stopping time τ for the filtration generated by X (and maybe some additional independent randomness) is said to be a **strong stationary time** for X if τ and X_τ are independent and X_τ is distributed according to π.

According to Diaconis and Fill [6], if $\mathfrak{t}$ is almost surely finite, then it has the same law as a strong stationary time for X, since it is possible to construct a coupling between X and $\mathfrak{X}$ such that $\mathfrak{t}$ is a strong stationary time for X (see also [16], where this coupling is explicitly constructed in terms of the random mappings).

Except when $a = 1/2$ and M is even, the $\mathfrak{t}$ we are to construct here and in Appendix A will be a.s. finite. Furthermore, when $a \in (0, 1/3]$, $\mathfrak{t}$ will be a **sharp** strong stationary time, in the sense that its law will be stochastically dominated by the law of any other strong stationary time. As a consequence, we get that

$$\forall\, n \in \mathbb{Z}_+, \qquad \mathfrak{s}(\mathcal{L}(X_n), \pi) \;=\; \mathbb{P}[\mathfrak{t} > n] \tag{4}$$

Indeed, this sharpness is a consequence of Remark 2.39 of Diaconis and Fill [6] and the fact that

$$\forall\, S \in \mathfrak{I}\backslash\{\mathbb{Z}_M\}, \qquad \Lambda(S, \lfloor M/2\rfloor) \;=\; 0 \tag{5}$$

This relation, with $S \in \mathfrak{V}\backslash\{\mathbb{Z}_M\}$, will not be satisfied by the constructions of Appendix A, so we will not be able to conclude to sharpness when $a \in (1/3, 1/2)$ or $a = 1/2$ and M odd.

For the remaining part of this section we assume $M \geqslant 3$ and $a \in (0, 1/3]$. Let us construct the desired random mappings $(\psi_S)_{S\in\mathfrak{I}}$. We distinguish $S = \{0\}$ from the other cases.

2.1 The random mapping $\psi_{\{0\}}$

The construction of $\psi_{\{0\}}$ is different from that of the other ψ_S, for $S \in \mathfrak{I}\backslash\{\{0\}\}$. Choose two mappings $\widetilde{\psi}, \widehat{\psi} : \mathbb{Z}_M \to \mathbb{Z}_M$ satisfying respectively $\widetilde{\psi}(0) = 0 = \widetilde{\psi}(-1) = \widetilde{\psi}(1)$ and $\widetilde{\psi}(x) \neq 0$ for $x \in \mathbb{Z}_M\backslash[\![-1, 1]\!]$, and $\widehat{\psi}(0) = 0$ and $\widehat{\psi}(x) \neq 0$ for $x \in \mathbb{Z}_M\backslash\{0\}$. Take $\psi_{\{0\}}$ to be equal to $\widetilde{\psi}$ with some probability $p \in [0, 1]$ and to $\widehat{\psi}$ with probability $1 - p$. Let us compute p so that Condition (1) is satisfied, which here amounts to the validity of

$$\mathbb{P}[\psi_{\{0\}}(x) = 0] \;=\; \frac{1}{\xi(\{0\})} P(x, 0) \tag{6}$$

for all $x \in \mathbb{Z}_M$ and for some $\xi(\{0\}) > 0$.

- When $x \notin [\![-1, 1]\!]$, both sides of (6) vanish.
- When $x = 0$, the l.h.s. of (6) is 1, while the r.h.s. is $(1 - 2a)/\xi(\{0\})$. This implies that $\xi(\{0\}) = 1 - 2a$.
- When $x \in \{-1, 1\}$, (6) is equivalent to

  $$p \;=\; \frac{a}{1 - 2a}$$

  and this number p does belongs to $(0, 1]$ for $a \in (0, 1/3]$.

Next we must check that for this random mapping $\psi_{\{0\}}$, (2) is satisfied, namely $\Psi(\{0\}) \in \mathfrak{I} = \mathfrak{I}$. This is true, because $\widetilde{\psi}^{-1}(\{0\}) = [\![-1, 1]\!] \in \mathfrak{I}$ and $\widehat{\psi}^{-1}(\{0\}) = \{0\} \in \mathfrak{I}$.

2.2 The other random mappings

We now come to the construction of the random mappings ψ_S, for $S \in \mathfrak{I}\backslash\{\{0\}\}$, which is valid for all $a \in (0, 1/2]$ and does not depend on the particular value of $S \in \mathfrak{I}\backslash\{\{0\}\}$. So let us call this random mapping ϕ. It will takes five values $\{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5\}$, and to describe them it is better to discriminate according the parity of M.

When M is odd. Here is the definition of the mappings ϕ_l, for $l \in [\![5]\!]$.

- ϕ_1 is defined by

$$\forall\, x \in \mathbb{Z}_M, \qquad \phi_1(x) \;\coloneqq\; \begin{cases} x+1 & \text{, if } x \in [\![-(M-1)/2, -1]\!] \\ 1 & \text{, if } x = 0 \\ x-1 & \text{, if } x \in [\![1, (M-1)/2]\!] \end{cases}$$

- ϕ_2 is defined by

$$\forall\, x \in \mathbb{Z}_M, \qquad \phi_2(x) \;\coloneqq\; \begin{cases} x+1 & \text{, if } x \in [\![-(M-1)/2, -1]\!] \\ -1 & \text{, if } x = 0 \\ x-1 & \text{, if } x \in [\![1, (M-1)/2]\!] \end{cases}$$

- ϕ_3 is defined by

$$\forall\, x \in \mathbb{Z}_M, \qquad \phi_3(x) \;\coloneqq\; \begin{cases} x-1 & \text{, if } x \in [\![-(M-1)/2, -1]\!] \\ 1 & \text{, if } x = 0 \\ x+1 & \text{, if } x \in [\![1, (M-1)/2]\!] \end{cases}$$

- ϕ_4 is defined by

$$\forall\, x \in \mathbb{Z}_M, \qquad \phi_4(x) \;\coloneqq\; \begin{cases} x-1 & \text{, if } x \in [\![-(M-1)/2, -1]\!] \\ -1 & \text{, if } x = 0 \\ x+1 & \text{, if } x \in [\![1, (M-1)/2]\!] \end{cases}$$

- ϕ_5 is just the identity mapping

The random mapping ϕ is taking each of the values ϕ_1, ϕ_2, ϕ_3 and ϕ_4 with the probability $a/2$ and the value ϕ_5 with the remaining probability $1-2a$. It is immediate to check (1) can be reinforced into

$$\forall\, x, y \in \mathbb{Z}_+, \qquad \mathbb{P}[\psi_S(x) = y] \;=\; P(x, y) \tag{7}$$

(called the strong association condition with $P^* = P$ in [16]). Furthermore, we have for any $r \in [\![(M-1)/2 - 1]\!]$,

$$\begin{cases} \phi_1^{-1}(B(0,r)) &=& B(0, r+1) \\ \phi_2^{-1}(B(0,r)) &=& B(0, r+1) \\ \phi_3^{-1}(B(0,r)) &=& B(0, r-1) \\ \phi_4^{-1}(B(0,r)) &=& B(0, r-1) \\ \phi_5^{-1}(B(0,r)) &=& B(0, r) \end{cases} \tag{8}$$

It follows that $\mathfrak{I}$ is left stable by the random mapping Ψ defined in (2) (since the remaining set $\mathbb{Z}_M = B(0, (M-1)/2)$ is left stable by any mapping from $\mathbb{Z}_M$ to $\mathbb{Z}_M$).

When M is even. The previous mappings have to be slightly modified, due to the special role of the point $M/2$.

More precisely, $\phi_1, \phi_2, \phi_3, \phi_4$ and ϕ_5 are defined in exactly the same way on $\mathbb{Z}_M \backslash \{M/2\}$ and in addition:

$$\begin{aligned}
\phi_1(M/2) &= M/2+1 \\
\phi_2(M/2) &= M/2-1 \\
\phi_3(M/2) &= M/2+1 \\
\phi_4(M/2) &= M/2-1 \\
\phi_5(M/2) &= M/2
\end{aligned}$$

The random mapping ϕ is taking each of the values $\phi_1, \phi_2, \phi_3, \phi_4$ and ϕ_5 with the same probabilities as in the case M odd. The strong association condition (7) as well as the stability of $\mathfrak{I}$ by Ψ are similarly verified ((8) is now true for $r \in [\![1, M/2-1]\!]$).

2.3 The Markov transition kernel $\mathfrak{P}$

To simplify the description of $\mathfrak{P}$ given in (3), let us identify $[\![-r, r]\!]$ with r, for $r \in [\![0, \lfloor M/2 \rfloor]\!]$. Then it appears that $\mathfrak{P}$ is the transition matrix of a birth and death chain:

$$\forall\ k,l \in [\![0, \lfloor M/2 \rfloor]\!], \qquad \mathfrak{P}(k,l) \;=\; \begin{cases} 1-3a & , \text{if } k=0=l \\ 3a & , \text{if } k=0 \text{ and } l=1 \\ a\frac{2l+1}{2k+1} & , \text{if } k \in [\![1, \lfloor M/2 \rfloor -1]\!] \text{ and } |k-l|=1 \\ 1-2a & , \text{if } k \in [\![1, \lfloor M/2 \rfloor -1]\!] \text{ and } k=l \\ 1 & , \text{if } k=\lfloor M/2 \rfloor = l \\ 0 & , \text{otherwise} \end{cases}$$

Since $\mathfrak{P}$ enables to reach the absorbing point $\lfloor M/2 \rfloor$ from all the other points, the absorbing time $\mathfrak{t}$ is a.s. finite and due to (5), its law is the distribution of a sharp strong stationary time for X, namely the tail probabilities of $\mathfrak{t}$ correspond exactly to the evolution of the separation distance between the time marginal distribution and π, see (4). Since the starting point $\mathfrak{X}_0 = \{0\}$, identified with 0, is the opposite boundary point of the absorbing point $\lfloor M/2 \rfloor$, Karlin and McGregor [12] described explicitly the law of $\mathfrak{t}$ in terms of the spectrum of $\mathfrak{P}$ (see also Fill [9] or [7] for probabilistic proofs via intertwining relations). In particular when this spectrum is non-negative, $\mathfrak{t}$ is a sum of independent geometric variables whose parameters are the eigenvalues (except 1) of $\mathfrak{P}$.

Remark 5 When $a = 1/3$, Diaconis and Fill [6] gives another illustrative example of a sharp strong stationary time for P, see also Section 4.1 of Pak [17].

□

Remark 6 We could have first projected $\mathbb{Z}_M$ on $[\![0, \lfloor M/2 \rfloor]\!]$ (sending 0 to 0, -1 and 1 to 1, etc.) and lump X to obtain a birth-and-death process $\widetilde{X}$. Its transition matrix $\widetilde{P}$ satisfies $\widetilde{P}(0,1) = 2a$, $\widetilde{P}(1,0) = a$, $\widetilde{P}(1,2) = a$, etc. (note that $P(\lfloor M/2 \rfloor, \lfloor M/2 \rfloor - 1)$ is equal to a or $2a$, depending on the parity of M). Constructing a corresponding set-valued intertwining dual, we would have ended with the same strong stationary time. According to Proposition 4.6 of Diaconis and Fill [6] (where we take into account that $\widetilde{P}$ is reversible and that $\widetilde{X}_0 = 0$), there exists a dual process to $\widetilde{X}$ taking values in $\{[\![0, x]\!] : x \in [\![0, \lfloor M/2 \rfloor]\!]\}$ if and only if $\widetilde{X}$ is monotone. It is easy to check that $\widetilde{X}$ is monotone if and only if $a \in (0, 1/3]$ (compare $\widetilde{P}(0, [\![1, \lfloor M/2 \rfloor]\!]) = 2a$ with $\widetilde{P}(1, [\![1, \lfloor M/2 \rfloor]\!]) = 1-a$, this special role of 0 is related to difference between Sections 2.1 and 2.2). This explains the critical position of $a = 1/3$ and justifies the different treatment of the case $a \in (1/3, 1/2]$ in Appendix A.

□

Remark 7 If we had chosen for $\psi_{\{0\}}$ a random mapping satisfying the strong association condition (7) instead of the weak association (1), then we could not have achieved the stability condition $\Psi(\mathfrak{I}) \subset \mathfrak{I}$. Indeed, the condition $\mathbb{P}[\psi_0^{-1}(0) = 1] = a$ would have implied that $\Psi(\{0\})$ must have taken values in the subsets of $\{-1, 0, 1\}$ not containing 0. Nevertheless, it is possible to choose a random mapping verifying (7) and such that the only additional value of $\Psi(\{0\})$ is the empty set, so that $\Psi(\{0\}) \in \{\emptyset, \{0\}, \{-1, 0, 1\}\}$. Due to the fact that necessarily $\mathfrak{P}(\{0\}, \cdot)\Lambda = \Lambda P(0, \cdot) = (1-2a)\delta_0 + a(\delta_{-1} + \delta_1)$ (where δ stands for the Dirac mass), we then end up with the same kernel $\mathfrak{P}$.

If with positive probability $\Psi(\{0\})$ takes other values than $\emptyset$, $\{0\}$ and $\{-1, 0, 1\}$, and if we keep the same ϕ for the other random mappings, then $\mathfrak{t}$ will not be sharp (if is a.s. finite at all, cf. Remark 31), as it can be deduced from Appendix A.

□

2.4 Illustration for $a = 1/3$ and M odd

When $a = 1/3$, the transition matrix $\mathfrak{P}$ is given on $[\![0, (M-1)/2]\!]$ by

$$\forall\ k, l \in [\![0, \lfloor M/2 \rfloor]\!], \qquad \mathfrak{P}(k, l) \quad = \quad \begin{cases} 0 & \text{, if } k = 0 = l \\ 1 & \text{, if } k = 0 \text{ and } l = 1 \\ \frac{1}{3}\frac{2l+1}{2k+1} & \text{, if } k \in [\![1, (M-3)/2]\!] \text{ and } |k - l| = 1 \\ \frac{1}{3} & \text{, if } k \in [\![1, (M-3)/2]\!] \text{ and } k = l \\ 1 & \text{, if } k = (M-1)/2 = l \\ 0 & \text{, otherwise} \end{cases}$$

Let $\mathfrak{t}$ be the time a Markov chain $\mathfrak{X}$ associated to $\mathfrak{P}$ and starting from 0 hits $(M-1)/2$. Consider W a random walk on $\mathbb{Z}$, starting from 0, whose transition probabilities of going one step upward, one step downward or to stay at the same position are all equal to $1/3$. Let ς be the hitting time by W of the set $\{-(M-1)/2, (M-1)/2\}$. Since for $k \in [\![1, (M-3)/2]\!]$, we have $\mathfrak{P}(k, k+1) \geqslant 1/3$ and $\mathfrak{P}(k, k-1) \leqslant 1/3$, a simple comparison with the random walk W enables us to see that $\mathfrak{t}$ is stochastically dominated by ς. This elementary observation leads to:

Corollary 8 *The strong stationary time $\mathfrak{t}$ for the random walk on the circle $\mathbb{Z}_M$ corresponding to $a = 1/3$, constructed as the absorption of the above Markov chain $\mathfrak{X}$, has tail distributions satisfying for M large enough,*

$$\forall\ r \geqslant 0, \qquad \mathbb{P}[\mathfrak{t} \geqslant rM^2] \quad \leqslant \quad 2\exp(-r/4)$$

Proof

It is sufficient to prove the same bound for ς: for M large enough,

$$\forall\ r \geqslant 0, \qquad \mathbb{P}[\varsigma \geqslant rM^2] \quad \leqslant \quad 2\exp(-r/4) \tag{9}$$

For any $\alpha \in \mathbb{R}$, define

$$\rho_\alpha \quad \coloneqq \quad \frac{1 + e^{-\alpha} + e^{\alpha}}{3}$$

We have for any $n \in \mathbb{Z}_+$,

$$\mathbb{E}[e^{\alpha W_{n+1}} | \sigma(W_n, W_{n-1}, ..., W_1, W_0 = 0)] \quad = \quad \rho_\alpha e^{\alpha W_n}$$

and as a consequence, the process $(M_n)_{n\in\mathbb{Z}_+}$ defined by

$$\forall\ n\in\mathbb{Z}_+,\qquad M_n\ \coloneqq\ e^{\alpha W_n-\ln(\rho_\alpha)n}$$

is a martingale.

Note that by symmetry and since ς is independent from $\mathrm{sgn}(W_\varsigma)$, we have

$$\begin{aligned}\mathbb{E}[M_\varsigma] &= \mathbb{E}[e^{\alpha(M-1)/2-\ln(\rho_\alpha)\varsigma}\mathbb{1}_{W_\varsigma=(M-1)/2}]+\mathbb{E}[e^{-\alpha(M-1)/2-\ln(\rho_\alpha)\varsigma}\mathbb{1}_{W_\varsigma=-(M-1)/2}]\\ &= \cosh(\alpha(M-1)/2)\mathbb{E}[e^{-\ln(\rho_\alpha)\varsigma}]\end{aligned}$$

Furthermore, $(M_{n\wedge\varsigma})_{n\in\mathbb{Z}_+}$ is a bounded martingale (since $\rho_\alpha\geqslant 1$), so the stopping theorem gives us $\mathbb{E}[M_\varsigma]=\mathbb{E}[M_0]=1$, and we get

$$\mathbb{E}[\rho_\alpha^{-\varsigma}]\ =\ \frac{1}{\cosh(\alpha(M-1)/2)}$$

By analytic extension, this equality is still valid if α is replaced by αi (where $i\in\mathbb{C}$, $i^2=-1$), as long as $|\alpha|(M-1)/2<\pi/2$, and we get

$$\mathbb{E}\left[\left(\frac{3}{1+2\cos(\alpha)}\right)^{\varsigma}\right]\ =\ \frac{1}{\cos(\alpha(M-1)/2)}$$

Apply this equality with $\alpha=1/M$, to deduce that for large M,

$$\mathbb{E}\left[\left(\frac{3}{1+2\cos(1/M)}\right)^{\varsigma}\right]\ \sim\ \frac{1}{\cos(1/2)}$$

For $r>0$, remarking that $\cos(1/M)<1$, we get

$$\begin{aligned}\mathbb{P}[\varsigma\geqslant rM^2] &\leqslant \left(\frac{1+2\cos(1/M)}{3}\right)^{rM^2}\mathbb{E}\left[\left(\frac{3}{1+2\cos(1/M)}\right)^{\varsigma}\right]\\ &\sim \frac{1}{\cos(1/2)}\left(1-\frac{1}{3M^2}\right)^{rM^2}\\ &= \frac{1}{\cos(1/2)}\exp(-r(1+o(1))/3)\end{aligned}$$

Taking into account that $1/\cos(1/2)\approx 1.139$, we see that (9) is satisfied for M large enough.

■

3 A first stopping time

Here we construct the first epoch of a set-valued dual process associated to the random walk $[X,Y,Z]$ on the Heisenberg group $\mathbb{H}_M$ described in the introduction, where the odd number $M\geqslant 3$ is fixed.

Denote by P the transition kernel of $[X,Y,Z]$. The uniform probability measure $\mathcal{U}$ on $\mathbb{H}_M$ is reversible with respect to P, so that $P^*=P$, where P^* is the adjoint operator of P in $\mathbb{L}^2(\mathcal{U})$.

As in the previous section, we are looking for a dual process $\mathfrak{X}\coloneqq(\mathfrak{X}_n)_{n\in\mathbb{Z}_+}$ taking values in a set $\mathfrak{V}$ of non-empty subsets of $\mathbb{H}_M$, whose transition kernel $\mathfrak{P}$ is intertwined with P through:

$$\mathfrak{P}\Lambda\ =\ \Lambda P \tag{10}$$

where the Markov kernel Λ from $\mathfrak{V}$ to $\mathbb{H}_M$ is given by

$$\forall\ \Omega\in\mathfrak{V},\ \forall\ u\in\mathbb{H}_M,\qquad \Lambda(\Omega,u)\ \coloneqq\ \frac{\mathcal{U}(u)}{\mathcal{U}(\Omega)}\mathbb{1}_\Omega(u)$$

Since $[X, Y, Z]$ is starting from $[0, 0, 0]$, we will require furthermore that $\mathfrak{X}_0 = \{[0, 0, 0]\}$.

The construction of $\mathfrak{P}$ will follow the general random mapping method described in [16] and already alluded to in the previous section. More precisely, for any $\Omega \in \mathfrak{V}$, we are looking for a **random mapping** $\psi_\Omega : \mathbb{H}_M \to \mathbb{H}_M$ satisfying two conditions:

- the **weak association** with $P^* = P$, namely

$$\forall\ u \in \mathbb{H}_M,\ \forall\ v \in \Omega, \qquad \mathbb{P}[\psi_\Omega(u) = v] \;=\; \frac{1}{\xi(\Omega)} P(u, v) \tag{11}$$

  where $\xi(\Omega) > 0$ is a positive number.

- the **stability** of $\mathfrak{V}$: the set

$$\Psi(\Omega) \;\coloneqq\; \psi_\Omega^{-1}(\Omega) \tag{12}$$

  belongs to $\mathfrak{V}$.

It is shown in [16] that the Markov kernel defined on $\mathfrak{V}$ by

$$\forall\ \Omega, \Omega' \in \mathfrak{V}, \qquad \mathfrak{P}(\Omega, \Omega') \;=\; \frac{\xi(\Omega)\mathcal{U}(\Omega')}{\mathcal{U}(\Omega)} \mathbb{P}[\Psi(\Omega) = \Omega'] \tag{13}$$

satisfies (10). Note that the whole state space $\mathbb{H}_M$ is absorbing for $\mathfrak{P}$, so if

$$\mathfrak{t} \;\coloneqq\; \inf\{n \in \mathbb{Z}_+ : \mathfrak{X}_n = \mathbb{Z}_M\} \;\in\; \mathbb{N} \sqcup \{\infty\}$$

is a.s. finite, then it has the same law as a strong stationary time for $[X, Y, Z]$, as a consequence of Diaconis and Fill [6].

Let us now describe $\mathfrak{V}$ and the corresponding random mappings $(\psi_\Omega)_{\Omega \in \mathfrak{V}}$.

The set $\mathfrak{V}$ consists of subsets $S \subset \mathbb{H}_M$ of the form

$$\Omega_{r,s,A} \;\coloneqq\; \{[x, y, z] \in \mathbb{H}_M : x \in B(r),\ y \in B(s),\ z \in A(x, y)\} \tag{14}$$

where $r \in [\![0, (M-1)/2]\!]$, $s \in [\![0, (M-1)/2]\!]$, $B(r) \coloneqq [\![-r, r]\!]$ seen as the closed ball of $\mathbb{Z}_M$ centered at 0 and of radius r, and A is a mapping from $B(r) \times B(s)$ to the non-empty subsets of $\mathbb{Z}_M$. It will be convenient to see A as a **restricted field** going from the base space $B(r) \times B(s)$ to the fiber space consisting of the non-empty subsets of $\mathbb{Z}_M$. A **field** will be a restricted field whose base space is $\mathbb{Z}_M \times \mathbb{Z}_M$, they will be mainly be used in the next section.

In order to construct our random mappings $(\psi_\Omega)_{\Omega \in \mathfrak{V}}$, we need to introduce the following 7 mappings, inspired by the considerations of the previous section. Denote $\mathbb{Z}_M^- \coloneqq [\![-(M-1)/2, -1]\!]$ and $\mathbb{Z}_M^+ \coloneqq [\![0, (M-1)/2]\!]$, seen as subsets of $\mathbb{Z}_M$. We define the mapping **sgn** on $\mathbb{Z}_M$ via

$$\forall\ x \in \mathbb{Z}_M, \qquad \mathrm{sgn}(x) \;\coloneqq\; \begin{cases} -1 & \text{, if } x \in \mathbb{Z}_M^- \\ 1 & \text{, if } x \in \mathbb{Z}_M^+ \end{cases}$$

Here are the mappings that will be the values of the random mappings $(\psi_\Omega)_{\Omega \in \mathfrak{V}}$:

- The mapping $\widetilde{\phi}^{(0)}$:

$$\forall\ [x, y, z] \in \mathbb{H}_M, \qquad \widetilde{\phi}^{(0)}([x, y, z]) \;\coloneqq\; \begin{cases} [0, y, z - xy] & \text{, if } x \in \{-1, 0, 1\} \\ [x, y, z] & \text{, if } x \notin \{-1, 0, 1\} \end{cases}$$

- The mapping $\widehat{\phi}^{(0)}$:

$$\forall\ [x, y, z] \in \mathbb{H}_M, \qquad \widehat{\phi}^{(0)}([x, y, z]) \;\coloneqq\; \begin{cases} [x, 0, z] & \text{, if } y \in \{-1, 0, 1\} \\ [x, y, z] & \text{, if } y \notin \{-1, 0, 1\} \end{cases}$$

- The mapping $\widetilde{\phi}^{(1)}$:

$$\forall\ [x,y,z] \in \mathbb{H}_M, \qquad \widetilde{\phi}^{(1)}([x,y,z]) \;\coloneqq\; [x-\mathrm{sgn}(x), y, z-\mathrm{sgn}(x)y]$$

- The mapping $\widehat{\phi}^{(1)}$:

$$\forall\ [x,y,z] \in \mathbb{H}_M, \qquad \widehat{\phi}^{(1)}([x,y,z]) \;\coloneqq\; [x, y-\mathrm{sgn}(y), z]$$

The mapping $\widetilde{\phi}^{(2)}$:

$$\forall\ [x,y,z] \in \mathbb{H}_M, \qquad \widetilde{\phi}^{(2)}([x,y,z]) \;\coloneqq\; [x+\mathrm{sgn}(x), y, z+\mathrm{sgn}(x)y]$$

- The mapping $\widehat{\phi}^{(2)}$:

$$\forall\ [x,y,z] \in \mathbb{H}_M, \qquad \widehat{\phi}^{(2)}([x,y,z]) \;\coloneqq\; [x, y+\mathrm{sgn}(y), z]$$

- $\widehat{\phi}^{(3)}$ is just the identity mapping on $\mathbb{H}_M$.

We can now define the family $(\psi_\Omega)_{\Omega\in\mathfrak{V}}$.

Again we fix a set $S \coloneqq \Omega_{r,s,A}$ as in (14). The underlying probability depends on S through the following cases.

- If $r = s = 0$. The random mapping ψ_Ω takes with the values $\widetilde{\phi}^{(0)}$ and $\widehat{\phi}^{(0)}$ with probability $1/2$ each. The weak association with P is satisfied with $\xi(\Omega) = 1/3$: for any $[x,y,z] \in \mathbb{H}_M$ and $[x',y',z'] \in \Omega$,

$$\mathbb{P}\big[\psi_\Omega([x,y,z]) = [x',y',z']\big] \;=\; 3P([x,y,z],[x',y',z']) \tag{15}$$

Indeed, first note that $[x',y',z'] \in \Omega_{0,0,A}$ implies that $x' = y' = 0$. Next, both sides vanish if we do not have $(x,y) \in \{(-1,0),(1,0),(0,1),(0,-1),(0,0)\}$, and $z' = z$.

Consider the case $(x,y) = (0,0)$, we have for any $z \in \mathbb{Z}_M$,

$$\begin{aligned} \mathbb{P}\left[\psi_\Omega([0,0,z]) = [0,0,z]\right] &= 1 \\ P([0,0,z],[0,0,z]) &= 1/3 \end{aligned}$$

so (15) is satisfied.

When $(x,y) = (-1,0)$, we have for any $z \in \mathbb{Z}_M$,

$$\begin{aligned} \mathbb{P}\left[\psi_\Omega([-1,0,z]) = [0,0,z]\right] &= \mathbb{P}[\psi_\Omega = \widetilde{\phi}^{(0)}] \;=\; 1/2 \\ P([-1,0,z],[0,0,z]) &= 1/6 \end{aligned}$$

so (15) is satisfied again. The other cases are treated in the same way.

- If $r = 0$ and $s \neq 0$. The random mapping ψ_Ω takes with the value $\widetilde{\phi}^{(0)}$ with probability p, $\widehat{\phi}^{(1)}$ and $\widehat{\phi}^{(2)}$ each with probability q, and $\widehat{\phi}^{(3)}$ with probability $1-p-2q$, where $p,q \in [0,1]$ are such that $1-p-2q \in [0,1]$. Let us find p,q such that furthermore the weak association with P is satisfied with some $\xi(\Omega) > 0$: for any $[x,y,z] \in \mathbb{H}_M$ and $[x',y',z'] \in S$,

$$\mathbb{P}\big[\psi_\Omega([x,y,z]) = [x',y',z']\big] \;=\; \frac{1}{\xi(\Omega)} P([x,y,z],[x',y',z']) \tag{16}$$

Indeed, first note that $[x',y',z'] \in \Omega_{0,s,A}$ implies that $x' = 0$. Next we have

$$\begin{aligned} \mathbb{P}\left[\psi_\Omega([x,y,z]) = [0,y',z']\right] &= p\mathbb{1}_{\{\widetilde{\phi}^{(0)}([x,y,z])=[0,y',z']\}} + q\mathbb{1}_{\{\widehat{\phi}^{(1)}([x,y,z])=[0,y',z']\}} \\ &\quad + q\mathbb{1}_{\{\widehat{\phi}^{(2)}([x,y,z])=[0,y',z']\}} + (1-p-2q)\mathbb{1}_{\{\widehat{\phi}^{(3)}([x,y,z])=[0,y',z']\}} \end{aligned}$$

Let us first investigate the possibility $\widetilde{\phi}^{(0)}([x,y,z]) = [0,y',z']$. Necessarily, $x \in \{-1,0,1\}$, $y = y'$, $z' = z - xy$. Whatever $x \in \{-1,0,1\}$, we have

$$\begin{aligned}\mathbb{P}\left[\psi_\Omega([x,y,z]) = [0,y,z]\right] &= p\mathbb{1}_{\{\widetilde{\phi}^{(0)}([x,y,z])=[0,y,z-xy]\}} + (1-p-2q)\mathbb{1}_{\{\widehat{\phi}^{(3)}([x,y,z])=[0,y,z-xy]\}} \\ &= \begin{cases} p & \text{, if } x \in \{-1,1\} \\ 1-2q & \text{, if } x = 0 \end{cases}\end{aligned}$$

On the other hand, we have

$$P([x,y,z],[0,y,z]) = \begin{cases} 1/6 & \text{, if } x \in \{-1,1\} \\ 1/3 & \text{, if } x = 0 \end{cases}$$

thus we end up with the conditions $1-2q = 2p$ and $\xi(\Omega) = 1/(6p)$.
Next we consider the possibility $\widehat{\phi}^{(1)}([x,y,z]) = [0,y',z']$ (the symmetric case $\widehat{\phi}^{(2)}([x,y,z]) = [0,y',z']$ is similarly treated). Necessarily $x = 0$, $z' = z$ and $y = y' \pm 1$, depending on $y \in \mathbb{Z}_M^+$ or $y \in \mathbb{Z}_M^-$. With these values, it follows that the l.h.s. of (16) is q and the r.h.s. is $1/(6\xi(\Omega))$, leading us to the equation $q = 1/(6\xi(\Omega))$. Putting together all the equations between p, q and $\xi(\Omega)$, we get that $p = q = 1/4$ and $\xi(\Omega) = 2/3$. It is then immediate to check that (16) is true.

- If $r \neq 0$ and $s = 0$. The random mapping ψ_Ω takes with the values $\widehat{\phi}^{(0)}, \widetilde{\phi}^{(1)}, \widetilde{\phi}^{(2)}$ and $\widehat{\phi}^{(3)}$, each with probability 1/4. The treatment of this case is similar to the previous one.
- If $r \neq 0$ and $s \neq 0$. The random mapping ψ_Ω takes with the value $\widehat{\phi}^{(3)}$ with probability 1/3 and each of the values $\widetilde{\phi}^{(1)}, \widehat{\phi}^{(1)}, \widetilde{\phi}^{(2)}$ and $\widehat{\phi}^{(2)}$, each with probability 1/6. This situation is the simplest one, we clearly have for any $[x,y,z] \in \mathbb{H}_M$ and $[x',y',z'] \in S$,

$$\mathbb{P}\big[\psi_\Omega([x,y,z]) = [x',y',z']\big] = P([x,y,z],[x',y',z'])$$

Our next task is to check that the random mapping Ψ associated to the family $(\psi_\Omega)_{\Omega\in\mathfrak{V}}$ lets $\mathfrak{V}$ stable and moreover to describe its action.

Fix some $S \coloneqq \Omega_{r,s,A} \in \mathfrak{V}$, we are wondering in which set it will be transformed by Ψ. Again we consider the previous situations.

- If $r = s = 0$. Then $\Psi(\Omega)$ is equal either to $(\widetilde{\phi}^{(0)})^{-1}(\Omega)$ or to $(\widehat{\phi}^{(0)})^{-1}(\Omega)$. Let us consider the first case. By definition, we have

$$\begin{aligned}(\widetilde{\phi}^{(0)})^{-1}(\Omega) &= \{[x',y',z'] \in \mathbb{H}_M \,:\, \widetilde{\phi}^{(0)}([x',y',z']) \in S\} \\ &= \{[x',y',z'] \in \mathbb{H}_M \,:\, \exists\, [x,y,z] \in S, \text{ with } \widetilde{\phi}^{(0)}([x',y',z']) = [x,y,z]\} \\ &= \{[x',y',z'] \in \mathbb{H}_M \,:\, \exists\, z \in A(0,0), \text{ with } \widetilde{\phi}^{(0)}([x',y',z']) = [0,0,z]\} \\ &= \{[x',y',z'] \in \mathbb{H}_M \,:\, \exists\, z \in A(0,0), \text{ with } x' \in \{-1,0,1\} \text{ and } [0,y',z'-x'y'] = [0,0,z]\} \\ &= \{[x',0,z'] \in \mathbb{H}_M \,:\, x' \in \{-1,0,1\} \text{ and } z' \in A(0,0)\} \\ &= \Omega_{1,0,A'}\end{aligned}$$

where A' is defined by:

$$\forall\ (x',y') \in B(1) \times \{0\}, \qquad A(x',y') \coloneqq A(0,0)$$

Similarly, we get $(\widehat{\phi}^{(0)})^{-1}(\Omega) = \Omega_{1,0,A'}$.

- If $r = 0$ and $s \neq 0$. There are three possibilities for $\Psi(\Omega)$: $(\widetilde{\phi}^{(0)})^{-1}(\Omega)$, $(\widehat{\phi}^{(1)})^{-1}(\Omega)$, $(\widehat{\phi}^{(2)})^{-1}(\Omega)$ or $(\widehat{\phi}^{(3)})^{-1}(\Omega)$. The same computation as above shows that

$$\begin{aligned}&(\widetilde{\phi}^{(0)})^{-1}(\Omega) \\ &= \{[x',y',z'] \in \mathbb{H}_M \,:\, \exists\, y \in B(s), \exists\, z \in A(0,y), \text{ with } x' \in \{-1,0,1\} \text{ and } [0,y',z'-x'y'] = [0,y,z]\} \\ &= \{[x',y',z'] \in \mathbb{H}_M \,:\, x' \in \{-1,0,1\},\ y' \in B(s) \text{ and } z' \in A(0,y') + x'y'\} \\ &= \Omega_{1,s,A'}\end{aligned}$$

where A' is defined by:

$$\forall\ (x',y')\in B(1)\times B(s),\qquad A'(x',y')\ \coloneqq\ A(0,y')+x'y'$$

Next consider $(\widehat{\phi}^{(1)})^{-1}(\Omega)$:

$$\begin{aligned}(\widehat{\phi}^{(1)})^{-1}(\Omega)\ &=\ \{[x',y',z']\in\mathbb{H}_M\ :\ \exists\ y\in B(s),\exists\ z\in A(0,y)\text{ with } x'=0,\ y'-\operatorname{sgn}(y')=y,\ z'=z\}\\ &=\ \Omega_{0,s-1,A'}\end{aligned}$$

where A' is defined by:

$$\forall\ (x',y')\in\{0\}\times B(s-1),\qquad A'(x',y')\ \coloneqq\ A(0,y'-\operatorname{sgn}(y'))$$

Similarly, $(\widehat{\phi}^{(2)})^{-1}(\Omega)=\Omega_{0,s-1,A'}$, with another set-valued mapping A':

$$\forall\ (x',y')\in\{0\}\times B(s+1),\qquad A'(x',y')\ \coloneqq\ A(0,y'+\operatorname{sgn}(y'))$$

Of course, we have $(\widehat{\phi}^{(3)})^{-1}(\Omega)=\Omega$.

- The other cases where $r\neq 0$ are treated in a similar way. For instance for $r\neq 0,(M-1)/2$ and $s\neq 0$, we have $(\widehat{\phi}^{(1)})^{-1}(\Omega)=\Omega_{r+1,s,A'}$ with

$$\forall\ (x',y')\in B((r+1)\wedge((M-1)/2))\times B(s),\qquad A'(x',y')\ \coloneqq\ A(x'-\operatorname{sgn}(x'),y')+\operatorname{sgn}(x')y'$$

Let $\mathfrak{P}$ the transition kernel induced by the above family of random mappings $(\psi_\Omega)_{\Omega\in\mathfrak{V}}$ and consider $\mathfrak{X}\coloneqq(\mathfrak{X}_n)_{n\in\mathbb{Z}_+}$ an associated Markov chain starting from $\{0,0,0\}$. For any $n\in\mathbb{Z}_+$, let us write $\mathfrak{X}_n=\Omega_{R_n,S_n,A_n}$ with the previous notation. Define

$$\sigma\ \coloneqq\ \inf\{n\in\mathbb{Z}_+\ :\ R_n=(M-1)/2=S_n\}$$

Taking into account the considerations of Section 2, σ is a.s. finite and we have

$$\forall\ n\geqslant\sigma,\qquad R_n\ =\ (M-1)/2\ =\ S_n$$

Nevertheless, this Markov chain has a serious drawback:

$$\forall\ n\in\mathbb{Z}_+,\ \forall\ x\in B(R_n),\ \forall\ y\in B(S_n),\qquad |A_n(x,y)|\ =\ 1 \tag{17}$$

Indeed, from the above construction, we deduce that

$$\forall\ n\in\mathbb{Z}_+,\ \forall\ x'\in B(R_{n+1}),\ \forall\ y'\in B(S_{n+1}),\ \exists\ x\in B(R_n),\ \exists\ y\in B(S_n)\ :\ |A_{n+1}(x',y')|\ =\ |A_n(x,y)|$$

This observation is true for any initial condition $\mathfrak{X}_0$. When $\mathfrak{X}_0=\{[0,0,0]\}$, the fiber-valued component of $\mathfrak{X}_0$ is only $\{0\}$ and has size 1. The latter property is inherited by all the following values of $\mathfrak{X}_n$ for $n\in\mathbb{Z}_+$, justifying (17).

For the fiber-valued components of $\mathfrak{X}$ to reach the whole state space $\mathbb{Z}_M$, we need to change our strategy after time σ. Before doing so in the next section, let us estimate the tail probabilities of σ, taking into account Corollary 8:

Lemma 9 *For M large enough, we have*

$$\forall\ r\geqslant 0,\qquad \mathbb{P}[\sigma\geqslant rM^2]\ \leqslant\ 5\exp(-r/10)$$

Proof

Let $\widetilde{X} := (\widetilde{X}_n)_{n\in\mathbb{Z}_+}$ and $\widetilde{Y} := (\widetilde{Y}_n)_{n\in\mathbb{Z}_+}$ be two independent random walks on $\mathbb{Z}_M$ as in Section 2.4. Let $(B_n)_{n\in\mathbb{Z}_+}$ be a family of independent Bernoulli variables of parameter 1/2 (independent from $(\widetilde{X},\widetilde{Y})$) and define

$$\forall\, n \in \mathbb{Z}_+, \qquad \theta_n \;:=\; \sum_{m\in[\![n]\!]} B_m$$

The chain (X, Y) has the same law as $(\widetilde{X}_{\theta_n}, \widetilde{Y}_{n-\theta_n})_{n\in\mathbb{Z}_+}$ and from the above construction, it appears that

$$\sigma \;=\; \inf\{n \in \mathbb{Z}_+ \,:\, \theta_n \geqslant \mathfrak{t}_1 \text{ and } n - \theta_n \geqslant \mathfrak{t}_2\}$$

where $\mathfrak{t}_1$ (respectively $\mathfrak{t}_2$) is the strong stationary time constructed as in Section 2.4 for $\widetilde{X}$ (resp. $\widetilde{Y}$).

It follows that for any $n \in \mathbb{Z}_+$,

$$\begin{aligned}\mathbb{P}[\sigma \geqslant n] &\leqslant \mathbb{P}[\theta_n \leqslant \mathfrak{t}_1] + \mathbb{P}[n - \theta_n \leqslant \mathfrak{t}_2]\\ &= 2\mathbb{P}[\theta_n \leqslant \mathfrak{t}_1]\end{aligned}$$

since $(\theta_n, \mathfrak{t}_1)$ and $(n - \theta_n, \mathfrak{t}_2)$ have the same law.

According to Corollary 8, we have for the conditional expectation knowing θ_n and for large M:

$$\mathbb{P}[\theta_n \leqslant \mathfrak{t}_1|\theta_n] \;\leqslant\; 2\exp(-\theta_n/(4M^2))$$

so that

$$\begin{aligned}\mathbb{P}[\theta_n \leqslant \mathfrak{t}_1] &\leqslant 2\mathbb{E}[\exp(-\theta_n/(4M^2))]\\ &= 2\mathbb{E}[\exp(-B_1/M^2)]^n\\ &= 2\left(\frac{1+\exp(-1/(4M^2))}{2}\right)^n\end{aligned}$$

It follows that if n is of the form $\lceil rM^2\rceil$ for some $r \geqslant 0$, then

$$\begin{aligned}\mathbb{P}[\sigma \geqslant rM^2] &\leqslant 4\left(\frac{1+\exp(-1/(4M^2))}{2}\right)^{rM^2}\\ &\leqslant 5\exp(-r/10)\end{aligned}$$

for M large enough (uniformly in $r \geqslant 0$).

■

4 A second stopping time

Here we modify the family of random mappings considered in the previous section, in order to construct the second epoch of a set-valued dual process associated to the random walk $[X, Y, Z]$.

Since we are to work after the stopping time σ, we only consider subsets of the form $\Omega_A := \Omega_{(M-1)/2,(M-1)/2,A}$, where A is a mapping from $\mathbb{Z}_M \times \mathbb{Z}_M$ to the subsets of $\mathbb{Z}_M$, what we called a field in Section 3. Let $\mathfrak{A}$ the set of fields, here it is more convenient to index all our objects of interest by $\mathfrak{A}$ instead of the set

$$\mathfrak{V}^\dagger \;:=\; \{\Omega_A \,:\, A \in \mathfrak{A}\}$$

Remark 10 Note that we do not require that the values of A are non-empty, thus $\mathfrak{V}^\dagger$ contains the empty set, contrary to $\mathfrak{V}$ (in particular $\mathfrak{V}^\dagger$ is not included in $\mathfrak{V}$). Nevertheless the Markov chains that we will consider on $\mathfrak{V}^\dagger$ will be forbidden to take $\emptyset$ as value and will stay in $\mathfrak{V}^\ddagger \coloneqq \mathfrak{V}^\dagger \setminus \{\emptyset\}$. We will denote $\mathfrak{A}^\ddagger$ the set of fields A such that $\Omega_A \neq \emptyset$, namely whose subset-valued fibers are all non-empty.

□

The main difference between our new family of random mappings $(\psi_A)_{A\in\mathfrak{A}}$ and that $(\psi_S)_{S\in\mathfrak{V}}$ of Section 3 consists in replacing the function sign acting on the first coordinates of $\mathbb{H}_M$ by a much more general mapping. More precisely, let us fix a field $A \in \mathfrak{A}$. Assume that for any $x, y \in \mathbb{Z}_M$, we are given two partitions of $\mathbb{Z}_M$ into two disjoint subsets respectively $\widetilde{B}^-_{A,x,y} \sqcup \widetilde{B}^+_{A,x,y}$ and $\widehat{B}^-_{A,x,y} \sqcup \widehat{B}^+_{A,x,y}$, that depend on A, x and y. We define corresponding mappings $\widetilde{\varphi}_A$ and $\widehat{\varphi}_A$ on $\mathbb{H}_M$ via

$$\forall\ [x,y,z] \in \mathbb{H}_M, \qquad \widetilde{\varphi}_A(x,y,z) \ \coloneqq \ \begin{cases} -1 & \text{, if } z \in \widetilde{B}^-_{A,x,y} \\ 1 & \text{, if } z \in \widetilde{B}^+_{A,x,y} \end{cases}$$

$$\forall\ [x,y,z] \in \mathbb{H}_M, \qquad \widehat{\varphi}_A(x,y,z) \ \coloneqq \ \begin{cases} -1 & \text{, if } z \in \widehat{B}^-_{A,x,y} \\ 1 & \text{, if } z \in \widehat{B}^+_{A,x,y} \end{cases}$$

Next we replace $\widetilde{\phi}^{(1)}$, $\widetilde{\phi}^{(2)}$, $\widehat{\phi}^{(1)}$ and $\widehat{\phi}^{(2)}$ respectively by

$$\begin{aligned}
\forall\ [x,y,z] \in \mathbb{H}_M, &\qquad \widetilde{\phi}^{(1)}_A([x,y,z]) \ \coloneqq\ [x - \widetilde{\varphi}_A(x,y,z), y, z - \widetilde{\varphi}_A(x,y,z)y] \\
\forall\ [x,y,z] \in \mathbb{H}_M, &\qquad \widetilde{\phi}^{(2)}_A([x,y,z]) \ \coloneqq\ [x + \widetilde{\varphi}_A(x,y,z), y, z + \widetilde{\varphi}_A(x,y,z)y] \\
\forall\ [x,y,z] \in \mathbb{H}_M, &\qquad \widehat{\phi}^{(1)}_A([x,y,z]) \ \coloneqq\ [x, y - \widehat{\varphi}_A(x,y,z), z] \\
\forall\ [x,y,z] \in \mathbb{H}_M, &\qquad \widehat{\phi}^{(2)}_A([x,y,z]) \ \coloneqq\ [x, y + \widehat{\varphi}_A(x,y,z), z]
\end{aligned}$$

The random mapping ψ_A is constructed as the corresponding ψ_S in the case $r \neq 0$ and $s \neq 0$. Namely, the random mapping ψ_A takes with the value $\widehat{\phi}^{(3)}$ with probability 1/3 and each of the values $\widetilde{\phi}^{(1)}_A, \widehat{\phi}^{(1)}_A, \widetilde{\phi}^{(2)}_A$ and $\widehat{\phi}^{(2)}_A$, each with probability 1/6. There is no difficulty in checking that ψ_A is associated to P:

$$\forall\ [x,y,z] \in \mathbb{H}_M,\ \forall\ [x',y',z'] \in S, \qquad \mathbb{P}\big[\psi_S([x,y,z]) = [x',y',z']\big] \ = \ P([x,y,z],[x',y',z'])$$

(note that $\widetilde{\phi}^{(0)}$ and $\widehat{\phi}^{(0)}$ are no longer required, they were only useful to initiate the spread of the evolving sets associated to $(\psi_S)_{s\in\mathfrak{V}}$ on the base space $\mathbb{Z}_M \times \mathbb{Z}_M$ corresponding to the two first coordinates of $\mathbb{H}_M$).

Let us, firstly check that the random mapping Ψ associated to the family $(\psi_A)_{A\in\mathfrak{A}}$ leaves $\mathfrak{V}^\dagger$ stable, and secondly describe its action.

Fix some $A \in \mathfrak{A}$, we are wondering what is $\Psi(\Omega_A)$, namely we have to compute $(\widetilde{\phi}^{(1)}_A)^{-1}(\Omega_A)$, $(\widehat{\phi}^{(1)})^{-1}(\Omega_A)$, $(\widetilde{\phi}^{(2)}_A)^{-1}(\Omega_A)$ and $(\widehat{\phi}^{(2)})^{-1}(\Omega_A)$. Let us start with

$$(\widetilde{\phi}^{(1)}_A)^{-1}(\Omega_A) \ = \ \{[x',y',z'] \in \mathbb{H}_M \,:\, \exists\ [x,y,z] \in \Omega_A, \text{ with } \widetilde{\phi}^{(1)}_A([x',y',z']) = [x,y,z]\}$$

The belonging of $[x,y,z]$ to Ω_A means that $z \in A(x,y)$, and the equality $\widetilde{\phi}^{(1)}_A([x',y',z']) = [x,y,z]$ is equivalent to

$$\begin{cases} x' - \varphi_A(x',y',z') & = \ x \\ y' & = \ y \\ z' - \varphi_A(x',y',z')y' & = \ z \end{cases}$$

Thus $[x',y',z']$ belongs to $(\widetilde{\phi}^{(1)}_A)^{-1}(S)$ if and only if

$$z' \ \in \ A(x' - \varphi_A(x',y',z'), y') + \varphi_A(x',y',z')y'$$

namely, either

$$\varphi_A(x', y', z') = -1 \quad \text{and} \quad z' \in A(x'+1, y') - y' \tag{18}$$

or

$$\varphi_A(x', y', z') = 1 \quad \text{and} \quad z' \in A(x'-1, y') + y' \tag{19}$$

Thus defining the new field $\widetilde{A}^{(1)}$ via

$$\forall\ (x', y') \in \mathbb{Z}_M^2, \qquad \widetilde{A}^{(1)}(x', y') \;\coloneqq\; \Big((A(x'+1, y') - y') \cap \widetilde{B}^-_{A,x',y'}\Big) \cup \Big((A(x'-1, y') + y') \cap \widetilde{B}^+_{A,x',y'}\Big)$$

we get that

$$(\widetilde{\phi}_A^{(1)})^{-1}(\Omega_A) = \Omega_{\widetilde{A}^{(1)}}$$

The other cases are treated in a similar way and we get

$$\begin{aligned}
(\widehat{\phi}_A^{(1)})^{-1}(\Omega_A) &= \Omega_{\widehat{A}^{(1)}} \\
(\widetilde{\phi}_A^{(2)})^{-1}(\Omega_A) &= \Omega_{\widetilde{A}^{(2)}} \\
(\widehat{\phi}_A^{(2)})^{-1}(\Omega_A) &= \Omega_{\widehat{A}^{(2)}}
\end{aligned}$$

where for any $(x', y') \in \mathbb{Z}_M^2$,

$$\begin{aligned}
\widehat{A}^{(1)}(x', y') &\coloneqq \Big(A(x', y'+1) \cap \widehat{B}^-_{A,x',y'}\Big) \cup \Big((A(x', y'-1) \cap \widehat{B}^+_{A,x',y'}\Big) \\
\widetilde{A}^{(2)}(x', y') &\coloneqq \Big((A(x'+1, y') - y') \cap \widetilde{B}^+_{A,x',y'}\Big) \cup \Big((A(x'-1, y') + y') \cap \widetilde{B}^-_{A,x',y'}\Big) \\
\widehat{A}^{(2)}(x', y') &\coloneqq \Big(A(x', y'+1) \cap \widehat{B}^+_{A,x',y'}\Big) \cup \Big((A(x', y'-1) \cap \widehat{B}^-_{A,x',y'}\Big)
\end{aligned}$$

Let $\mathfrak{P}^\ddagger$ the transition kernel induced by the above family of random mappings $(\psi_A)_{A\in\mathfrak{A}}$ and consider $\mathfrak{X}^\ddagger \coloneqq (\mathfrak{X}_n^\ddagger)_{n\in\mathbb{Z}_+}$ an associated Markov chain starting from Ω_{A_0}, for some initial field $A_0 \in \mathfrak{A}^\ddagger$. For any $n \in \mathbb{Z}_+$, let us write $\mathfrak{X}_n^\ddagger = \Omega_{A_n}$. The chain $(A_n)_{n\in\mathbb{Z}_+}$ is Markovian and its transition matrix $\mathfrak{Q}$ is given by

$$\mathfrak{Q}(A, A') \;\coloneqq\; \frac{|A'|}{|A|}\left(\frac{1}{6}\mathbb{1}_{\widetilde{A}^{(1)}}(A') + \frac{1}{6}\mathbb{1}_{\widehat{A}^{(1)}}(A') + \frac{1}{6}\mathbb{1}_{\widetilde{A}^{(2)}}(A') + \frac{1}{6}\mathbb{1}_{\widehat{A}^{(2)}}(A') + \frac{1}{3}\mathbb{1}_{A}(A')\right)$$

for any fields A, A', where $A \in \mathfrak{A}^\ddagger$ and where for any field A, the **thickness** of A is defined by

$$|A| \;\coloneqq\; \prod_{x,y\in\mathbb{Z}_M} |A(x, y)|$$

Note in particular that a transition to a field with an empty value (equivalently, whose thickness is 0) has the probability 0.

Our next task is to make an appropriate choice of the partitions $\mathbb{Z}_M = \widetilde{B}^-_{A,x,y} \sqcup \widetilde{B}^+_{A,x,y}$ and $\mathbb{Z}_M = \widehat{B}^-_{A,x,y} \sqcup \widehat{B}^+_{A,x,y}$ so that the Markov chain $(A_n)_{n\in\mathbb{Z}_+}$ ends up at the full field A_∞, defined by

$$\forall\ x, y \in \mathbb{Z}_M, \qquad A_\infty(x, y) = \mathbb{Z}_M$$

(note that this field is absorbing).

A guiding principle behind such a choice should be that there is a chance to get a "big" field (measured through its thickness). It leads us to following choice:

$$\forall\ A\in\mathfrak{A},\ \forall\ x,y\in\mathbb{Z}_M,\qquad \begin{cases} \widetilde{B}^-_{A,x,y} &:= & A(x+1,y)-y\\ \widetilde{B}^+_{A,x,y} &:= & \mathbb{Z}_M\backslash\widetilde{B}^-_{A,x,y}\\ \widehat{B}^-_{A,x,y} &:= & A(x,y+1)\\ \widehat{B}^+_{A,x,y} &:= & \mathbb{Z}_M\backslash\widehat{B}^-_{A,x,y}\end{cases}$$

We get that for any $A\in\mathfrak{A}$ and any $(x,y)\in\mathbb{Z}_M^2$,

$$\begin{cases} \widetilde{A}^{(1)}(x,y) &:= & \Big(A(x+1,y)-y\Big)\cup\Big(A(x-1,y)+y\Big)\\ \widetilde{A}^{(2)}(x,y) &:= & \Big(A(x+1,y)-y\Big)\cap\Big(A(x-1,y)+y\Big)\\ \widehat{A}^{(1)}(x,y) &:= & A(x,y+1)\cup A(x,y-1)\\ \widehat{A}^{(2)}(x,y) &:= & A(x,y+1)\cap A(x,y-1)\end{cases}\tag{20}$$

Remark 11 The fact that $\widetilde{A}^{(1)}(x,y)$ (respectively $\widehat{A}^{(1)}(x,y)$) is the biggest possible has to be compensated by the fact $\widetilde{A}^{(2)}(x,y)$ (resp. $\widehat{A}^{(2)}(x,y)$) is the smallest possible. But we should not worry so much about this feature, as $\mathfrak{Q}$ promotes bigger fields.

□

Let us check that this choice of dual process goes in the direction of our purposes.

Proposition 12 *The Markov kernel $\mathfrak{P}^{\ddagger}$ associated to* (20) *admits only one recurrence class which is $\{A_\infty\}$, i.e. the Markov chain $(A_n)_{n\in\mathbb{Z}_+}$ ends up being absorbed in finite time at the full field.*

Proof

Let be given any $A_0\in\mathfrak{A}^{\ddagger}\backslash\{A_\infty\}$. It is sufficient to find a finite sequence $(A_l)_{l\in[\![L]\!]}$ with $L\in\mathbb{N}$, $A_L=A_\infty$ and

$$\forall\ l\in[\![0,L-1]\!],\qquad \mathfrak{Q}(A_l,A_{l+1})\ >\ 0$$

Here is a construction of such a sequence.

Denote $\widetilde{T}$ (respectively $\widehat{T}$) the mapping on fields corresponding to the transition $A\to\widetilde{A}^{(1)}$ (resp. $A\to\widehat{A}^{(1)}$).

We begin by constructing A_1, A_2, A_3 and A_4 by successively applying $\widetilde{T}$, $\widehat{T}$, $\widetilde{T}$ and $\widehat{T}$. Fix $(x,y)\in\mathbb{Z}_M^2$ as well as $z\in A_0(x,y)$. Applying $\widetilde{T}$, we get that $z+y\in A_1(x+1,y)$ and $z-y\in A_1(x-1,y)$. Applying $\widehat{T}$, we have that $z+y\in A_2(x+1,y+1)$ and $z-y\in A_2(x-1,y+1)$. Next $\widetilde{T}$ insures that $z-1=z+y-(y+1)\in A_3(x,y+1)$ and $z+1=z-y+(y+1)\in A_3(x,y+1)$. Finally, under $\widehat{T}$, we get that $z-1\in A_4(x,y)$ and $z+1\in A_4(x,y)$.

Successively applying again $\widetilde{T}$, $\widehat{T}$, $\widetilde{T}$ and $\widehat{T}$, we construct A_5, A_6, A_7 and A_8. By the above considerations, we deduce that $z-2,z$ and $z+2$ belong to $A_8(x,y)$. Let us successively apply $M-3$ more times $\widetilde{T}$, $\widehat{T}$, $\widetilde{T}$ and $\widehat{T}$, to get A_9, ..., $A_{4(M-1)}$. It appears that $A_{4(M-1)}(x,y)$ contains $z-M+1,z-M+3,...,z+M-3,z+M-1$. Due to the fact that M is odd, the latter set is just $\mathbb{Z}_M$.

Thus we get that for any $(x,y)\in\mathbb{Z}_M^2$, $A_{4(M-1)}=\mathbb{Z}_M$, namely $A_{4(M-1)}=A_\infty$. It provides the desired finite sequence with $L=4(M-1)$.

■

Remark 13 The successive applications of $\widetilde{T}$, $\widehat{T}$, $\widetilde{T}$ and $\widehat{T}$ is not without recalling the construction of the bracket of two vector fields in differential geometry. The latter is used to investigate hypoellipticity, see for instance the book of Hörmander [11], the continuous Heisenberg group being a famous instance. Our objective of showing that the full space is covered by the dual process is a discrete analogue of the property of hypoelliptic diffusions to admit a positive density at any positive time (see also [15] for another link between hypoellipticity and intertwining dual processes).

□

From the above results, we can construct a strong stationary time for the random walk on $\mathbb{H}_M$. Heuristically, we first consider the stopping time σ considered in Section 3. Call $\bar{A}_0$ the field obtained at time σ (whose subset fibers are all of cardinal 1). Let ζ be the the hitting time of the full field, starting from $\bar{A}_0$. The time $\sigma + \zeta$ is a strong stationary time for the random walk on $\mathbb{H}_M$. Rigorously, we should check that $\bar{A}_0$ and ζ are independent. There is a simpler way: note that the random mappings of the previous section and of this section can be put together in a unique family $(\psi_\Omega)_{\Omega \in \mathfrak{V}}$, where depending if the base part of $\Omega \in \mathfrak{V}$ is equal to $\mathbb{Z}_M^2$ or not, ψ_Ω is defined as in this section or as in the previous section. With respect to this family, we can apply the result of [16] to get that $\sigma + \zeta$ is indeed a strong stationary time for the random walk on $\mathbb{H}_M$.

This ends the qualitative construction of a strong stationary time. To go quantitative, the hitting time ζ has to be investigated more thoroughly. More precisely, our goal is to prove Theorem 1.

Fix $A \in \mathfrak{A}^{\ddagger}$ for the two following lemmas.

Lemma 14 *We have*

$$\left(|\widetilde{A}^{(1)}| = |A| = |\widehat{A}^{(1)}|\right) \Rightarrow A = A_\infty$$

Proof

So let us assume that

$$|\widetilde{A}^{(1)}| = |A| = |\widehat{A}^{(1)}| \tag{21}$$

For any $(x, y) \in \mathbb{Z}_M^2$, we have

$$\begin{aligned} |\widetilde{A}^{(1)}(x,y)| &= \left|\left(A(x+1,y) - y\right) \cup \left(A(x-1,y) + y\right)\right| \\ &\geqslant |A(x+1,y) - y| \\ &= |A(x+1,y)| \end{aligned}$$

and we get

$$\begin{aligned} |\widetilde{A}^{(1)}| &\geqslant \prod_{(x,y)\in\mathbb{Z}_M^2} |A(x+1,y)| \\ &= \prod_{(x,y)\in\mathbb{Z}_M^2} |A(x,y)| \\ &= |A| \end{aligned}$$

Due to (21), the previous inequality must be an equality, and we deduce that

$$\forall\ (x,y) \in \mathbb{Z}_M^2, \qquad \left|\left(A(x+1,y) - y\right) \cup \left(A(x-1,y) + y\right)\right| = |A(x+1,y) - y|$$

namely

$$\forall\ (x,y) \in \mathbb{Z}_M^2, \qquad \left(A(x+1,y) - y\right) \cup \left(A(x-1,y) + y\right) = A(x+1,y) - y$$

Similarly, we get

$$\forall\ (x,y) \in \mathbb{Z}_M^2, \qquad \Big(A(x+1,y)-y\Big) \cup \Big(A(x-1,y)+y\Big) \;=\; A(x-1,y)+y$$

so that

$$\forall\ (x,y) \in \mathbb{Z}_M^2, \qquad A(x+1,y)-y \;=\; A(x-1,y)+y \tag{22}$$

The same reasoning with $\widehat{A}^{(1)}$ instead of $\widetilde{A}^{(1)}$, leads us to

$$\forall\ (x,y) \in \mathbb{Z}_M^2, \qquad A(x,y+1) \;=\; A(x,y-1)$$

or equivalently

$$\forall\ (x,y) \in \mathbb{Z}_M^2, \qquad A(x,y+2) \;=\; A(x,y)$$

Since M is odd, the mapping $\mathbb{Z}_M \ni y \mapsto y+2$ has only one orbit, which by consequence covers $\mathbb{Z}_M$. It follows that for any fixed $x \in \mathbb{Z}_M$, the set $A(x,y)$ does not depend on y, let us call it $A(x)$.

Coming back to (22), we get

$$\forall\ (x,y) \in \mathbb{Z}_M^2, \qquad A(x+2) \;=\; A(x)+2y$$

Since any element $z \in \mathbb{Z}_M$ can be written under the form $2y$ for some $y \in \mathbb{Z}_M$, we deduce

$$\forall\ (x,z) \in \mathbb{Z}_M^2, \qquad A(x+2) \;=\; A(x)+z$$

Iterating M times this relation in x, we obtain

$$\forall\ (x,z) \in \mathbb{Z}_M^2, \qquad A(x) \;=\; A(x)+z$$

and this relations implies that $A(x) = \mathbb{Z}_M$.

This amounts to say that for any $(x,y) \in \mathbb{Z}_M$, $A(x,y) = \mathbb{Z}_M$, namely $A = A_\infty$.

■

Here is a quantitative version of the previous lemma:

Lemma 15 *When $A \neq A_\infty$, then either*

$$\frac{|\widetilde{A}^{(1)}|}{|A|} \;\geqslant\; 1+\frac{1}{M} \quad \text{or} \quad \frac{|\widehat{A}^{(1)}|}{|A|} \;\geqslant\; 1+\frac{1}{M}$$

Proof

When $A \neq A_\infty$, then either $|\widetilde{A}^{(1)}| > |A|$ or $|\widehat{A}^{(1)}| > |A|$, since the proof of Lemma 14 shows that we always have $|\widetilde{A}^{(1)}| \geqslant |A|$ and $|\widehat{A}^{(1)}| \geqslant |A|$, and that $|\widetilde{A}^{(1)}| = |A| = |\widehat{A}^{(1)}|$ implies that $A = A_\infty$.

Consider the situation where $|\widetilde{A}^{(1)}| > |A|$, then there exists $(x_0,y_0) \in \mathbb{Z}_M^2$ such that $|\widetilde{A}^{(1)}(x_0,y_0)| > |A(x_0+1,y_0)|$ (otherwise $|A^{(1)}| \leqslant |A|$, according to the first part of the proof of Lemma 14). We deduce

$$\begin{aligned}
\ln(|\widetilde{A}^{(1)}|) - \ln(|A|) &= \sum_{(x,y)\in\mathbb{Z}_M^2} \ln(|\widetilde{A}^{(1)}(x,y)|) - \ln(|A(x+1,y)|) \\
&\geqslant \ln(|\widetilde{A}^{(1)}(x_0,y_0)|) - \ln(|A(x_0+1,y_0)|)
\end{aligned} \tag{23}$$

Since $|A(x_0+1,y_0)|+1 \leqslant |\widetilde{A}^{(1)}(x_0,y_0)| \leqslant M$, by concavity of the logarithm, we get that

$$\begin{aligned}
\ln(|\widetilde{A}^{(1)}(x_0,y_0)|) - \ln(|A(x_0+1,y_0)|) &\geqslant \ln(M) - \ln(M-1) \\
&= -\ln\left(1-\frac{1}{M}\right)
\end{aligned}$$

We deduce that

$$\ln(|\widetilde{A}^{(1)}|) - \ln(|A|) \quad \geqslant \quad -\ln\left(1 - \frac{1}{M}\right)$$

which implies

$$\frac{|\widetilde{A}^{(1)}|}{|A|} \quad \geqslant \quad \frac{1}{1-\frac{1}{M}} \quad \geqslant \quad 1 + \frac{1}{M} \tag{24}$$

In the situation where $|\widehat{A}^{(1)}| > |A|$, then there exists $(x_0, y_0) \in \mathbb{Z}_M^2$ such that $|\widehat{A}^{(1)}(x_0, y_0)| > |A(x_0, y_0 + 1)|$. Considerations similar to the previous ones then lead to (24), where $\widetilde{A}^{(1)}$ has been replaced by $\widehat{A}^{(1)}$.

■

Define the stochastic chain $R \coloneqq (R_n)_{n\in\mathbb{Z}_+}$ via

$$\forall\ n \in \mathbb{Z}_+, \qquad R_n \quad \coloneqq \quad \ln(|A_n|)$$

The following result is the crucial element of the proof of Theorem 1:

Lemma 16 *We have*

$$\forall\ n \in \mathbb{Z}_+, \qquad \mathbb{E}[R_{n+1}|\mathcal{A}_n] \quad \geqslant \quad R_n + \frac{1}{37M^2} \quad \text{on } \{\zeta > n\}$$

(where the filtration $(\mathcal{A}_n)_{n\in\mathbb{Z}_+}$ *is generated by* $(A_n)_{n\in\mathbb{Z}_+}$*).*

Proof

By the Markov property, for any $n \in \mathbb{Z}_+$, we have $\mathbb{E}[R_{n+1}|\mathcal{A}_n] = \mathbb{E}[R_{n+1}|A_n]$. Furthermore, for any $A \in \mathfrak{A}^\ddagger$, we have

$$\begin{aligned}\mathbb{E}[R_{n+1}|A_n = A] \quad &= \quad \sum_{A'\in\mathfrak{A}^\ddagger} \mathfrak{Q}(A, A')\ln(|A'|) \\ &= \quad \frac{1}{|A|}\sum_{A'\in\mathfrak{A}} \mathfrak{K}(A, A')|A'|\ln(|A'|)\end{aligned} \tag{25}$$

where $\mathfrak{K}$ is the kernel on $\mathfrak{A}$ defined by

$$\mathfrak{K}(A, A') \quad \coloneqq \quad \frac{1}{6}\mathbb{1}_{\widetilde{A}^{(1)}}(A') + \frac{1}{6}\mathbb{1}_{\widehat{A}^{(1)}}(A') + \frac{1}{6}\mathbb{1}_{\widetilde{A}^{(2)}}(A') + \frac{1}{6}\mathbb{1}_{\widehat{A}^{(2)}}(A') + \frac{1}{3}\mathbb{1}_{A}(A')$$

In (25) is enforced the usual convention that $\varphi(0) = 0$, where φ is the mapping $\mathbb{R}_+ \ni r \mapsto r\ln(r) \in \mathbb{R}$. This mapping is convex, so we can apply Jensen's inequality to get that for any $A \in \mathfrak{A}^\ddagger$,

$$\begin{aligned}\sum_{A'\in\mathfrak{A}} \mathfrak{K}(A, A')|A'|\ln(|A'|) \quad &\geqslant \quad \sum_{A'\in\mathfrak{A}} \mathfrak{K}(A, A')|A'|\ln\left(\sum_{A'\in\mathfrak{A}} \mathfrak{K}(A, A')|A'|\right) \\ &= \quad |A|\ln(|A|)\end{aligned}$$

where we took into account that

$$\forall\ A \in \mathfrak{A}^\ddagger, \qquad \sum_{A'\in\mathfrak{A}} \mathfrak{K}(A, A')|A'| \quad = \quad |A| \tag{26}$$

as a consequence of the Markovianity of $\mathfrak{Q}$.

But in the case of the mapping φ, Jensen's inequality can strengthened, as shown in [14]. Indeed, we have

$$\forall\, s,t \geqslant 0, \qquad t\ln(t) \;\geqslant\; s\ln(s) + (1+\ln(s))(t-s) + (\sqrt{t}-\sqrt{s})^2 \tag{27}$$

Replacing s by $|A|$, t by $|A'|$, multiplying by $\mathfrak{K}(A,A')$ and summing over $A'\in\mathfrak{A}$, we get

$$\begin{aligned}
&\sum_{A'\in\mathfrak{A}} \mathfrak{K}(A,A')|A'|\ln(|A'|)\\
&\geqslant\; |A|\ln(|A|) + (1+\ln(|A|))\sum_{A'\in\mathfrak{A}} \mathfrak{K}(A,A')(|A'|-|A|) + \sum_{A'\in\mathfrak{A}} \mathfrak{K}(A,A')(\sqrt{|A'|}-\sqrt{|A|})^2\\
&=\; |A|\ln(|A|) + \sum_{A'\in\mathfrak{A}} \mathfrak{K}(A,A')(\sqrt{|A'|}-\sqrt{|A|})^2
\end{aligned}$$

where we used (26) again.

Assume now that $A \neq A_\infty$. With the above notation, it means that $A_n \neq A_\infty$, i.e. $\zeta > n$.

According to Lemma 15, either $\frac{|\widetilde{A}^{(1)}|}{|A|} \geqslant 1+\frac{1}{M}$ or $\frac{|\widehat{A}^{(1)}|}{|A|} \geqslant 1+\frac{1}{M}$. Whatever the case, we deduce that

$$\begin{aligned}
\frac{1}{|A|}\sum_{A'\in\mathfrak{A}} \mathfrak{K}(A,A')(\sqrt{|A'|}-\sqrt{|A|})^2 \;&\geqslant\; \sum_{A'\in\{\widetilde{A}^{(1)},\widehat{A}^{(1)}\}} \mathfrak{K}(A,A')\left(\sqrt{\frac{|A'|}{|A|}}-1\right)^2\\
&\geqslant\; \frac{1}{6}\left(\sqrt{1+\frac{1}{M}}-1\right)^2\\
&=\; \frac{1}{6}\left(2+\frac{1}{M}-2\sqrt{1+\frac{1}{M}}\right)\\
&\geqslant\; \frac{\sqrt{3}}{64M^2}\\
&\geqslant\; \frac{1}{37M^2}
\end{aligned}$$

where we used the elementary bound (recall that $M \geqslant 3$):

$$\forall\, x\in[0,1/3], \qquad \sqrt{1+x} \;\leqslant\; 1+\frac{1}{2}x-\frac{3\sqrt{3}}{64}x^2 \tag{28}$$

(coming from $(d/dx)^2\sqrt{1+x} \leqslant -3\sqrt{3}/32$ for $x\in[0,1/3]$).

■

The next result goes in the direction of Theorem 1, by proving in a weak sense that ζ is of order $M^4\ln(M)$.

Proposition 17 *We have*

$$\mathbb{E}[\zeta] \;\leqslant\; 37M^4\ln(M)$$

Proof

According to Lemma 16, the stochastic chain $(R_{\zeta\wedge n}-\frac{1}{37M^2}(\zeta\wedge n))_{n\in\mathbb{Z}_+}$ is a submartingale. It follows that

$$\forall\, n\in\mathbb{Z}_+, \qquad \mathbb{E}[R_{\zeta\wedge n}] \;\geqslant\; \frac{1}{37M^2}\mathbb{E}[\zeta\wedge n]$$

Letting n go to infinity, we get, by dominated convergence in the l.h.s. and by monotone convergence in the r.h.s.,

$$\mathbb{E}[R_\zeta] \geqslant \frac{1}{37M^2}\mathbb{E}[\zeta]$$

To conclude to the desired bound, note that

$$\begin{aligned}\mathbb{E}[R_\zeta] &= \ln(|A_\infty|)\\ &= \ln(M^{M^2})\end{aligned}$$

■

We can now come to the

Proof of Theorem 1

Traditional Markov arguments enable to strengthen the weak estimate of Proposition 17 into a stronger one about the tail probabilities of ζ. More precisely the previous computations did not take into account that the Markov chain $(A_n)_{n\in\mathbb{Z}_+}$ starts from a field A_0 whose fibers are singletons. In fact they are valid for any initial field A_0. So whatever A_0, we have

$$\begin{aligned}\mathbb{P}[\zeta \geqslant e37M^4\ln(M)] &\leqslant \frac{\mathbb{E}[\zeta]}{e37M^4\ln(M)}\\ &\leqslant \frac{1}{e}\end{aligned}$$

By the Markov property we deduce

$$\forall\, n\in\mathbb{Z}+,\qquad \mathbb{P}[\zeta \geqslant ne37M^4\ln(M)] \leqslant e^{-n}$$

For any $r \geqslant 0$, writing

$$r \geqslant \left\lfloor\frac{r}{37eM^4\ln(M)}\right\rfloor 37eM^4\ln(M)$$

(where $\lfloor\cdot\rfloor$ stands for the integer part), we get

$$\begin{aligned}\mathbb{P}[\zeta \geqslant r] &\leqslant \mathbb{P}\left[\zeta > \left\lfloor\frac{r}{37eM^4\ln(M)}\right\rfloor 37eM^4\ln(M)\right]\\ &\leqslant \exp\left(-\left\lfloor\frac{r}{37eM^4\ln(M)}\right\rfloor\right)\\ &\leqslant e\exp\left(-\frac{r}{37eM^4\ln(M)}\right)\end{aligned}$$

Theorem 1 is a simple consequence of this bound and of Lemma 9, since $\tau = \sigma + \zeta$ and we have

$$\begin{aligned}\forall\, n\in\mathbb{Z}_+,\qquad \mathbb{P}[\tau \geqslant n] &\leqslant \mathbb{P}[\sigma \geqslant n] + \mathbb{P}[\zeta \geqslant n]\\ &\leqslant 5\exp\left(-\frac{r}{10M^2}\right) + 3\exp\left(-\frac{r}{37eM^4\ln(M)}\right)\\ &\leqslant 8\exp\left(-\frac{r}{101M^4\ln(M)}\right)\end{aligned}$$

uniformly over $r \geqslant 0$, for M large enough.

■

Remark 18 Lemma 16 cannot be essentially improved under its present form, because it is almost an equality when A_n is very close to A_∞. But away from the latter end, there is a lot of room for improvements. Hopefully they could lead to bound on ζ of order M up to logarithmic corrections. This is the kind of results we are looking for. Furthermore, we believe that measuring the size of fields through $|\cdot|$ is not sufficient for this purpose, this is illustrated by the proof of Lemma 14, where the translations of the fibers of the field A induced by $\widetilde{A}^{(1)}$ played an important role for the "diversification" of the fibers. But this feature is lost in Lemmas 15 and 16, where only the size is taken into account. Better estimates in Theorem 1 (and by consequence in Theorem 2 and Theorem 3, whose proofs will follow the same pattern) would require to investigate more carefully this point.

□

5 A reduced strong stationary time

In this section, we indicate the changes in the above arguments needed to prove Theorem 2. It will give us the opportunity to give a broad view of the whole approach by revisiting it.

First note that (Y, Z) is indeed a Markov chain, whose state space is $\mathbb{Z}_M^2$ and whose generic elements will be denoted $[y, z]$. The associated transition matrix P is given by

$$\forall\ [y,z],\ [y',z'] \in \mathbb{Z}_M^2, \qquad P([y,z],[y',z']) \;=\; \begin{cases} 1/6 & \text{, if } [y',z'] \in \{[y\pm 1, z], [y, z\pm y]\} \\ 1/3 & \text{, if } [y',z'] = [y,z] \\ 0 & \text{, otherwise} \end{cases}$$

and the corresponding reversible probability is the uniform distribution on $\mathbb{Z}_M^2$. To construct a corresponding set-valued intertwining dual $\mathfrak{X}$ as in [16], we are to specify a set $\mathfrak{V}$ of non-empty subsets of $\mathbb{Z}_M^2$ and a family of random mappings $(\psi_\Omega)_{\Omega\in\mathfrak{V}}$ compatible with P, namely satisfying the weak association and stability recalled in Section 2.

Every non-empty subset $\Omega \subset \mathbb{Z}_M^2$ can be uniquely determined by a subset $B \subset \mathbb{Z}_M$ and a family $(A(y))_{y\in B}$ of non-empty subsets of $\mathbb{Z}_M$ (that will be referred to as the **fibers** in the sequel) such that

$$[y,z] \in \Omega \quad \Leftrightarrow \quad y \in B \text{ and } z \in A(y)$$

This description is similar in spirit to the decomposition of a probability measure on $\mathbb{Z}_M^2$ into its marginal law on the first coordinate and into the Markov kernel corresponding to the distribution of the second coordinate knowing the first one (in fact it is the same decomposition if uniform distributions are considered). By analogy with the terminology of Section 3, the representation of Ω given by $B \ni y \mapsto A(y)$ is called a **restricted field** with **base** B and a **field** when $B = \mathbb{Z}_M$.

We take for $\mathfrak{V}$ the set of subsets of $\mathbb{Z}_M^2$ whose base is a closed ball centered at $0 \in \mathbb{Z}_M$, denoted $B(r)$ where $r \in [\![0, (M-1)/2]\!]$ is the radius. The elements Ω of $\mathfrak{V}$ are of three types:

- type 0 if the base of Ω is the singleton $\{0\}$,
- type 1 if the base of Ω is different from $\{0\}$ and $\mathbb{Z}_M$, i.e. with a radius $0 < r < (M-1)/2$,
- type 2 if Ω is described by a field $\mathbb{Z}_M \ni y \mapsto A(y)$.

The description of the random mappings ψ_Ω depends on the type of the subset $\Omega \in \mathfrak{V}$.

• When Ω is of type 0, the random mapping ψ_Ω takes the values $\widetilde{\phi}^{(0)}$ with probability 4/5 and the value $\widehat{\phi}^{(0)}$ with probability 1/5, where $\widetilde{\phi}^{(0)}$ is the identity mapping and $\widehat{\phi}^{(0)}$ is defined by

$$\forall\ [y,z] \in \mathbb{Z}_M^2, \qquad \widehat{\phi}^{(0)}([y,z]) \;\coloneqq\; \begin{cases} [0,z] & \text{, if } y \in \{-1,0,1\} \\ [y,z] & \text{, if } y \notin \{-1,0,1\} \end{cases}$$

Let us check the weak association property with $\xi(\Omega) = 5/6$, i.e. we have to show that

$$\forall\ [y,z] \in \mathbb{Z}_M^2,\ [y',z'] \in \Omega, \qquad \mathbb{P}[\psi_\Omega([y,z]) = [y',z']] \ = \ \frac{6}{5} P([y,z],[y',z']) \tag{29}$$

Indeed, since Ω is of type 0, we have $y' = 0$, so that if the l.h.s. does not vanish, we must have $y \in \{0, \pm 1\}$ and $z = z'$. Consider first the case where $y \in \{\pm 1\}$. We have on one hand

$$\begin{aligned} \mathbb{P}[\psi_\Omega([\pm 1, z]) = [0,z]] &= \mathbb{P}[\psi_\Omega = \widehat{\phi}^{(0)}] \\ &= \frac{1}{5} \end{aligned}$$

On the other hand,

$$P([\pm 1, z],[0,z]) \ = \ \frac{1}{6}$$

so that (29) is satisfied. Consider next the case where $y = 0$. We have

$$\begin{aligned} \mathbb{P}[\psi_\Omega([0,z]) = [0,z]] &= \mathbb{P}[\psi_\Omega = \widehat{\phi}^{(0)} \text{ or } \psi_\Omega = \widetilde{\phi}^{(0)}] \\ &= 1 \end{aligned}$$

while

$$P([0,z],[0,z]) \ = \ \frac{1}{3} + \frac{2}{6} \ = \ \frac{5}{6}$$

so (29) equally holds. When the l.h.s. of (29) vanishes, it is immediate to check the r.h.s. is also null.

The stability property is satisfied, since $(\widetilde{\psi}^{(0)})^{-1}(\Omega) = \Omega$ and $(\widehat{\psi}^{(0)})^{-1}(\Omega) = B(1) \times A(0)$, where $A(0)$ is the unique fiber in the restrictive field representation of Ω.

• When Ω is of type 1, the random mapping ψ_Ω takes either one of the values $\widetilde{\phi}^{(1,+)}, \widetilde{\phi}^{(1,-)}, \widehat{\phi}^{(1,+)}, \widehat{\phi}^{(1,-)}$ with probability 1/6 each or the identity mapping $\widetilde{\phi}^{(0)}$ with probability 1/3, where

$$\forall\ [y,z] \in \mathbb{Z}_M^2, \qquad \begin{cases} \widetilde{\phi}^{(1,\pm)}([y,z]) &:= \ [y, z \pm y] \\ \widehat{\phi}^{(1,\pm)}([y,z]) &:= \ [y \pm \operatorname{sgn}(y), z] \end{cases}$$

As in Section 3, the weak association (with $\xi(\Omega) = 1$) and the stability properties are not difficult to check. Note that during this phase, the size of the base is evolving, but the size of the fiber remains equal to 1.

• When Ω is of type 2, the description of the random mapping ψ_Ω is more involved and follows the pattern given in Section 4. More precisely, it corresponds to forgetting the x component there. Denote $A : \mathbb{Z}_M \ni y \mapsto A(y)$ the field defining Ω. We consider the following sign functions

$$\begin{aligned} \forall\ [y,z] \in \mathbb{Z}_M^2, \qquad \widetilde{\varphi}_\Omega(y,z) &:= \begin{cases} -1 & \text{, if } z \in A(y) - y \\ 1 & \text{, if } z \notin A(y) - y \end{cases} \\ \forall\ [y,z] \in \mathbb{Z}_M^2, \qquad \widehat{\varphi}_\Omega(y,z) &:= \begin{cases} -1 & \text{, if } z \in A(y+1) \\ 1 & \text{, if } z \notin A(y+1) \end{cases} \end{aligned}$$

as well as the corresponding mappings acting on $\mathbb{Z}_M^2$

$$\begin{aligned} \forall\ [y,z] \in \mathbb{Z}_M^2, \qquad \widetilde{\phi}_\Omega^{(2,-)}([y,z]) &:= \ [y, z - \widetilde{\varphi}_\Omega(y,z)y] \\ \forall\ [y,z] \in \mathbb{Z}_M^2, \qquad \widetilde{\phi}_\Omega^{(2,+)}([y,z]) &:= \ [y, z + \widetilde{\varphi}_\Omega(y,z)y] \\ \forall\ [y,z] \in \mathbb{Z}_M^2, \qquad \widehat{\phi}_\Omega^{(2,-)}([y,z]) &:= \ [y - \widehat{\varphi}_\Omega(y,z), z] \\ \forall\ [y,z] \in \mathbb{Z}_M^2, \qquad \widehat{\phi}_\Omega^{(2,+)}([y,z]) &:= \ [y + \widehat{\varphi}_\Omega(y,z), z] \end{aligned}$$

The random mapping ψ_Ω takes each of the $\widetilde{\phi}^{(2,-)}, \widetilde{\phi}^{(2,+)}, \widehat{\phi}^{(2,-)}, \widehat{\phi}^{(2,+)}$ with probability $1/6$ and the identity mapping $\widetilde{\phi}^{(0)}$ with the remaining probability $1/3$.

As in Section 4, we check that ψ_Ω is weakly associated to P, with $\xi(\Omega) = 1$. It is also stable. More precisely, consider the fields $\widetilde{A}^{(2,-)}, \widetilde{A}^{(2,+)}, \widehat{A}^{(2,-)}$ and $\widehat{A}^{(2,+)}$ defined by

$$\forall\, y \in \mathbb{Z}_M, \qquad \begin{cases} \widetilde{A}^{(2,-)}(y) & := \Big(A(y) - y\Big) \cup \Big(A(y) + y\Big) \\ \widetilde{A}^{(2,+)}(y) & := \Big(A(y) - y\Big) \cap \Big(A(y) + y\Big) \\ \widehat{A}^{(2,-)}(y) & := A(y+1) \cup A(y-1) \\ \widehat{A}^{(2,+)}(y) & := A(y+1) \cap A(y-1) \end{cases}$$

Then $(\widetilde{\phi}_\Omega^{(2,-)})^{-1}(\Omega)$, $(\widetilde{\phi}_\Omega^{(2,+)})^{-1}(\Omega)$, $(\widehat{\phi}_\Omega^{(2,-)})^{-1}(\Omega)$ and $(\widehat{\phi}_\Omega^{(2,+)})^{-1}(\Omega)$ are respectively the elements of $\mathfrak{V}$ of type 2 associated to the fields $\widetilde{A}^{(2,-)}, \widetilde{A}^{(2,+)}, \widehat{A}^{(2,-)}$ and $\widehat{A}^{(2,+)}$.

From the family of random mappings $(\psi_\Omega)_{\Omega\in\mathfrak{V}}$, construct the set-valued dual $\mathfrak{X} := (\mathfrak{X}(n))_{n\in\mathbb{Z}_+}$, as in [16]. Define two corresponding stopping times

$$\begin{aligned} \sigma &:= \inf\{n \in \mathbb{Z}_+ \,:\, \mathfrak{X}(n) \text{ is of type 2}\} \\ \tau &:= \inf\{n \in \mathbb{Z}_+ \,:\, \mathfrak{X}(n) = \mathbb{Z}_M^2\} \end{aligned}$$

According to the general intertwining theory of Diaconis and Fill [6], the absorption time τ has the same law as a strong stationary time for (Y, Z). It remains to investigate the tail probabilities of τ to prove Theorem 2.

The first step is to evaluate σ. This is easy, since by considering the radius of the base of the restricted field associated to $\mathfrak{X}$ while it is of type 0 or 1, it amounts to quantify the absorption at $(M-1)/2$ of the birth and death chain W on $[\![0, (M-1)/2]\!]$ starting from 0 and whose transition kernel is the matrix Q defined by

$$\forall\, k \neq l \in [\![0, (M-1)/2]\!],$$
$$Q(k,l) \quad := \quad \begin{cases} 1/2 & \text{, if } k = 0 \text{ and } l = 1 \\ (2l+1)/(6(2k+1)) & \text{, if } k \in [\![1, (M-1)/2 - 1]\!] \text{ and } |l-k| = 1 \\ 0 & \text{, otherwise} \end{cases}$$

(see [16] for the general principles and Section 2 for this kind of birth-and-death dual chain, we took into account that when the radius of the base is $k \in [\![1, (M-1)/2]\!]$ the cardinal of the corresponding set is $2k+1$, because all the fibers are of cardinal 1). This absorbing chain is twice slower than the one considered in Section 2.4: if $\widetilde{W}$ is a Markov chain starting from 0 and whose transition kernel is the matrix $\mathfrak{P}$ defined at the beginning of Section 2.4, then W has the same law as

$$\left(\widetilde{W}\left(\sum_{l=0}^{n} B_l\right)\right)_{n\in\mathbb{Z}_+}$$

where $(B_l)_{l\in\mathbb{Z}_+}$ is a sequence of independent Bernoulli random variables of parameter $1/2$ (and independent from $\widetilde{W}$). From the proof of Lemma 9, we deduce

$$\forall\, r \geqslant 0, \qquad \mathbb{P}[\sigma \geqslant r] \quad \leqslant \quad \frac{5}{2}\exp\left(-\frac{r}{10M^2}\right)$$

It remains to estimate $\tau - \sigma$. This random time has the same law as the absorption time ζ at the full field $A_\infty := (\mathbb{Z}_M)_{y\in\mathbb{Z}_M}$ of the field-valued Markov chain $(A_n)_{n\in\mathbb{Z}_+}$ whose transition kernel $\mathfrak{Q}$ is given, for any fields A, A' by

$$\mathfrak{Q}(A, A') \quad := \quad \frac{|A'|}{|A|}\left(\frac{1}{6}\mathbb{1}_{\widetilde{A}^{(2,-)}}(A') + \frac{1}{6}\mathbb{1}_{\widetilde{A}^{(2,+)}}(A') + \frac{1}{6}\mathbb{1}_{\widehat{A}^{(2,-)}}(A') + \frac{1}{6}\mathbb{1}_{\widehat{A}^{(2,+)}}(A') + \frac{1}{3}\mathbb{1}_{A}(A')\right)$$

where for any field A, the **thickness** of A is defined by

$$|A| \;\coloneqq\; \prod_{y\in\mathbb{Z}_M} |A(y)|$$

Our first task is to check that $\mathfrak{Q}$ leads to the absorption at A_∞ from any starting field. In this direction the proof of Lemma 14 is still valid and even simpler: it is sufficient to remove the first component x in the fields. It follows that for any field A,

$$\left(|\widetilde{A}^{(2,-)}| \;=\; |A| \;=\; |\widehat{A}^{(2,-)}|\right) \;\Rightarrow\; A \;=\; A_\infty$$

As a consequence, Lemmas 15 and 16 provide exactly the same estimates:
When $A \neq A_\infty$, then either

$$\frac{|\widetilde{A}^{(2,-)}|}{|A|} \;\geqslant\; 1+\frac{1}{M} \quad \text{or} \quad \frac{|\widehat{A}^{(2,-)}|}{|A|} \;\geqslant\; 1+\frac{1}{M}$$

and

$$\forall\; n<\zeta, \qquad \mathbb{E}[R_{n+1}|\mathcal{A}_n] \;\geqslant\; R_n+\frac{1}{37M^2}$$

where

$$\forall\; n\in\mathbb{Z}_+, \qquad R_n \;\coloneqq\; \ln(|A_n|)$$

and where the filtration $(\mathcal{A}_n)_{n\in\mathbb{Z}_+}$ is generated by $(A_n)_{n\in\mathbb{Z}_+}$.

The main difference comes with Proposition 17: since $|A_\infty| = M^M$ instead of M^{M^2}, we get

$$\mathbb{E}[\zeta] \;\leqslant\; 37M^3\ln(M)$$

The proof of Theorem 1 can then be transposed to show Theorem 2.

Remark 19 Coming back to the whole finite Heisenberg Markov chain $[X,Y,Z]$, we could think that after the strong stationary time τ defined above, $[X,Y,Z]$ will reach equilibrium after a new strong stationary time of order M^2. This is not clear from our approach, since at time τ we don't know how X and (Y,Z) are linked. To get a result in this direction in the spirit of this paper, we should try to introduce fields describing a subset of values of $x\in\mathbb{Z}_M$, for any given $(y,z)\in\mathbb{Z}_M^2$, and see if we would be able to deduce a corresponding strong stationary time with a better order than $M^4\ln(M)$.

□

6 Extension to higher dimensional Heisenberg walks

Here we explain how the constructions of the two previous sections can be extended to deal with higher dimensional Heisenberg walks.

Again we apply the random mapping method described in [16] and recalled in Sections 2 and 3. So our main ingredients are a set $\mathfrak{V}$ of non-empty subsets of $\mathbb{H}_{N,M}$, and for any $\Omega\in\mathfrak{V}$, a random mapping $\psi_\Omega : \mathbb{H}_{N,M} \to \mathbb{H}_{N,M}$.

To present them, recall that in the introduction we associated to any $[x] \coloneqq [x_{k,l}]_{1\leqslant k<l\leqslant N} \in \mathbb{H}_{N,M}$ and to any $l\in[\![N-1]\!]$, the l^{th} upper diagonal $d_l[x] \coloneqq (x_{k,k+l})_{k\in[\![N-l]\!]}$. Denote $\mathbb{D}_l$ the set of such elements, i.e.

$$\mathbb{D}_l \;\coloneqq\; \mathbb{Z}_M^{\{(k,k+l)\,:\,k\in[\![N-l]\!]\}}$$

Let d_0 be the usual diagonal consisting only of 1, and set $\mathbb{D}_0 = \{d_0\}$. We also write

$$d_{[\![0,l]\!]}[x] \;\; := \;\; (d_k[x])_{k\in[\![0,l]\!]} \;\in\; \mathbb{D}_{[\![0,l]\!]} \;:=\; \prod_{k\in[\![0,l]\!]} \mathbb{D}_k$$

A set $\Omega \in \mathfrak{V}$ is described by a family of numbers $(r_k)_{k\in[\![N-1]\!]}$ belonging to $[\![0,(M-1)/2]\!]$ and a family of characteristic subsets $(A_l(d_{[\![0,l-1]\!]}))_{l\in[\![2,N-1]\!],\, d_{[\![0,l-1]\!]}\in\mathbb{D}_{[\![0,l-1]\!]}}$, in the following way: an element $[x]$ belongs to Ω if and only if

$$\begin{cases} & d_1[x] \;\in\; \prod_{k\in[\![N-1]\!]} B(r_k) \\ \forall\, l \in [\![2,N-1]\!], & d_l[x] \;\in\; A_l(d_{[\![0,l-1]\!]}[x]) \end{cases} \tag{30}$$

(recall that $B(r)$ is the closed ball of $\mathbb{Z}_M$ centered at 0 and of radius $r \geqslant 0$). For each $l \in [\![2,N-1]\!]$ and $d_{[\![0,l-1]\!]} \in \mathbb{D}_{[\![0,l-1]\!]}$, $A_l(d_{[\![0,l-1]\!]})$ is a subset of $\mathbb{D}_l$. To simplify the already heavy notations, from now on, it will be denoted $A(d_{[\![0,l-1]\!]})$, since the index $l \in [\![N-1]\!]$ can be extracted from the argument $d_{[\![0,l-1]\!]} \in \mathbb{D}_{[\![0,l-1]\!]}$.

We say that $\Omega \in \mathfrak{V}$ is of **type** 1, if there exists $k \in [\![N-1]\!]$ such that $r_k \neq (M-1)/2$. It is of **type** $b \in [\![2,N-1]\!]$, if:

- it is not of type 1,
- for any $l \in [\![2,b-1]\!]$ and any $d_{[\![0,l-1]\!]} \in \mathbb{D}_{[\![0,l-1]\!]}$, the subset $A(d_{[\![0,l-1]\!]})$ is the full set $\mathbb{D}_l$,
- there exists $d_{[\![0,b-1]\!]} \in \mathbb{D}_{[\![0,b-1]\!]}$ such that $A(d_{[\![0,b-1]\!]}) \neq \mathbb{D}_b$.

As a natural extension, the set $\Omega \in \mathfrak{V}$ is said to be of **type** N if it is equal to $\mathbb{H}_{N,M}$, which is the set such that all the radiuses are all equal to $(M-1)/2$ and whose characteristic subsets are all equal to $\mathbb{Z}_M$.

The idea behind the following construction, is to first wait for the equilibrium to be reached on $\mathbb{D}_1 = \mathbb{D}_{[\![1]\!]}$, next on $\mathbb{D}_{[\![2]\!]}$, then on $\mathbb{D}_{[\![3]\!]}$, etc., each time adding the next upper diagonal, up to $\mathbb{D}_{[\![N-1]\!]}$, which corresponds to the full $\mathbb{H}_{N,M}$.

The types will be useful to describe the random mappings ψ_Ω, for $\Omega \in \mathfrak{V}$. For this purpose, another notation is required. For $I \in [\![N-1]\!]$ and $\epsilon \in \{\pm 1\}$, let $F_{I,\epsilon}$ be the mapping acting on $\mathbb{H}_{N,M}$ by adding (respectively subtracting) the $(I+1)^{\text{th}}$ row to the I^{th} row, if $\epsilon = 1$ (resp. $\epsilon = -1$). We will also see $F_{I,\epsilon}$ as a mapping acting on the $\mathbb{D}_{[\![0,l]\!]}$, for $l \in [\![N-1]\!]$ (and this is the only reason for the addition of the diagonal d_0 in $d_{[\![0,l]\!]}$).

For fixed $\Omega \in \mathfrak{V}$, ψ_Ω is defined as follows.

• We start with the situation where Ω is not of type 1, so let $b \in [\![2,N-1]\!]$ be its type. The characteristic subsets of Ω are denoted $(A(d_{[\![0,l-1]\!]}))_{l\in[\![2,N-1]\!],\, d_{[\![0,l-1]\!]}\in\mathbb{D}_{[\![0,l-1]\!]}}$. As a consequence of Ω not being of type 1, its radiuses r_k, for $k \in [\![N-1]\!]$, are all equal to $(M-1)/2$.

Aside from the identity, the values of ψ_Ω are the $\phi_{\Omega,I,\epsilon}$, for $I \in [\![N-1]\!]$ and $\epsilon \in \{\pm 1\}$, where

$$\forall\, [x] \in \mathbb{H}_{N,M}, \qquad \phi_{\Omega,I,\epsilon}([x]) \;\; := \;\; F_{I,\epsilon\varphi(I,d_{[\![0,b]\!]}[x])}([x])$$

where $\varphi(I, d_{[\![0,b]\!]}[x]) \in \{-1,1\}$ will be defined below. Note that the dependence on Ω only goes through its type b.

Each $\phi_{\Omega,I,\epsilon}$ will be chosen with probability $1/(3(N-1))$ and the identity with the remaining probability $1/3$. It remains to define the quantity $\varphi(I, d_{[\![0,b]\!]}[x])$. The index $I \in [\![N-1]\!]$ is equally assumed to be fixed now.

Let be given a family $(B_{I,d_{[\![0,b-1]\!]}})_{d_{[\![0,b-1]\!]}\in\mathbb{D}_{[\![0,d-1]\!]}}$ of subsets from $\mathbb{D}_b$, that will be specified later on. Consider an element $d_{[\![0,b]\!]} \in \mathbb{D}_{[\![0,b]\!]}$, that can be naturally decomposed into $d_{[\![0,b-1]\!]} \in \mathbb{D}_{[\![0,d-1]\!]}$ and $d_b \in \mathbb{D}_b$. The quantity $\varphi(I, d_{[\![0,b]\!]})$ has the form:

$$\varphi(I, d_{[\![0,b]\!]}) \;\; := \;\; \begin{cases} 1 & \text{, if } d_b \in B_{I,d_{[\![0,b-1]\!]}} \\ -1 & \text{, otherwise} \end{cases}$$

• When the type of Ω is 1, the construction of the random mapping ψ_Ω is rather extending that of Section 3. As above, we consider the mappings $\phi_{\Omega,I,\epsilon}$, for $I \in [\![N-1]\!]$ and $\epsilon \in \{\pm 1\}$, where now $\varphi(I, d_{[\![0,b]\!]}[x])$ is replaced by $\mathrm{sgn}(x_{I,I+1})$.

Due to the particularity of the case where one of the r_I vanishes for $I \in [\![N-1]\!]$, we need to introduce the mappings $\phi_{\Omega,I,0}$:

$$\forall\ [x] \in \mathbb{H}_{N,M}, \qquad \phi_{\Omega,I,0}([x]) \quad = \quad \begin{cases} \phi_{\Omega,I,-1}([x]) & \text{, if } x_{I,I+1} \in \{-1,1\} \\ [x] & \text{, otherwise} \end{cases}$$

(with the notations of Section 3, when $N = 3$, $\phi_{\Omega,1,0}$ corresponds to $\widetilde{\phi}^{(0)}$ and $\phi_{\Omega,2,0}$ to $\widehat{\phi}^{(0)}$).

The description of the underlying probabilities are as follows. First we sample $I \in [\![N-1]\!]$ uniformly. Next,

- if $r_I \neq 0$, then ψ_Ω is equal to $\phi_{\Omega,I,1}$, $\phi_{\Omega,I,-1}$ or the identity, each with probability 1/3.
- if $r_I = 0$, then $\psi_\Omega = \phi_{\Omega,I,0}$ (for the variable $x_{I,I+1} \in \mathbb{Z}_M$, this corresponds to the situation described in Section 2.1, where $a = 1/3$ and $p = 1$).

Our next task is to check the stability of $\mathfrak{V}$ by the mappings $\phi_{\Omega,I,\epsilon}$, for $\Omega \in \mathfrak{V}$, $I \in [\![N-1]\!]$ and $\epsilon \in \{-1, 0, 1\}$. Here is a first case:

Lemma 20 *When the type of Ω is $b \in [\![2, N-1]\!]$, for $I \in [\![N-1]\!]$, $\Omega' := \phi^{-1}_{\Omega,I,1}(\Omega)$ belongs to $\mathfrak{V}$, with a type $b' \in [\![b, N-1]\!]$ and with characteristic subsets given by*

$$\forall\ l \in [\![2, N-1]\!],\ \forall\ d_{[\![0,l-1]\!]} \in \mathbb{D}_{[\![0,l-1]\!]},$$

$$A'(d_{[\![0,l-1]\!]}) \quad = \quad \begin{cases} \mathbb{D}_l & \text{, if } l < b \\ \Big([A(F_{I,1}(d_{[\![0,b-1]\!]})) - \theta_I[d_{b-1}]] \cap B_{I,d_{[\![0,b-1]\!]}}\Big) \cup & \\ \qquad \Big([A(F_{I,-1}(d_{[\![0,b-1]\!]})) + \theta_I[d_{b-1}]] \cap B^{\mathrm{c}}_{I,d_{[\![0,b-1]\!]}}\Big) & \text{, if } l = b \\ A(F_{I,\varphi(I,d_{[\![0,b]\!]})}(d_{[\![0,l-1]\!]})) - \varphi(I, d_{[\![0,b]\!]})\theta_I[d_{l-1}] & \text{, if } l > b \end{cases}$$

where $\theta_I[d_{l-1}]$ is the element of $\mathbb{D}_l$ whose coordinates vanish, except the I^{th} one, which is equal to the $(I+1)^{\text{th}}$ coordinate of d_{l-1} (with the convention that $\theta_I[d_{l-1}] = 0$ if this coordinate does not exist, i.e. $I + 1 > N - l + 1$).

Proof

An element $[x'] \in \mathbb{H}_{N,M}$ belongs to Ω' if and only if there exists $[x] \in \Omega$ such that $\phi_{\Omega,I}([x']) = [x]$. Namely, $[x]$ being defined by

$$\forall\ 1 \leqslant k < l \leqslant N, \qquad x_{k,l} \quad = \quad \begin{cases} x'_{I,l} + \varphi(I, d_{[\![0,b]\!]}[x'])x'_{I+1,l} & \text{, if } k = I \\ x'_{k,l} & \text{, otherwise} \end{cases} \tag{31}$$

it must satisfy (30). In particular, there is no restriction on the $x_{k,k+1}$, for $k \in [\![N-1]\!]$, and by consequence the same is true for the $x'_{k,k+1}$, leading to the fact that all the r'_k are also equal to $(M-1)/2$. It already shows that if $\Omega' \in \mathfrak{V}$, its type is at least 2.

Note that (31) can written in terms of the upper diagonals:

$$\forall\ l \in [\![N-1]\!], \qquad d_l[x] \quad = \quad d_l[x'] + \varphi(I, d_{[\![0,b]\!]}[x'])\theta_{I,l}[d_{l-1}[x']] \tag{32}$$

Consider an index $l \in [\![2, N-1]\!]$, we distinguish several cases.

• When $l < b$, we have $A(d_{[\![0,l-1]\!]}) = \mathbb{D}_l$ for all $d_{[\![0,l-1]\!]} \in \mathbb{D}_{[\![0,l-1]\!]}$. Thus Ω induces no restriction on $d_l[x]$ nor on $d_l[x']$. We can take $A'(d_{[\![0,l-1]\!]}) = \mathbb{D}_l$ for all $d_{[\![0,l-1]\!]} \in \mathbb{D}_{[\![0,l-1]\!]}$. This proves that if $\Omega' \in \mathfrak{V}$, as it will be shown below, then its type is at least equal to b.

- When $l = b$, we consider two subcases.
  - If $d_b[x'] \in B_{I,d_{[\![0,b-1]\!]}[x']}$, then (32) implies

$$d_b[x] \quad = \quad d_b[x'] + \theta_{I,b}[d_{b-1}[x']]$$

Taking into account that

$$d_{[\![0,b-1]\!]}[x] \quad = \quad F_{I,\varphi(I,d_{[\![0,b]\!]}[x'])}(d_{[\![0,b-1]\!]}[x']) \quad = \quad F_{I,1}(d_{[\![0,b-1]\!]}[x'])$$

the condition $d_b[x] \in A(d_{[\![0,b-1]\!]}[x])$ translates into

$$d_b[x'] \quad \in \quad A(F_{I,1}(d_{[\![0,b-1]\!]}[x'])) - \theta_{I,b}[d_{b-1}[x']]$$

and we get

$$d_b[x'] \quad \in \quad \big(A(F_{I,1}(d_{[\![0,b-1]\!]}[x']) - \theta_{I,b}[d_{b-1}[x']]\big) \cap B_{I,d_{[\![0,b-1]\!]}[x']}$$

Conversely, this inclusion implies $d_b[x] \in A(d_{[\![0,b-1]\!]}[x])$, since the above arguments can be reversed.

  - If $d_b[x'] \notin B_{I,d_{[\![0,b-1]\!]}[x']}$, then similar considerations lead to the equivalence of $d_b[x] \in A(d_{[\![0,b-1]\!]}[x])$ with

$$d_b[x'] \quad \in \quad \big(A(F_{I,-1}(d_{[\![0,b-1]\!]}[x'])) + \theta_{I,b}[d_{b-1}[x']]\big) \cap B^{\mathrm{c}}_{I,d_{[\![0,b-1]\!]}[x']}$$

It follows that when $l = b$, we can take $A'(d_{[\![0,b-1]\!]}[x'])$ equal to

$$\Big([A(F_{I,1}(d_{[\![0,b-1]\!]}[x'])) - \theta_I[d_{b-1}[x']]] \cap B_{I,d_{[\![0,b-1]\!]}[x']}\Big) \cup \\ \Big([A(F_{I,-1}(d_{[\![0,b-1]\!]}[x'])) + \theta_I[d_{b-1}[x']]] \cap B^{\mathrm{c}}_{I,d_{[\![0,b-1]\!]}[x']}\Big)$$

- When $l > b$, from (32) and

$$d_{[\![0,l-1]\!]}[x] \quad = \quad F_{I,\varphi(I,d_{[\![0,b]\!]}[x'])}(d_{[\![0,l-1]\!]}[x'])$$

the condition $d_b[x] \in A(d_{[\![0,b-1]\!]}[x])$ translates into

$$d_l[x'] \quad \in \quad A(F_{I,\varphi(I,d_{[\![0,b]\!]}[x'])}(d_{[\![0,l-1]\!]}[x'])) - \varphi(I, d_{[\![0,b]\!]}[x'])\theta_I[d_{l-1}[x']]$$

whose r.h.s. is a function of $d_{[\![0,l-1]\!]}[x']$, since $l > b$. It follows that $A'(d_{[\![0,l-1]\!]}[x'])$ can be defined as the above r.h.s.

■

Similar arguments, or replacing the sets $B_{I,d_{[\![0,b-1]\!]}}$ by their complementary sets, leads to

Lemma 21 *When the type of Ω is $b \in [\![2, N-1]\!]$, for $I \in [\![N-1]\!]$, $\Omega' := \phi^{-1}_{\Omega,I,-1}(\Omega)$ belongs to $\mathfrak{V}$, with a type $b' \in [\![b, N-1]\!]$ and with characteristic subsets given by*

$$\forall\ l \in [\![2, N-1]\!],\ \forall\ d_{[\![0,l-1]\!]} \in \mathbb{D}_{[\![0,l-1]\!]},$$

$$A'(d_{[\![0,l-1]\!]}) \quad = \quad \begin{cases} \mathbb{D}_l & ,\ \text{if } l < b \\ \Big([A(F_{I,-1}(d_{[\![0,b-1]\!]})) + \theta_I[d_{b-1}]] \cap B_{I,d_{[\![0,b-1]\!]}}\Big) \cup \\ \qquad \Big([A(F_{I,1}(d_{[\![0,b-1]\!]})) - \theta_I[d_{b-1}]] \cap B^{\mathrm{c}}_{I,d_{[\![0,b-1]\!]}}\Big) & ,\ \text{if } l = b \\ A(F_{I,-\varphi(I,d_{[\![0,b]\!]})}(d_{[\![0,l-1]\!]})) + \varphi(I, d_{[\![0,b]\!]})\theta_I[d_{l-1}] & ,\ \text{if } l > b \end{cases}$$

Due to the guideline recalled in Remark 11, when the type of Ω is $b \in [\![2, N-1]\!]$, we are lead to choose

$$\forall\, I \in [\![N-1]\!],\ \forall\, d_{[\![0,b-1]\!]} \in \mathbb{D}_{[\![0,b-1]\!]}, \qquad B_{I,d_{[\![0,b-1]\!]}} \;=\; A(F_{I,1}(d_{[\![0,b-1]\!]})) - \theta_I[d_{b-1}] \tag{33}$$

It follows from Lemma 20, that when Ω is of type $b \in [\![2, N-1]\!]$, the characteristic subsets of $\Omega' \coloneqq \phi^{-1}_{\Omega,I,1}(\Omega)$ are given by

$$\begin{aligned}
&\forall\, l \in [\![2, N-1]\!],\ \forall\, d_{[\![0,l-1]\!]} \in \mathbb{D}_{[\![0,l-1]\!]},\\
A'(d_{[\![0,l-1]\!]}) \;&=\; \begin{cases} \mathbb{D}_l & \text{, if } l < b\\ \big(A(F_{I,1}(d_{[\![0,b-1]\!]})) - \theta_I[d_{b-1}]\big) \cup \big(A(F_{I,-1}(d_{[\![0,b-1]\!]})) + \theta_I[d_{b-1}]\big) & \text{, if } l = b\\ A(F_{I,\varphi(I,d_{[\![0,b]\!]})}(d_{[\![0,l-1]\!]})) - \varphi(I, d_{[\![0,b]\!]})\theta_I[d_{l-1}] & \text{, if } l > b \end{cases}
\end{aligned}$$

From Lemma 21, we deduce that when Ω is of type $b \in [\![2, N-1]\!]$, the characteristic subsets of $\Omega' \coloneqq \phi^{-1}_{\Omega,I,-1}(\Omega)$ are given by

$$\begin{aligned}
&\forall\, l \in [\![2, N-1]\!],\ \forall\, d_{[\![0,l-1]\!]} \in \mathbb{D}_{[\![0,l-1]\!]},\\
A'(d_{[\![0,l-1]\!]}) \;&=\; \begin{cases} \mathbb{D}_l & \text{, if } l < b\\ \big(A(F_{I,1}(d_{[\![0,b-1]\!]})) - \theta_I[d_{b-1}]\big) \cap \big(A(F_{I,-1}(d_{[\![0,b-1]\!]})) + \theta_I[d_{b-1}]\big) & \text{, if } l = b\\ A(F_{I,\varphi(I,d_{[\![0,b]\!]})}(d_{[\![0,l-1]\!]})) - \varphi(I, d_{[\![0,b]\!]})\theta_I[d_{l-1}] & \text{, if } l > b \end{cases}
\end{aligned}$$

Remark 22 Note that if the $A(d_{[\![0,l-1]\!]})$ are all singletons for $l > b$, then the same will be true for the $A'(d_{[\![0,l-1]\!]})$. Since we will start our dual process from the singleton $\{[0]\}$, it follows that when the dual process takes as values subsets of type b, then all the characteristic subsets corresponding to the upper diagonals strictly above the b^{th} upper diagonal are singletons. Only the characteristic subsets of the b^{th} upper diagonal will have a certain tendency to increase, as in Section 4.

□

When the type of $\Omega \in \mathfrak{V}$ is 1, similar results hold for $\phi^{-1}_{\Omega,I,-1}(\Omega)$, $\phi^{-1}_{\Omega,I,0}(\Omega)$ and $\phi^{-1}_{\Omega,I,1}(\Omega)$, in particular these subsets belong to $\mathfrak{V}$. We let the reader writes them down as an exercise. More interestingly, note that while the set-valued dual process stays of type 1, the components of d_1 behave as the random walks described in Section 2.4 (see also Section 3), except that they have to share time.

More precisely, let $\mathfrak{X} \coloneqq (\mathfrak{X}(n))_{n\in\mathbb{Z}_+}$ be a set-valued dual associated to the random mappings $(\psi_\Omega)_{\Omega\in\mathfrak{V}}$ and starting from $\{[0]\}$ (when $[X](0) = [0]$). Define for any $b \in [\![N]\!]$, $\mathfrak{V}_b$ the subset of $\mathfrak{V}$ consisting of elements whose type is larger than or equal to b.

Corollary 23 *For any $b \in [\![N]\!]$, $\mathfrak{V}_b$ is absorbing for $\mathfrak{X}$.*

Proof

The cases $b \geqslant 2$ is a consequence of Lemma 20 and 21. For $b = 1$, $\mathfrak{V}_1 = \mathfrak{V}$, so the result comes from the exercise left to the reader.

■

For any $b \in [\![N]\!]$, define

$$\mathfrak{t}_b \;=\; \inf\{n \in \mathbb{Z}_+ \;:\; \mathfrak{X}(n) \in \mathfrak{V}_b\}$$

Note that $\mathfrak{t}_1 = 0$ and $\mathfrak{t}_N$ is the absorption time of $\mathfrak{X}$ at $\mathfrak{V}_N = \{\mathbb{H}_{N,M}\}$. In particular, $\mathfrak{t}_N$ has the same law as a strong stationary time for $[X]$, it is the τ of Theorem 3. The proof of Theorem 3 is based on the analysis of the differences $\mathfrak{t}_{b+1} - \mathfrak{t}_b$ conditioned by the past of $\mathfrak{X}$ up to time $\mathfrak{t}_b$, for $b \in [\![1, N-1]\!]$. For $b = 1$, this conditioning is void, since we have $[X](0) = [0]$ and $\mathfrak{X}_0 = \{[0]\}$.

Lemma 24 *For M large enough, we have*

$$\forall\ N \geqslant 3,\ \forall\ r \geqslant 0, \qquad \mathbb{P}[\mathfrak{t}_2 \geqslant r] \quad\leqslant\quad 5\frac{N-1}{2}\exp\left(-\frac{r}{5(N-1)M^2}\right)$$

(the factor 5/2 is here just to recover Lemma 9 when $N = 3$).

Proof

The arguments are very close to those of Lemma 9. For $k \in [\![N-1]\!]$, let $\widetilde{X}_k \coloneqq (\widetilde{X}_k(n))_{n\in\mathbb{Z}_+}$ be $N-1$ independent random walks on $\mathbb{Z}_M$ as in Section 2.4. Let $(B_n)_{n\in\mathbb{Z}_+}$ be a family of independent variables uniformly distributed on $[\![N-1]\!]$ (and independent from the $\widetilde{X}_k$, for $k \in [\![N-1]\!]$) and define

$$\forall\ k \in [\![N-1]\!],\ \forall\ n \in \mathbb{Z}_+, \qquad \theta_k(n) \quad\coloneqq\quad \sum_{m\in[\![n]\!]} \mathbb{1}_{\{B_m=k\}}$$

The chain $[X_{k,k+1}(n)]_{k\in[\![N-1]\!],\, n\in\mathbb{Z}_+} = (d_1[X](n))_{n\in\mathbb{Z}_+}$ has the same law as $(\widetilde{X}_k(\theta_k(n)))_{k\in[\![N-1]\!],\, n\in\mathbb{Z}_+}$ and from the above construction, it appears that

$$\mathfrak{t}_2 \quad=\quad \inf\{n \in \mathbb{Z}_+ \,:\, \forall\ k \in [\![N-1]\!],\, \theta_k(n) \geqslant \widetilde{\mathfrak{t}}_k\}$$

where for any $k \in [\![N-1]\!]$, $\widetilde{\mathfrak{t}}_k$ is the strong stationary time constructed as in Section 2.4 for $\widetilde{X}_k$.

It follows that for any $n \in \mathbb{Z}_+$,

$$\begin{aligned}\mathbb{P}[\mathfrak{t}_2 \geqslant n] &\leqslant \sum_{k\in[\![N-1]\!]} \mathbb{P}[\theta_k(n) \leqslant \widetilde{\mathfrak{t}}_k]\\ &= (N-1)\mathbb{P}[\theta_1(n) \leqslant \widetilde{\mathfrak{t}}_1]\end{aligned}$$

since the $(\theta_k(n), \widetilde{\mathfrak{t}}_k)$ have the same law for all $k \in [\![N-1]\!]$.

According to Corollary 8, we have for the conditional expectation knowing $\theta_k(n)$ and for large M:

$$\mathbb{P}[\theta_1(n) \leqslant \widetilde{\mathfrak{t}}_1 | \theta_1(n)] \quad\leqslant\quad 2\exp(-\theta_1(n)/(4M^2))$$

so that

$$\begin{aligned}\mathbb{P}[\theta_1(n) \leqslant \widetilde{\mathfrak{t}}_1] &\leqslant 2\mathbb{E}[\exp(-\theta_1(n)/(4M^2))]\\ &= 2\mathbb{E}[\exp(-\mathbb{1}_{\{B_1=1\}}/M^2)]^n\\ &= 2\left(\frac{N-2+\exp(-1/(4M^2))}{N-1}\right)^n\\ &= 2\left(1+\frac{\exp(-1/(4M^2))-1}{N-1}\right)^n\\ &\leqslant 2\exp\left(-\frac{n}{5(N-1)M^2}\right)\end{aligned}$$

for M large enough, uniformly in $n \in \mathbb{Z}_+$ and in $N \in \mathbb{N}$, $N \geqslant 3$.

As a consequence, for any $r \geqslant 0$, we have

$$\begin{aligned}\mathbb{P}[\mathfrak{t}_2 \geqslant r] &\leqslant \mathbb{P}[\mathfrak{t}_2 \geqslant \lfloor r \rfloor]\\ &\leqslant 2(N-1)\exp\left(-\frac{\lfloor r\rfloor}{5(N-1)M^2}\right)\\ &\leqslant 2(N-1)\exp\left(-\frac{r-1}{5(N-1)M^2}\right)\\ &\leqslant 2(N-1)\exp\left(-\frac{1}{5(N-1)M^2}\right)\exp\left(-\frac{r}{5(N-1)M^2}\right)\\ &\leqslant 5\frac{N-1}{2}\exp\left(-\frac{r}{5(N-1)M^2}\right)\end{aligned}$$

since we have $2\exp(-1/(5\times 2\times 3^2)) \simeq 2.02234613753 < 5/2$.

■

The above estimate implies the more telling bound, for M large enough and uniformly in $N\in\mathbb{N}$, $N\geqslant 3$,

$$\forall\ r\geqslant 0,\qquad \mathbb{P}[\mathfrak{t}_2\geqslant N\ln(N)M^2+rNM^2]\ \leqslant\ 5\exp(-r/5)$$

but for our purposes the formulation of Lemma 24 will be more convenient.

The important step is the following result, whose proof will follow the arguments of Section 4.

Proposition 25 *For $b\in[\![2,N-1]\!]$ and M large enough (uniformly in b and N), we have*

$$\forall\ r\geqslant 0,\qquad \mathbb{P}[\mathfrak{t}_{b+1}-\mathfrak{t}_b\geqslant r|\mathfrak{X}_{[\![0,\mathfrak{t}_b]\!]}]\ \leqslant\ 3\exp\left(-\frac{2r}{101(N-1)^2M^{N(N-1)/2+1}\ln(M)}\right)$$

As in Section 4, this result is to be proven by getting an estimate on the tendency of $\mathfrak{X}(n)$ to grow, while $\mathfrak{X}(n)$ says in $\mathfrak{V}_b$. With this respect, introduce the quantity

$$\forall\ \Omega\in\mathfrak{V}_b,\qquad R[\Omega]\ \coloneqq\ \sum_{d_{[\![0,b-1]\!]}\in\mathbb{D}_{[\![0,b-1]\!]}}\ln(|A(d_{[\![0,b-1]\!]})|)\tag{34}$$

where $b\in[\![2,N-1]\!]$ is fixed.

To any family $A\coloneqq(A(d_{[\![0,b-1]\!]}))_{d_{[\![0,b-1]\!]}\in\mathbb{D}_{[\![0,b-1]\!]}}$ of subsets of $\mathbb{D}_b$ and to any $I\in[\![N-1]\!]$, associate the new families $A^{\cup,I}\coloneqq(A^{\cup,I}(d_{[\![0,b-1]\!]}))_{d_{[\![0,b-1]\!]}\in\mathbb{D}_{[\![0,b-1]\!]}}$ and $A^{\cap,I}\coloneqq(A^{\cap,I}(d_{[\![0,b-1]\!]}))_{d_{[\![0,b-1]\!]}\in\mathbb{D}_{[\![0,b-1]\!]}}$ defined by taking for any $d_{[\![0,b-1]\!]}\in\mathbb{D}_{[\![0,b-1]\!]}$,

$$\begin{aligned}A^{\cup,I}(d_{[\![0,b-1]\!]})\ &\coloneqq\ \big(A(F_{I,1}(d_{[\![0,b-1]\!]}))-\theta_I[d_{b-1}]\big)\cup\big(A(F_{I,-1}(d_{[\![0,b-1]\!]}))+\theta_I[d_{b-1}]\big)\\ A^{\cap,I}(d_{[\![0,b-1]\!]})\ &\coloneqq\ \big(A(F_{I,1}(d_{[\![0,b-1]\!]}))-\theta_I[d_{b-1}]\big)\cap\big(A(F_{I,-1}(d_{[\![0,b-1]\!]}))+\theta_I[d_{b-1}]\big)\end{aligned}$$

Note that the definition (34) can be extended to a family such as A, let us denote $R[A]$ the corresponding quantity. The following result is the generalization of Lemma 14:

Lemma 26 *We have*

$$\forall\ I\in[\![N-1]\!],\qquad R[A^{\cup,I}]\ \geqslant\ R[A]$$

and if for all $I\in[\![N-1]\!]$, $R[A^{\cup,I}]=R[A]$, then $A=A_{b,\infty}$, where $A_{b,\infty}\coloneqq(A_{b,\infty}(d_{[\![0,b-1]\!]}))_{d_{[\![0,b-1]\!]}\in\mathbb{D}_{[\![0,b-1]\!]}}$ is defined via

$$\forall\ d_{[\![0,b-1]\!]}\in\mathbb{D}_{[\![0,b-1]\!]},\qquad A_{b,\infty}(d_{[\![0,b-1]\!]})\ \coloneqq\ \mathbb{D}_b$$

Proof

Concerning the first point, for any $d_{[\![0,b-1]\!]}\in\mathbb{D}_{[\![0,b-1]\!]}$ and $I\in[\![N-1]\!]$, we have

$$\begin{aligned}|A^{\cup,I}(d_{[\![0,b-1]\!]})|\ &=\ |\big(A(F_{I,1}(d_{[\![0,b-1]\!]}))-\theta_I[d_{b-1}]\big)\cup\big(A(F_{I,-1}(d_{[\![0,b-1]\!]}))+\theta_I[d_{b-1}]\big)|\\ &\geqslant\ |A(F_{I,1}(d_{[\![0,b-1]\!]}))-\theta_I[d_{b-1}]|\\ &=\ |A(F_{I,1}(d_{[\![0,b-1]\!]}))|\end{aligned}\tag{35}$$

so that

$$\begin{aligned}R[A^{\cup,I}]\ &=\ \sum_{d_{[\![0,b-1]\!]}\in\mathbb{D}_{[\![0,b-1]\!]}}\ln(|A^{\cup,I}(d_{[\![0,b-1]\!]})|)\\ &\geqslant\ \sum_{d_{[\![0,b-1]\!]}\in\mathbb{D}_{[\![0,b-1]\!]}}\ln(|A(F_{I,1}(d_{[\![0,b-1]\!]}))|)\\ &=\ \sum_{d_{[\![0,b-1]\!]}\in\mathbb{D}_{[\![0,b-1]\!]}}\ln(|A(d_{[\![0,b-1]\!]})|)\\ &=\ R[A]\end{aligned}$$

where we used that the mapping $F_{I,1}$ is a bijection in $\mathbb{D}_{[\![0,b-1]\!]}$, with inverse mapping given by $F_{I,-1}$.

Assume next that the family A is such that $R[A^{\cup,I}] = R[A]$, for any $I \in [\![N-1]\!]$. According to the above computation, we must have for any $d_{[\![0,b-1]\!]} \in \mathbb{D}_{[\![0,b-1]\!]}$,

$$|\left(A(F_{I,1}(d_{[\![0,b-1]\!]})) - \theta_I[d_{b-1}]\right) \cup \left(A(F_{I,-1}(d_{[\![0,b-1]\!]})) + \theta_I[d_{b-1}]\right)| = |A(F_{I,1}(d_{[\![0,b-1]\!]})) - \theta_I[d_{b-1}]|$$

namely

$$\left(A(F_{I,1}(d_{[\![0,b-1]\!]})) - \theta_I[d_{b-1}]\right) \cup \left(A(F_{I,-1}(d_{[\![0,b-1]\!]})) + \theta_I[d_{b-1}]\right) = A(F_{I,1}(d_{[\![0,b-1]\!]})) - \theta_I[d_{b-1}]$$

Similarly, replacing (35) by

$$|\left(A(F_{I,1}(d_{[\![0,b-1]\!]})) - \theta_I[d_{b-1}]\right) \cup \left(A(F_{I,-1}(d_{[\![0,b-1]\!]})) + \theta_I[d_{b-1}]\right)| \geqslant |A(F_{I,-1}(d_{[\![0,b-1]\!]})) + \theta_I[d_{b-1}]|$$

we get for any $d_{[\![0,b-1]\!]} \in \mathbb{D}_{[\![0,b-1]\!]}$,

$$\left(A(F_{I,1}(d_{[\![0,b-1]\!]})) - \theta_I[d_{b-1}]\right) \cup \left(A(F_{I,-1}(d_{[\![0,b-1]\!]})) + \theta_I[d_{b-1}]\right) = A(F_{I,-1}(d_{[\![0,b-1]\!]})) + \theta_I[d_{b-1}]$$

and we deduce

$$A(F_{I,1}(d_{[\![0,b-1]\!]})) - \theta_I[d_{b-1}] = A(F_{I,-1}(d_{[\![0,b-1]\!]})) + \theta_I[d_{b-1}]$$

i.e.

$$\begin{aligned} A(d_{[\![0,b-1]\!]}) &= A(F_{I,-1} \circ F_{I,1}^{-1}(d_{[\![0,b-1]\!]})) + 2\theta_I[(F_{I,1}^{-1}(d_{[\![0,b-1]\!]}))_{b-1}] \\ &= A(F_{I,-1}^2(d_{[\![0,b-1]\!]})) + 2\theta_I[(F_{I,-1}(d_{[\![0,b-1]\!]}))_{b-1}] \end{aligned}$$

where $F_{I,-1}^2$ is the composition of $F_{I,-1}$ with itself.

Recall from Lemma 20 that if $I \geqslant N-b+1$, then the $\theta_I[(F_{I,-1}(d_{[\![0,b-1]\!]}))_{b-1}]$ vanishes. First consider the case $I = N-1$, where this condition is satisfied, so that

$$A(d_{[\![0,b-1]\!]}) = A(F_{N-1,-1}^2(d_{[\![0,b-1]\!]})) \tag{36}$$

Let us identify $d_{[\![0,b-1]\!]}$ as an incomplete $N \times N$ matrix from $\mathbb{H}_{N,M}$ whose b^{th}, $(b+1)^{\text{th}}$, ... $(N-1)^{\text{th}}$ upper diagonals have been removed, i.e. let us write it $[x_{k,l}]_{1 \leqslant k < l \leqslant k+b-1}$. Similarly, identify $F_{N-1,-1}^2(d_{[\![0,b-1]\!]}) \in \mathbb{D}_{[\![0,b-1]\!]}$ with $[x'_{k,l}]_{1 \leqslant k < l \leqslant k+b-1}$. Then we have $x'_{k,l} = x_{k,l}$, except for $(k,l) = (N-1,N)$, where $x'_{N-1,N} = x_{N-1,N} - 2$. Since the mapping $\mathbb{Z}_M \ni z \mapsto z-2 \in \mathbb{Z}_M$ is a bijection, it follows from (36) that $A(d_{[\![0,b-1]\!]})$ does not depend on the coordinate $x_{N-1,N}$ of $d_{[\![0,b-1]\!]}$.

Next assume that $b \geqslant 3$ and take $I = N-2$. We have

$$A(d_{[\![0,b-1]\!]}) = A(F_{N-2,-1}^2(d_{[\![0,b-1]\!]})) \tag{37}$$

Writing $[x_{k,l}]_{1 \leqslant k < l \leqslant k+b-1}$ the l.h.s. and $[x'_{k,l}]_{1 \leqslant k < l \leqslant k+b-1}$ the r.h.s., these coordinates coincide, except that $x'_{N-2,N-1} = x_{N-2,N-1} - 2$ and $x'_{N-2,N} = x_{N-2,N} - 2x_{N-1,N}$. Since both side of (37) do not depend on $x_{N-1,N}$, it follows that they also do not depend on the coordinate $x_{N-2,N}$. Resorting again to the bijectivity of the mapping $\mathbb{Z}_M \ni z \mapsto z-2 \in \mathbb{Z}_M$, we see they equally do not depend on $x_{N-2,N-1}$. By iteration, considering successively $I = N-2$, ..., $I = N-b+1$, it appears that $A(d_{[\![0,b-1]\!]})$ does not depend on the coordinates $x_{k,l}$, where $k > N-b$ (and $k < l \leqslant k+b-1$).

For $I = N-b$, we have

$$A(d_{[\![0,b-1]\!]}) = A(F_{N-b,-1}^2(d_{[\![0,b-1]\!]})) + 2\theta_{N-b}[(F_{N-b,-1}(d_{[\![0,b-1]\!]}))_{b-1}] \tag{38}$$

Note that the diagonal $F_{N-b,-1}(d_{[\![0,b-1]\!]}))_{b-1} \in \mathbb{D}_{b-1}$ is different from d_{b-1} only in the last-but-one of its coordinates. It follows that $\theta_{N-b}[(F_{N-b,-1}(d_{[\![0,b-1]\!]}))_{b-1}] = \theta_{N-b}(d_{b-1}) = (0,0,...,0,x_{N-b+1,N}) \in \mathbb{D}_{[\![0,b-1]\!]}$, with the above notation. Denote $y \coloneqq (x_{N-b+1,N-b+2}, x_{N-b+1,N-b+3}, ..., x_{N-b+1,N-1})$. We have that

- the set $A(d_{\llbracket 0,b-1\rrbracket})$ does not depend on y nor on $x_{N-b+1,N}$,
- the set $A(F^2_{N-b,-1}(d_{\llbracket 0,b-1\rrbracket}))$ a priori depends on y, but not on $x_{N-b+1,N}$,
- the vector $2\theta_{N-b}[(F_{N-b,-1}(d_{\llbracket 0,b-1\rrbracket}))_{b-1}]$ only depends on $x_{N-b+1,N}$.

It follows that $A(d_{\llbracket 0,b-1\rrbracket})$ is preserved by the translations by vectors of the form $(0,0,...,0,z) \in \mathbb{D}_{\llbracket 0,b-1\rrbracket}$ with $z \in \mathbb{Z}_M$. Namely, we can write $A(d_{\llbracket 0,b-1\rrbracket}) = A^{(1)}(d_{\llbracket 0,b-1\rrbracket}) \times \mathbb{Z}_M$, where $A^{(1)}(d_{\llbracket 0,b-1\rrbracket})$ is a subset of $\mathbb{Z}_M^{\{(k,b+k)\,:\,k\in\llbracket N-b-1\rrbracket\}}$. Coming back to (38), we deduce that $A^{(1)}(d_{\llbracket 0,b-1\rrbracket})$ do not depend on y (nor on the row indexed by $\llbracket N-b+1,N\rrbracket$ of $d_{\llbracket 0,b-1\rrbracket}$.

The previous arguments can be iterated with $I = N-b-1$, ..., $I = 1$. At the end we get that $A(d_{\llbracket 0,b-1\rrbracket}) = \mathbb{D}_b$, as desired.

■

The next result is the generalization of Lemma 15.

Lemma 27 *When* $A \coloneqq (A(d_{\llbracket 0,b-1\rrbracket}))_{d_{\llbracket 0,b-1\rrbracket}\in\mathbb{D}_{\llbracket 0,b-1\rrbracket}} \neq A_{b,\infty}$, *there exist* $I \in \llbracket N-1\rrbracket$ *such that*

$$R(A^{\cup,I}) \quad \geqslant \quad R(A) - \ln\left(1 - \frac{1}{M^{N-b}}\right)$$

so that $\exp(R(A^{\cup,I}) - R(A)) \geqslant 1 + 1/M^{N-b}$.

Proof

Lemma 26 shows that when $A \neq A_{b,\infty}$, there exists $\widetilde{I} \in \llbracket N-1\rrbracket$ such that $R(A^{\cup,\widetilde{I}}) > R(A)$. Furthermore, from the beginning of the proof of Lemma 26, there exists $\widetilde{d}_{\llbracket 0,b-1\rrbracket} \in \mathbb{D}_{\llbracket 0,b-1\rrbracket}$ such that

$$|A^{\cup,\widetilde{I}}(\widetilde{d}_{\llbracket 0,b-1\rrbracket})| \quad > \quad |A(F_{\widetilde{I},1}(\widetilde{d}_{\llbracket 0,b-1\rrbracket}))|$$

otherwise we would end up with the contradiction $R(A^{\cup,\widetilde{I}}) \leqslant R(A)$.

It follows that

$$\begin{aligned} R(A^{\cup,\widetilde{I}}) - R(A) &= \sum_{d_{\llbracket 0,b-1\rrbracket}\in\mathbb{D}_{\llbracket 0,b-1\rrbracket}} \ln(|A^{\cup,\widetilde{I}}(d_{\llbracket 0,b-1\rrbracket})|) - \ln(|A(d_{\llbracket 0,b-1\rrbracket})|) \\ &\geqslant \ln(|A^{\cup,\widetilde{I}}(\widetilde{d}_{\llbracket 0,b-1\rrbracket})|) - \ln(|A(\widetilde{d}_{\llbracket 0,b-1\rrbracket})|) \end{aligned}$$

Since $|A(\widetilde{d}_{\llbracket 0,b-1\rrbracket})| + 1 \leqslant |A^{\cup,\widetilde{I}}(\widetilde{d}_{\llbracket 0,b-1\rrbracket})| \leqslant \mathrm{card}(\mathbb{D}_b) = M^{N-b}$, by concavity of the logarithm, we get that

$$\begin{aligned} \ln(|A^{\cup,\widetilde{I}}(\widetilde{d}_{\llbracket 0,b-1\rrbracket})|) - \ln(|A(\widetilde{d}_{\llbracket 0,b-1\rrbracket})|) &\geqslant \ln(M^{N-b}) - \ln(M^{N-b} - 1) \\ &= -\ln\left(1 - \frac{1}{M^{N-b}}\right) \end{aligned}$$

The first desired bound follows:

$$R(A^{\cup,I}) - R(A) \quad \geqslant \quad -\ln\left(1 - \frac{1}{M^{N-b}}\right)$$

and the last inequality is deduced as in the proof of Lemma 15.

■

Let us keep following the path of Section 4 by presenting the generalization of Lemma 16. We need the following notations:

$$\forall\ n \in \mathbb{Z}_+, \qquad \left\{ \begin{array}{lcl} \mathcal{A}_n & \coloneqq & \sigma(\mathfrak{X}(0), \mathfrak{X}(1), ..., \mathfrak{X}(n)) \\ R_n & \coloneqq & R[\mathfrak{X}(n)] \end{array} \right.$$

Lemma 28 *We have for any $n \in \mathbb{Z}_+$ such that $\mathfrak{X}(n)$ is of type b, with fixed $b \in [\![2, N-1]\!]$,*

$$\mathbb{E}[R_{n+1}|\mathcal{A}_n] \quad \geqslant \quad R_n + \frac{2}{37(N-1)M^{2(N-b)}}$$

Proof

From the general theory developed in [16], from Lemmas 20 and 21 and from the choice (33), the conditional law of the characteristic subsets A' of $\mathfrak{X}(n+1)$ knowing $\mathcal{A}_n$, in particular knowing the characteristic subsets A of $\mathfrak{X}(n)$, is given by

$$\mathfrak{Q}(A, A') \quad \coloneqq \quad \frac{1}{3}\delta_A(A') + \frac{1}{3(N-1)} \sum_{I \in [\![N-1]\!]} \frac{|A'|}{|A|}(\delta_{A^{\cup,I}}(A') + \delta_{A^{\cap,I}}(A'))$$

where

$$|A| \quad \coloneqq \quad \prod_{d_{[\![0,b-1]\!]} \in \mathbb{D}_{[\![0,b-1]\!]}} |A(d_{[\![0,b-1]\!]})|$$

This is a Markov kernel on $\mathbb{A}_b$, the set of families of non-empty subsets of $\mathbb{D}_b$ indexed by $\mathbb{D}_{[\![0,b-1]\!]}$. As in the proof of Lemma 16, the kernel $\mathfrak{Q}$ is the modification through the cardinal weights of the kernel $\mathfrak{K}$ defined by

$$\mathfrak{K}(A, A') \quad \coloneqq \quad \frac{1}{3}\delta_A(A') + \frac{1}{3(N-1)} \sum_{I \in [\![N-1]\!]} (\delta_{A^{\cup,I}}(A') + \delta_{A^{\cap,I}}(A'))$$

Note that since $\mathfrak{Q}$ is a Markov kernel, we have for any $A \in \mathbb{A}_b$,

$$\begin{aligned} \sum_{A' \in \mathbb{A}_b} \mathfrak{K}(A, A')|A'| &= |A| \sum_{A' \in \mathbb{A}_b} \mathfrak{K}(A, A')\frac{|A'|}{|A|} \\ &= |A| \sum_{A' \in \mathbb{A}_b} \mathfrak{Q}(A, A') \\ &= |A| \end{aligned} \tag{39}$$

With the above notations, it follows that while $\mathfrak{X}(n)$ is of type b,

$$\begin{aligned} \mathbb{E}[R_{n+1}|\mathcal{A}_n] &= \sum_{A' \in \mathbb{A}_b} \mathfrak{Q}(A, A') \ln(|A'|) \\ &= \sum_{A' \in \mathbb{A}_b} \mathfrak{K}(A, A')\frac{|A'|}{|A|} \ln(|A'|) \end{aligned}$$

From (27), we have

$$|A'| \ln(|A'|) \quad \geqslant \quad |A| \ln(|A|) + (1 + \ln(|A|))(|A'| - |A|) + (\sqrt{|A'|} - \sqrt{|A|})^2$$

so we deduce

$$\begin{aligned}
\mathbb{E}[R_{n+1}|\mathcal{A}_n] &\geqslant \frac{1}{|A|}\sum_{A'\in\mathbb{A}_b}\mathfrak{K}(A,A')|A'|\ln(|A'|)\\
&\geqslant \frac{1}{|A|}\sum_{A'\in\mathbb{A}_b}\mathfrak{K}(A,A')\left(|A|\ln(|A|)+(1+\ln(|A|))(|A'|-|A|)+(\sqrt{|A'|}-\sqrt{|A|})^2\right)\\
&= \ln(|A|)+\frac{1+\ln(|A|)}{|A|}\sum_{A'\in\mathbb{A}_b}\mathfrak{K}(A,A')(|A'|-|A|)+\frac{1}{|A|}\sum_{A'\in\mathbb{A}_b}\mathfrak{K}(A,A')(\sqrt{|A'|}-\sqrt{|A|})^2\\
&= \ln(|A|)+\frac{1}{|A|}\sum_{A'\in\mathbb{A}_b}\mathfrak{K}(A,A')(\sqrt{|A'|}-\sqrt{|A|})^2\\
&= \ln(|A|)+\frac{1}{3(N-1)|A|}\sum_{I\in[\![N-1]\!]}((\sqrt{|A^{\cup,I}|}-\sqrt{|A|})^2+(\sqrt{|A^{\cap,I}|}-\sqrt{|A|})^2)\\
&\geqslant \ln(|A|)+\frac{1}{3(N-1)|A|}\sum_{I\in[\![N-1]\!]}(\sqrt{|A^{\cup,I}|}-\sqrt{|A|})^2\\
&= \ln(|A|)+\frac{1}{3(N-1)}\sum_{I\in[\![N-1]\!]}\left(\sqrt{\frac{|A^{\cup,I}|}{|A|}}-1\right)^2\\
&= \ln(|A|)+\frac{1}{3(N-1)}\sum_{I\in[\![N-1]\!]}\left(\exp\left(\frac{R(A^{\cup,I})-R(A)}{2}\right)-1\right)^2
\end{aligned}$$

where we took into account (39) in the second equality. From Lemma 27, when $A\neq A_{b,\infty}$, there exists $I\in[\![N-1]\!]$ such that

$$\begin{aligned}
\left(\exp\left(\frac{R(A^{\cup,I})-R(A)}{2}\right)-1\right)^2 &\geqslant \left(\sqrt{1+\frac{1}{M^{N-b}}}-1\right)^2\\
&= 2+\frac{1}{M^{N-b}}-2\sqrt{1+\frac{1}{M^{N-b}}}\\
&\geqslant \frac{6\sqrt{3}}{64M^{2(N-b)}}
\end{aligned}$$

due to (28). So we get when $\mathfrak{X}(n)$ is of type b (and so the corresponding A is not equal to $A_{b,\infty}$, otherwise we would have $\mathfrak{X}(n)\in\mathfrak{V}_{b+1}$),

$$\begin{aligned}
\mathbb{E}[R_{n+1}|\mathcal{A}_n] &\geqslant \ln(|A|)+\frac{2}{37(N-1)M^{2(N-b)}}\\
&= R_n+\frac{2}{37(N-1)M^{2(N-b)}}
\end{aligned}$$

■

We deduce a weak estimate on $\mathfrak{t}_{b+1}-\mathfrak{t}_b$, as in Proposition 17

Proposition 29 *We have, for* $b\in[\![2,N-1]\!]$,

$$\begin{aligned}
\mathbb{E}[\mathfrak{t}_{b+1}-\mathfrak{t}_b] &\leqslant 37\frac{N-1}{2}(N-b)M^{N(b+1)-b(b+3)/2}\ln(M)\\
&\leqslant 37\frac{(N-1)(N-2)}{2}M^{N(N-1)/2+1}\ln(M)
\end{aligned}$$

Proof

According to Lemma 28, the stochastic chain

$$\left(R_{\mathfrak{t}_{b+1}\wedge(\mathfrak{t}_b+n)}-\frac{2}{37(N-1)M^{2(N-b)}}((\mathfrak{t}_{b+1}-\mathfrak{t}_b)\wedge n)\right)_{n\in\mathbb{Z}_+}$$

is a submartingale. It follows that

$$\forall\ n\in\mathbb{Z}_+,\qquad \mathbb{E}[R_{\mathfrak{t}_{b+1}\wedge(\mathfrak{t}_b+n)}]\ \geqslant\ \frac{2}{37(N-1)M^{2(N-b)}}\mathbb{E}[(\mathfrak{t}_{b+1}-\mathfrak{t}_b)\wedge n]$$

Letting n go to infinity, we get, by dominated convergence in the l.h.s. and by monotone convergence in the r.h.s.,

$$\mathbb{E}[R_{\mathfrak{t}_{b+1}}]\ \geqslant\ \frac{2}{37(N-1)M^{2(N-b)}}\mathbb{E}[\mathfrak{t}_{b+1}-\mathfrak{t}_b]$$

To get the first announced bound, note that

$$\begin{aligned}\mathbb{E}[R_{\mathfrak{t}_{b+1}}] &= \ln(|A_{b,\infty}|)\\ &= \ln(|\mathbb{D}_b|^{|\mathbb{D}_{[\![0,b-1]\!]}|})\\ &= \ln(M^{(N-b)M^{(N-1)+(N-2)+\cdots(N-b+1)}})\\ &= (N-b)M^{(2N-b)(b-1)/2}\ln(M)\end{aligned}$$

so that

$$\begin{aligned}\mathbb{E}[\mathfrak{t}_{b+1}-\mathfrak{t}_b] &\leqslant 37\frac{N-1}{2}M^{2(N-b)}(N-b)M^{(2N-b)(b-1)/2}\ln(M)\\ &= 37\frac{N-1}{2}(N-b)M^{N(b+1)-b(b+3)/2}\ln(M)\end{aligned}$$

To conclude to the second bound, note that the quadratic mapping $\mathbb{R}\ni b\mapsto N(b+1)-b(b+3)/2$ attains its maximum value at $b=N-3/2$ and on $\mathbb{Z}_+$ its maximum value is attained at $b=N-1$ and at $b=N-2$. It follows that

$$\begin{aligned}\mathbb{E}[\mathfrak{t}_{b+1}-\mathfrak{t}_b] &\leqslant 37\frac{N-1}{2}(N-b)M^{N(N-1)/2+1}\ln(M)\\ &\leqslant 37\frac{(N-1)(N-2)}{2}M^{N(N-1)/2+1}\ln(M)\end{aligned}$$

■

Proposition 25 is now obtained via the Markovian arguments recalled in the proof of Theorem 1.

Note that Proposition 25 provides the justification of the assertion made after the statement of Theorem 3. Following the indication given there, we come to the

Proof of Theorem 3

Since $\mathfrak{X}$ is a set-valued process deduced from a family of random mappings satisfying the conditions of [16], it is an intertwining dual for $[X]$ and its absorbing time $\mathfrak{t}_N$ at $\mathbb{H}_{N,M}$ has the same law as a strong stationary time fo $[X]$.

It remains to write that for any $r \geqslant 0$,

$$\begin{aligned}
\mathbb{P}[\mathfrak{t}_N \geqslant r] &= \mathbb{P}\left[\sum_{b\in[\![N-1]\!]} \mathfrak{t}_{b+1}-\mathfrak{t}_b \geqslant r\right] \\
&\leqslant \mathbb{P}\left[\exists\, b\in[\![N-1]\!] \,:\, \mathfrak{t}_{b+1}-\mathfrak{t}_b \geqslant r\right] \\
&\leqslant \sum_{b\in[\![N-1]\!]} \mathbb{P}\left[\mathfrak{t}_{b+1}-\mathfrak{t}_b \geqslant r\right] \\
&= \mathbb{P}\left[\mathfrak{t}_2 \geqslant r\right] + \sum_{b\in[\![2,N-1]\!]} \mathbb{P}\left[\mathfrak{t}_{b+1}-\mathfrak{t}_b \geqslant r\right] \\
&\leqslant 5\frac{N-1}{2}\exp\left(-\frac{r}{5(N-1)M^2}\right) + 3(N-2)\exp\left(-\frac{2r}{101(N-1)(N-2)M^{N(N-1)/2+1}\ln(M)}\right) \\
&\leqslant \frac{11N-17}{2}\exp\left(-\frac{2r}{101(N-1)(N-2)M^{N(N-1)/2+1}\ln(M)}\right)
\end{aligned}$$

where we used Lemma 24 and Proposition 25 in the last-but-one bound.

■

To end this section, let us mention the modifications required by the proof of Theorem 4. They extend to higher dimensions the arguments of Section 5.

First note that the last column $C_N[X] \coloneqq [X_{k,N}]_{k\in[\![N-1]\!]}$ is indeed a Markov chain, whose state space is $\mathbb{Z}_M^{N-1}$ and whose generic elements will be denoted $[x] \coloneqq [x_k]_{k\in[\![N-1]\!]}$. The associated transition matrix P is given by

$$\forall\ [x],\,[x']\in\mathbb{Z}_M^{N-1},\quad P([x],[x']) = \begin{cases} 1/(6(N-1)) & \text{, if } [x'] = F_{I,\epsilon}[x] \text{ for some } I\in[\![N-1]\!] \text{ and } \epsilon\in\{\pm1\} \\ 1/3 & \text{, if } [x']=[x] \\ 0 & \text{, otherwise} \end{cases}$$

where for any$[x] \coloneqq [x_k]_{k\in[\![N-1]\!]}\in\mathbb{Z}_M^{N-1}$, $k\in[\![N-1]\!]$, $I\in[\![N-1]\!]$ and $\epsilon\in\{\pm1\}$, the k-th coordinate of $F_{I,\epsilon}[x]$ is given by

$$(F_{I,\epsilon}[x])_k \coloneqq \begin{cases} x_k & \text{, if } k > I \\ x_k+\epsilon x_{k+1} & \text{, if } k\leqslant I \end{cases}$$

with the convention $x_N = 1$.

The transition kernel P admits the uniform distribution on $\mathbb{Z}_M^{N-1}$ as reversible probability.

Let us describe the subset $\mathfrak{V}$ of non-empty subsets of $\mathbb{Z}_M^{N-1}$ and the family of random mappings $(\psi_\Omega)_{\Omega\in\mathfrak{V}}$ which are the primal ingredients to apply the method of [16].

Any non-empty subset Ω of $\mathbb{Z}_M^{N-1}$ is uniquely determined by a non-empty subset $B\subset\mathbb{Z}_M$ and a family $(A_l(x_{[\![l+1,N-1]\!]}))_{l\in[\![1,N-2]\!],\,x_{[\![l+1,N-1]\!]}\in\mathbb{Z}_M^{[\![l+1,N-1]\!]}}$ of non-empty subsets of $\mathbb{Z}_M$ (called **characteristic subsets** in the sequel) such that

$$[x]\in\Omega \quad\Leftrightarrow\quad x_{N-1}\in B \text{ and for all } l\in[\![l+1,N-1]\!],\ x_l\in A_l(x_{[\![l+1,N-1]\!]})$$

(where $x_{[\![l+1,N-1]\!]} = (x_k)_{k\in[\![l+1,N-1]\!]}$).

We take for $\mathfrak{V}$ the set of subsets Ω of $\mathbb{Z}_M^{N-1}$ such that B is a closed ball $B(r)$ with radius $r\in[\![0,(M-1)/2]\!]$. As above, such Ω will of several types, defined iteratively:

- type 0, if B is the singleton $\{0\}$,
- type 1, if the radius of B satisfies $0<r<(M-1)/2$,

- type $l \in [\![2, N-1]\!]$, if it is not of type k for any $k \in [\![0, l-1]\!]$ and if there exists at least one $x_{[\![N-l+1,N-1]\!]} \in \mathbb{Z}_M^{[\![N-l+1,N-1]\!]}$ such that $A_{N-l}(x_{[\![N-l+1,N-1]\!]}) \neq \mathbb{Z}_M$.
- type N, if $\Omega = \mathbb{Z}_M^{N-1}$.

The construction of the random mappings is similar to the one presented earlier in this section, just keeping the effects on the last column. The most important feature to retain is the action on the subsets of type b, for any fixed $b \in [\![2, N-1]\!]$, and even only on the characteristic subsets $A_{N-b}(x_{[\![N-b+1,N-1]\!]})$, for $x_{[\![N-b+1,N-1]\!]} \in \mathbb{Z}_M^{[\![N-b+1,N-1]\!]}$. More precisely, given such a family

$$A \quad \coloneqq \quad (A_{N-b}(x_{[\![N-b+1,N-1]\!]}))_{x_{[\![N-b+1,N-1]\!]} \in \mathbb{Z}_M^{[\![N-b+1,N-1]\!]}} \tag{40}$$

and $I \in [\![N-1]\!]$, associate two other families of the same nature $A^{\cup,I}$ and $A^{\cap,I}$ defined by taking for any $x_{[\![N-b+1,N-1]\!]} \in \mathbb{Z}_M^{[\![N-b+1,N-1]\!]}$,

$$\begin{aligned} A^{\cup,I}(x_{[\![N-b+1,N-1]\!]}) \quad &\coloneqq \quad \big(A(F_{I,+}(x_{[\![N-b+1,N-1]\!]})) - \delta_b(I)x_{N-b+1}\big) \cup \big(A(F_{I,-}(x_{[\![N-b+1,N-1]\!]}))) + \delta_b(I)x_{N-b+1}\big) \\ A^{\cap,I}(x_{[\![N-b+1,N-1]\!]}) \quad &\coloneqq \quad \big(A(F_{I,+}(x_{[\![N-b+1,N-1]\!]})) - \delta_b(I)x_{N-b+1}\big) \cap \big(A(F_{I,-}(x_{[\![N-b+1,N-1]\!]}))) + \delta_b(I)x_{N-b+1}\big) \end{aligned}$$

where $\delta_b(I)$ is the Kronecker symbol whose value is 1 if $I = b$ and 0 otherwise.

Remark that for $I < b$, we end up with $A^{\cup,I} = A^{\cap,I} = A$.

Let $\mathfrak{A}_b$ be the set of families of the form (40). Following meticulously the method described in the first part of this section, we are led to investigate Markov chains $(A_n)_{n \in \mathbb{Z}_+}$ on $\mathfrak{A}_b$ whose transition kernel $\mathfrak{Q}$ is given by

$$\forall\ A, A' \in \mathfrak{A}_b, \qquad \mathfrak{Q}(A, A') \quad \coloneqq \quad \frac{1}{3}\delta_A(A') + \frac{1}{3(N-1)} \sum_{I \in [\![N-1]\!]} \frac{|A'|}{|A|}(\delta_{A^{\cup,I}}(A') + \delta_{A^{\cap,I}}(A'))$$

where

$$\forall\ A \in \mathfrak{A}_b, \qquad |A| \quad \coloneqq \quad \prod_{x_{[\![N-b+1,N-1]\!]} \in \mathbb{Z}_M^{[\![N-b+1,N-1]\!]}} |A(x_{[\![N-b+1,N-1]\!]})|$$

Specifying Lemma 26 to the last column of the objects considered there, it appears that $(A_n)_{n \in \mathbb{Z}_+}$ ends up being absorbed into $A_{b,\infty}$, the element of $\mathfrak{A}_b$ whose fibers are all equal to $\mathbb{Z}_M$. Denote τ_b the corresponding absorbing time. Our approach relies on the possibility to estimate the tail probabilities of τ_b. Here is the equivalent of Proposition 25:

Proposition 30 *For $b \in [\![2, N-1]\!]$ and for any initial distribution of A_0, we have for M large enough (uniformly in b and N),*

$$\forall\ r \geqslant 0, \qquad \mathbb{P}[\tau_b \geqslant r] \quad \leqslant \quad 3\exp\left(-\frac{2r}{101(N-1)M^{N+1}\ln(M)}\right)$$

The proof of these bounds is similar to that of Proposition 25. The only differences are:

- Since the fibers are included into $\mathbb{Z}_M$ (instead of $\mathbb{D}_b$), we can replace $1/M^{N-b}$ by $1/M$, in Lemmas 27 and 28.
- Since the base is $\mathbb{Z}_M^{[\![N-b+1,N-1]\!]}$ (instead of $\mathbb{D}_{[\![0,b-1]\!]}$), we can replace

$$\ln(|\mathbb{D}_b|^{|\mathbb{D}_{[\![0,b-1]\!]}|}) \quad = \quad (N-b)M^{(2N-b)(b-1)/2}\ln(M)$$

by $\ln(|\mathbb{Z}_M^{\mathbb{Z}_M^{[\![N-b+1,N-1]\!]}}|) = M^b \ln(M) \leqslant M^{N-1}\ln(M)$, in Proposition 29.

In addition to Proposition 30, we furthermore need an estimate on the strong stationary time required by the coordinate $X_{N-1,N}$. It is the analogue of Lemma 24, where the factor $N-1$ in the r.h.s. can be removed (since we don't have to wait for the whole first upper diagonal to reach equilibrium).

Putting together all these estimates as in the proof of Theorem 3, we end up with a strong stationary time $\widetilde{\tau}$ whose tail probabilities satisfies for M large enough,

$$\begin{aligned}\forall\, r \geqslant 0, \qquad \mathbb{P}[\widetilde{\tau} \geqslant r] &\leqslant \frac{5}{2}\exp\left(-\frac{r}{5(N-1)M^2}\right) + 3(N-2)\exp\left(-\frac{2r}{101(N-1)M^{N-1+2}\ln(M)}\right)\\ &\leqslant \frac{6N-7}{2}\exp\left(-\frac{2r}{101(N-1)M^{N+1}\ln(M)}\right)\end{aligned}$$

This ends the proof of Theorem 4.

A The finite circle: remaining cases

In the context of the beginning of Section 2, we deal here with the remaining cases where $a \in (1/3, 1/2]$. To construct the sets $\mathfrak{V}$ and the corresponding random mappings $(\psi_S)_{S\in\mathfrak{V}}$ satisfying the conditions of weak association with P and of stability of $\mathfrak{V}$, we distinguish two situations, depending on the parity of $M \in \mathbb{N}\backslash\{1,2\}$.

A.1 When M is even

For $a \in (1/3, 1/2]$, we need to add new kinds of sets in $\mathfrak{V}$, in addition to the segments from $\mathfrak{I}$. More precisely, for $r \in [\![0, M/2]\!]$, let $B_-(0,r)$ be the set of $x \in B(0,r)$ which have the same parity as r (there is no ambiguity in the definition of the parity in $\mathbb{Z}_M$, since M is even). Consider

$$\begin{aligned}\mathfrak{I}_- &:= \{B_-(0,r)\;:\; r \in [\![1, M/2]\!]\}\\ \mathfrak{V} &:= \mathfrak{I} \sqcup \mathfrak{I}_-\end{aligned}$$

Note that the only subset of the form $B_-(0,r)$ that belongs to $\mathfrak{I}$ is $B_-(0,0) = \{0\}$, which does not belong to $\mathfrak{I}_-$.

A.1.1 The random mapping $\psi_{\{0\}}$

When $a \in (1/3, 1/2]$, the construction of $\psi_{\{0\}}$ given in Section 2.1 is no longer valid. So here is another construction (an alternative one will be provided in Section A.3.1). Choose two mappings $\widetilde{\psi}, \widehat{\psi} : \mathbb{Z}_M \to \mathbb{Z}_M$ satisfying respectively $\widetilde{\psi}(0) = 0 = \widetilde{\psi}(-1) = \widetilde{\psi}(1)$ and $\widetilde{\psi}(x) \neq 0$ for $x \in \mathbb{Z}_M\backslash[\![-1,1]\!]$, and $\widehat{\psi}(-1) = 0 = \widehat{\psi}(1)$ and $\widehat{\psi}(x) \neq 0$ for $x \in \mathbb{Z}_M\backslash\{-1,1\}$. Take $\psi_{\{0\}}$ to equal to $\widetilde{\psi}$ with some probability $p \in [0,1]$ and to $\widehat{\psi}$ with probability $1-p$. Let us compute p so that Condition (1) is satisfied, which here still amounts to (6).

- When $x \notin [\![-1,1]\!]$, both sides of (6) vanish.
- When $x \in \{-1,1\}$, the l.h.s. of (6) is 1, while the r.h.s. is $a/\zeta(\{0\})$. This implies that $\zeta(\{0\}) = a$.
- When $x = 0$, (6) is equivalent to

  $$p = \frac{1-2a}{a}$$

  and this number p does belongs to $[0,1]$ for $a \in (1/3, 1/2]$.

Next we must check that for this random mapping $\psi_{\{0\}}$, (2) is satisfied, namely $\Psi(\{0\}) \in \mathfrak{V} = \mathfrak{I}\sqcup\mathfrak{I}_-$. This is true, because $\widetilde{\psi}^{-1}(\{0\}) = [\![-1,1]\!] \in \mathfrak{I}$ and $\widehat{\psi}^{-1}(\{0\}) = \{-1,1\} = B_-(0,1) \in \mathfrak{I}_-$.

A.1.2 The other random mappings and the Markov transition kernel $\mathfrak{P}$

For $S \in \mathfrak{I} \sqcup \mathfrak{I}_- \backslash \{0\}$, take the same random mapping $\psi_S = \phi$ defined in Section 2.2. It is clear that (7) is still satisfied, since the proof is valid for any $a \in (0, 1/2]$ (and any $M \geqslant 3$). Concerning the stability of $\mathfrak{I} \sqcup \mathfrak{I}_-$ by ϕ, note that in addition to (8), we also have for any $r \in [\![1, M/2]\!]$,

$$\left\{ \begin{array}{lcl} \phi_1^{-1}(B_-(0,r)) & = & B_-(0,r+1) \\ \phi_2^{-1}(B_-(0,r)) & = & B_-(0,r+1) \\ \phi_3^{-1}(B_-(0,r)) & = & B_-(0,r-1) \\ \phi_4^{-1}(B_-(0,r)) & = & B_-(0,r-1) \\ \phi_5^{-1}(B_-(0,r)) & = & B_-(0,r) \end{array} \right. \tag{41}$$

(where $M/2 + 1$ has to be understood as $M/2 - 1$).

As in Section 2.3, we identify $B(0, r)$ with r, for $r \in [\![0, M/2]\!]$, and furthermore, for $r \in [\![1, M/2]\!]$, we identify $B_-(0, r)$ with $-r$.

It appears that $\mathfrak{P}$ is also the transition matrix of a birth and death chain, but this time on $[\![-M/2, M/2]\!]$:

$$\forall\ k, l \in [\![-M/2, M/2]\!], \qquad \mathfrak{P}(k, l) \quad = \quad \begin{cases} 1 - 2a & \text{, if } k = 0 \text{ and } l = 1 \\ 3a - 1 & \text{, if } k = 0 \text{ and } l = -1 \\ a\frac{2l+1}{2k+1} & \text{, if } k \geqslant 1,\ k \neq M/2 \text{ and } |k - l| = 1 \\ a\frac{|l|+1}{|k|+1} & \text{, if } k \leqslant -1, \text{ and } |k - l| = 1 \\ 1 - 2a & \text{, if } |k| \geqslant 1,\ k \neq M/2 \text{ and } k = l \\ 1 & \text{, if } k = M/2 = l \\ 0 & \text{, otherwise} \end{cases}$$

(we used that $|B_-(0, r)| = r + 1$, for $r \in [\![0, M/2]\!]$).

When $p \in (1/3, 1/2)$, $\mathfrak{P}$ enables to reach the absorbing point $M/2$ from all the other points, thus the absorbing time $\mathfrak{t}$ is a.s. finite and its law is the distribution of a strong stationary time for X. A different feature is that the starting point $\mathfrak{X}_0 = \{0\}$, identified with 0, is at the middle of the discrete segment $[\![-M/2, M/2]\!]$ and the left boundary is not absorbing.

When $p = 1/2$, the transition from 0 to 1 is forbidden: $\mathfrak{P}(0, 1) = 0$. Starting from 0, the Markov chain $\mathfrak{X}$ stays on the irreducible state space $[\![-M/2, 0]\!]$ and never reaches $M/2$, i.e. $\mathfrak{t} = \infty$ a.s. This result could have been guessed, as due to the periodicity of order 2, X does not converge to π in large times. The Markov chain $-\mathfrak{X}$ is a finite equivalent of the process on $\mathbb{Z}_+$ introduced by Pitman in [18] (see also [16] for an approach via random mappings).

Remark 31 It is important that $\psi_{\{0\}}$ is different from the random mapping ϕ considered in Sections 2.2 and A.1.2. Indeed, whatever $a \in (0, 1/2]$, if we had taken $\psi_{\{0\}} = \phi$, we would have ended up with $\mathfrak{X}_1 \in \{\{-1, 1\}, \{0\}\}$ and from (41), we can deduce that for any $n \in \mathbb{Z}_+$, we would have $\mathfrak{X}_n \in \{\{0\}\} \sqcup \mathfrak{I}_-$. In particular $\mathfrak{t} = +\infty$ when M is even.

□

A.2 When M is odd

In this situation, we enrich the set $\mathcal{I}_-$. For $r \in [\![0, (M-1)/2]\!]$, $B_-(0, r)$ is defined as at the beginning of Section A.1. Now the parity of an element $x \in \mathbb{Z}_M$ is the parity of its representative in $[\![-(M - 1)/2, (M-1)/2)]\!]$. Furthermore, for $r \in [\![(M+1)/2, M-1]\!]$, we consider

$$B_-(0, r) \quad = \quad B_-(0, (M-1)/2) \cup B(-(M-1)/2, r - (M-1)/2) \cup B((M-1)/2, r - (M-1)/2)$$

namely this subset contains all the points encountered when going clock-wise from $r-(M-1)$ to $M-1-r$, and all the other points which have the parity of $r-(M-1)$. In particular when $r=M-1$, we get $B_-(0,M-1)=\mathbb{Z}_M$. We take

$$\begin{aligned}\mathcal{I}_- &\coloneqq \{B_-(0,r) \,:\, r\in[\![1,M-1]\!]\}\\ \mathfrak{V} &\coloneqq \mathfrak{I}\cup\mathfrak{I}_-\end{aligned}$$

Note that the only element in the intersection of $\mathcal{I}$ and $\mathcal{I}_-$ is the whole state space $\mathbb{Z}_M=B(0,(M-1)/2)=B_-(0,M-1)$, nevertheless, it will be convenient to see $B(0,(M-1)/2)$ and $B_-(0,M-1)$ as different (i.e. to interpret $\mathfrak{V}$ as a multiset, with $\mathbb{Z}_M$ of multiplicity 2), namely to write $\mathfrak{V}=\mathfrak{I}\sqcup\mathfrak{I}_-$.

We consider the same random mappings as those constructed in Section A.1: The random mapping $\psi_{\{0\}}$ is the one of Section A.1.1 and for $S\in\mathfrak{V}\backslash\{\{0\}\}$, $\psi_S=\phi$, defined in Sections 2.2 and A.1.2.

It follows that (1) holds (with $\zeta(\{0\})=a$ and $\zeta(S)=1$, for $S\in\mathfrak{V}\backslash\{\{0\}\}$). Furthermore, due to the fact that M is odd, we get that (41) is still true for $r\in[\![1,M-1]\!]$, with the convention that $B_-(0,M))=\mathbb{Z}_M$.

Now we identify $B(0,r)$ with r, for $r\in[\![0,(M-1)/2]\!]$, and $B_-(0,r)$ with $-r$, for $r\in[\![1,M-1]\!]$. In accordance with the multiplicity 2 of $\mathbb{Z}_M$ mentioned above, the whole state space $\mathbb{Z}_M$ is seen as the two points $(M-1)/2$ and $-(M-1)$.

This identification enables us to see $\mathfrak{P}$ as the transition matrix of a birth and death chain on $[\![-(M-1),(M-1)/2]\!]$:

$$\forall\ k,l\in[\![-(M-1),(M-1)/2]\!],\quad \mathfrak{P}(k,l) \;=\; \begin{cases} 1-2a & \text{, if } k=0 \text{ and } l=1\\ 3a-1 & \text{, if } k=0 \text{ and } l=-1\\ a\frac{2l+1}{2k+1} & \text{, if } k\geqslant 1,\ k\neq M/2 \text{ and } |k-l|=1\\ a\frac{|l|+1}{|k|+1} & \text{, if } k\leqslant -1,\ k\neq -(M-1) \text{ and } |k-l|=1\\ 1 & \text{, if } k=l\in\{-(M-1),(M-1)/2\}\\ 0 & \text{, otherwise}\end{cases}$$

(we used that $|B_-(0,r)|=r+1$, for $r\in[\![0,M-1]\!]$).

When $p\in(1/3,1/2)$, $\mathfrak{P}$ enables to reach the two absorbing points $(M-1)/2$ and $-(M-1)$ from all the other points, thus the absorbing time $\mathfrak{t}$ is a.s. finite and its law is the distribution of a strong stationary time for X. The Markov chain $\mathfrak{X}$ still starts from 0 and ends up being absorbed in one of boundary points $(M-1)/2$ or $-(M-1)$.

When $p=1/2$, the transition from 0 to 1 is still forbidden: $\mathfrak{P}(0,1)=0$. Starting from 0, the Markov chain $\mathfrak{X}$ stays on the irreducible state space $[\![-(M-1),0]\!]$ and ends up being absorbed at $-(M-1)$. Thus $\mathfrak{t}$ is a.s. finite and X admits a strong stationary time, it was expected as there is no problem of periodicity when M is odd.

A.3 Alternative random mappings, still for $a\in[1/3,1/2)$

The constructions of the previous subsections could also have been obtained by first lumping X (see Remark 6, whose "projection" is valid for all $a\in(0,1/2]$). Here we propose another construction which is no longer compatible with this procedure. We take for $\mathfrak{V}$ the set of all balls $B(x,r)$, for $x\in\mathbb{Z}_M$ and $r\in[\![0,\lfloor M/2\rfloor]\!]$. All these balls are different, except that $B(x,\lfloor M/2\rfloor)=\mathbb{Z}_M$ for any $x\in\mathbb{Z}_M$. The space $\mathfrak{V}$ can be seen as a wheel: the tyre is the discrete circle consisting of the $B(x,0)=\{x\}$ for $x\in\mathbb{Z}_M$. For any fixed $x\in\mathbb{Z}_M$, the set $\{B(x,r)\,:\,r\in[\![0,\lfloor M/2\rfloor\}$ is a ray going from the tyre to the center of the wheel, represented by $\mathbb{Z}_M$. The Markov kernel $\mathfrak{P}$ that we are to construct will respect this wheel graph.

A.3.1 The alternative random mappings $\psi_{\{x\}}$, for $x \in \mathbb{Z}_M$

Fix some $x \in \mathbb{Z}_M$. We sightly modify the random mapping considered in Section A.1.1 (after rotating $\mathbb{Z}_M$ by $-x$). Choose three mappings $\widetilde{\psi}_x, \widehat{\psi}_x, \check{\psi}_x : \mathbb{Z}_M \to \mathbb{Z}_M$ satisfying respectively

- $\widetilde{\psi}_x(x) = x = \widetilde{\psi}_x(x-1) = \widetilde{\psi}_x(x+1)$ and $\widetilde{\psi}_x(y) \neq x$ for $y \in \mathbb{Z}_M \backslash [\![x-1, x+1]\!]$
- $\widehat{\psi}_x(x-1) = x$ and $\widehat{\psi}_x(y) \neq x$ for $y \in \mathbb{Z}_M \backslash \{x-1\}$
- $\check{\psi}_x(x+1) = x$ and $\check{\psi}_x(y) \neq x$ for $y \in \mathbb{Z}_M \backslash \{x+1\}$

Take $\psi_{\{x\}}$ to equal to $\widetilde{\psi}_x$ with some probability $p \in [0,1]$ and to each of $\widehat{\psi}_x$ and $\check{\psi}_x$ with probability $(1-p)/2$. Let us compute p so that Condition (1) is satisfied, which here amounts to

$$\forall\, y \in \mathbb{Z}_M, \qquad \mathbb{P}[\psi_{\{x\}}(y) = x] \;=\; \frac{1}{\zeta(\{x\})} P(y,x) \tag{42}$$

- When $y \notin [\![x-1, x+1]\!]$, both sides of (42) vanish.
- When $y \in \{x-1, x+1\}$, the l.h.s. of (42) is $1-(1-p)/2$, while the r.h.s. is $a/\zeta(\{x\})$. This implies that $\zeta(\{x\}) = 2a/(1+p)$.
- When $y = x$, (42) is equivalent to

  $$p \;=\; \frac{(1-2a)(1+p)}{2a}$$

  namely $p = (1-2a)/(4a-1)$, which belongs to $[0,1)$ for $a \in (1/3, 1/2]$.

For the computations of the next section, note that according to (3),

$$\begin{aligned}
\mathfrak{P}(\{x\}, \{x-1, x, x+1\}) &= 3\zeta(\{x\})p \\
&= 3\frac{2a}{1+p}\frac{(1-2a)(1+p)}{2a} \\
&= 3(1-2a)
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{P}(\{x\}, \{x-1\}) &= \mathfrak{P}(\{x\}, \{x+1\}) \\
&= \frac{1-\mathfrak{P}(\{x\}, \{x-1, x, x+1\})}{2} \\
&= 3a-1
\end{aligned}$$

Next we must check that for this random mapping $\psi_{\{x\}}$, (2) is satisfied, namely $\psi_x(\{x\}) \in \mathfrak{V}$. This is true, because $\widetilde{\psi}_x^{-1}(\{x\}) = B(x,1)$, $\widehat{\psi}_x^{-1}(\{x\}) = B(x-1,0)$ and $\check{\psi}_x^{-1}(\{x\}) = B(x+1,0)$.

A.3.2 The other random mappings and the Markov transition kernel $\mathfrak{P}$

For any $x \in \mathbb{Z}_M$, the mappings $\phi_{1,x}, \phi_{2,x}, \phi_{3,x}, \phi_{4,x}$ and $\phi_{5,x}$, as well as the random mapping ϕ_x, are constructed as $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5$ and ϕ in Sections 2.2 and A.1.2, but are centered at x instead of 0. Then we take $\psi_S = \phi_x$, for any $S = B(x,r)$, with $r \in [\![1, \lfloor M/2 \rfloor]\!]$. By the same proofs as before ("rotated" by $-x$), we get that these random mappings are strongly associated to P and that (2) is satisfied, since we have for any $r \in [\![1, \lfloor M/2 \rfloor]\!]$,

$$\left\{\begin{aligned}
\phi_{1,x}^{-1}(B(x,r)) &= B(x,r+1) \\
\phi_{2,x}^{-1}(B(x,r)) &= B(x,r+1) \\
\phi_{3,x}^{-1}(B(x,r)) &= B(x,r-1) \\
\phi_{4,x}^{-1}(B(x,r)) &= B(x,r-1) \\
\phi_{5,x}^{-1}(B(x,r)) &= B(x,r)
\end{aligned}\right.$$

(where $B(x, \lfloor M/2 \rfloor) = B(x, \lfloor M/2 \rfloor + 1) = \mathbb{Z}_M$).

The corresponding Markov kernel $\mathfrak{P}$ is compatible with the wheel structure of $\mathfrak{V}$ and we have for any $S, S' \in \mathfrak{V}$ which are neighbors in this graph, and where x is the center of S,

$$\mathfrak{P}(S,S') = \begin{cases} 3a-1 & \text{, if } S = \{x\} \text{ and } S' = \{x+1\} \text{ or } S' = \{x-1\} \\ 3-6a & \text{, if } S = \{x\} \text{ and } S' = \{x-1, x, x+1\} \\ a\frac{2l+1}{2k+1} & \text{, if } S = B(x,k) \text{ and } S = B(x,l) \text{ with } k \in [\![1, \lfloor M/2 \rfloor - 1]\!] \text{ and } |k-l| = 1 \\ 1-2a & \text{, if } S = B(x,k) = S' \text{ with } x \in \mathbb{Z}_M \text{ and } k \in [\![1, \lfloor M/2 \rfloor - 1]\!] \\ 1 & \text{, if } S = S' = \mathbb{Z}_M \end{cases}$$

For a corresponding Markov chain, $\mathfrak{X}$ starting from $\{0\}$, we are interested in the absorption time $\mathfrak{t}$ in $\mathbb{Z}_M$, since its distribution is the law of a strong stationary time for X. Note that we can again come back to a birth and death chain: for any ball $S \in \mathfrak{V}$, denote $\rho(S)$ its radius (with $\rho(\mathbb{Z}_M) = \lfloor M/2 \rfloor$). Remark that $\rho(\mathfrak{X})$ is a birth and death chain, starting from 0, absorbed at $\lfloor M/2 \rfloor$ and whose transition matrix is:

$$\forall\ k,l \in [\![0, \lfloor M/2 \rfloor]\!], \qquad \mathfrak{Q}(k,l) = \begin{cases} 3a-1 & \text{, if } k = 0 = l \\ 3-6a & \text{, if } k = 0 \text{ and } l = 1 \\ a\frac{2l+1}{2k+1} & \text{, if } k \in [\![1, \lfloor M/2 \rfloor - 1]\!] \text{ and } |k-l| = 1 \\ 1-2a & \text{, if } k \in [\![1, \lfloor M/2 \rfloor - 1]\!] \text{ and } k = l \\ 1 & \text{, if } k = \lfloor M/2 \rfloor = l \\ 0 & \text{, otherwise} \end{cases}$$

The absorption time of $\rho(\mathfrak{X})$ at $\lfloor M/2 \rfloor$ has the same law as $\mathfrak{t}$ and Karlin and McGregor [12] enable to compute it in terms of the spectrum of $\mathfrak{Q}$.

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miclo@math.cnrs.fr

Institut de Mathématiques de Toulouse
Université Paul Sabatier, 118, route de Narbonne
31062 Toulouse cedex 9, France

Toulouse School of Economics
1, Esplanade de l'Université
31080 Toulouse Cedex 06, France