CONSTRUCTION OF SET-VALUED DUAL PROCESSES ON MANIFOLDS

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ABSTRACT. The purpose of this paper is to construct a Brownian motion $X := (X_t)_{t \geqslant 0}$ taking values in a Riemannian manifold M, together with a compact valued process $D := (D_t)_{t \geqslant 0}$ such that, at least for small enough $\mathscr{F}^D$-stopping time $\tau \geqslant 0$ and conditioned to $\mathscr{F}^D_\tau$, the law of X_τ is the normalized Lebesgue measure on D_τ. This intertwining result is a generalization of Pitman theorem. We first construct regular intertwined processes related to Stokes' theorem. Then using several limiting procedures we construct synchronous intertwined, free intertwined, mirror intertwined processes. The local times of the Brownian motion on the (morphological) skeleton or the boundary of D plays an important role. Several example with moving intervals, discs, annulus, symmetric convex sets are investigated.

KEYWORDS. Brownian motions on Riemannian manifolds, intertwining relations, set-valued dual processes, couplings of primal and dual processes, stochastic mean curvature evolutions, boundary and skeleton local times, generalized Pitman theorem.

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1. INTRODUCTION AND MAIN RESULTS

Let M be a d-dimensional complete Riemannian manifold. We fix a point $o \in M$ for convenience. Denote respectively by ρ, μ and $\underline{\mu}$, the Riemannian distance, the Lebesgue measure on M and the corresponding $(d-1)$-Hausdorff measure. The main objective of this paper is to construct intertwined processes and to solve Conjecture 6 in [6] in the case of Brownian motion $(X_t)_{t \geqslant 0}$ and stochastic modified mean curvature flow $(D_t)_{t \geqslant 0}$. This conjecture says that an intertwined construction in the sense of Definition 1.1 is always possible.

Definition 1.1. We say that a Brownian motion $X = (X_t)_{t \geqslant 0}$ in M and a Markov process $D = (D_t)_{t \geqslant 0}$, with values in subsets of M and continuous with respect to the Hausdorff topology, are τ-intertwined where τ is a positive stopping time in the filtration $\mathscr{F}^D$ of D if for all bounded $\mathscr{F}^D$-stopping time τ' smaller than τ, conditioned on $\mathscr{F}^D_{\tau'}$, $X_{\tau'}$ has uniform law in $D_{\tau'}$ (and in particular $X_{\tau'} \in \bar{D}_{\tau'}$). We say that X and D are intertwined if they are τ-intertwined, τ being the lifetime of (X, D), assumed to be a.s. positive and $\mathscr{F}^D$-measurable.

This is a generic definition, below stronger topologies on subsets of M will be considered.

Our main results are Theorems 2.8, 3.5 and 4.1 presenting such joint constructions of the primal Brownian motion $(X_t)_{t \geqslant 0}$ and the dual domain-valued $(D_t)_{t \geqslant 0}$ processes. The coupling of Theorem 2.8 consists in the infinite-dimensional stochastic differential equation (2.10), based on a function $f : (x, D) \mapsto f(x, D)$ which is a deformation of the signed distance from $x \in M$ to the boundary of the domain D (see Assumption (2.2) for

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the precise requirements). Theorem 3.5 is obtained by specifying some approximating functions f. Given the trajectory $(X_t)_{t\geqslant 0}$ of the Brownian motion, we construct the domain evolution $(D_t)_{t\geqslant 0}$ using the local time of $(X_t)_{t\geqslant 0}$ on the skeletons of $(D_t)_{t\geqslant 0}$ and the mean curvatures of the normal foliations of these domains (see (3.30)). Other approximating functions f lead to Theorem 4.1, where the prominent role is played by the local time at the boundary. This situation is in some sense opposite to the previous one, since the driving Brownian motion is now independent from $(X_t)_{t\geqslant 0}$, while it was as correlated as it can be in Theorem 3.5. These theoretical results are illustrated by the fundamental examples of Section 5. First we recover the intertwining relation between the real Brownian motion and the three-dimensional Bessel process. Next we deal with rotationally symmetric manifolds. Finally we present the application of our results to symmetric convex domains in the plane, even if the detailed proofs are deferred to a forthcoming paper.

A first motivation for such constructions comes from the quantitative investigation of convergence to equilibrium of diffusions on manifold. Assume that X and D are τ-intertwined, where τ is the hitting time by D of the whole state space M. If furthermore τ is finite (typically true when M is compact), then the Riemannian measure can be normalized into a probability and τ is a strong stationary time for the Brownian motion X, i.e. a stopping time such that τ and X_τ are independent and X_τ is uniformly distributed. In this situation, the tail distributions of τ provide quantitative estimates for the speed of convergence of the Brownian motion toward equilibrium, in the separation sense, see Diaconis and Fill [7] for the general description of this approach, in the case of finite state space. This probabilistic method is an alternative to the functional inequality approach, see e.g. the book [3] of Bakry, Gentil and Ledoux, and is based on other geometric considerations, as we will see in the sequel. Other motivations for the couplings of primal and dual processes in the context of diffusions can be found in Machida [11] and [14].

2. Intertwined dual processes: existence in connection with Stoke's formula

In this section we make a construction of intertwined processes X and D based on the Stokes' Formula (2.1) below. Consider a relatively compact domain D in M with C^2 boundary. Let $f : \bar{D} \to \mathbb{R}$ a C^2 function such that $\nabla f|_{\partial D} = N^D$ the normal inward vector on boundary. Then by Stoke's formula, for any C^2 function $g : \bar{D} \to \mathbb{R}$,

$$-\int_{\partial D} g d\underline{\mu} = \int_{\partial D} g\langle \nabla f, -N^D\rangle \, d\underline{\mu} = \int_D g\Delta f \, d\mu + \int_D \langle \nabla g, \nabla f\rangle \, d\mu. \tag{2.1}$$

For $\alpha \in (0,1)$, denote by $\mathscr{D}^{2+\alpha}$ the set of relatively compact connected open subsets D of M with $C^{2+\alpha}$ boundary.

Definition 2.1. For a given $\alpha \in (0,1)$, $\varepsilon > 0$, we denote by $\mathcal{F}^{\alpha,\varepsilon}$ the set of $D \in \mathscr{D}^{2+\alpha}$ such that

- $D \subset B(o,R)$ the Riemannian ball centered at o with radius $R = 1/\varepsilon$;
- $\rho(\partial D, S(D)) \geqslant \varepsilon$, where $S = S(D)$ is the skeleton of D (see appendix A for details);
- $\rho(\partial D, S^{\mathrm{out}}(D)) \geqslant \varepsilon$, where $S^{\mathrm{out}}(D)$ is the outer skeleton of D, i.e. the skeleton of $(\bar{D})^c$.

On $\mathcal{F}^{\alpha,\varepsilon}$, $(\widetilde{D}_t)_{t\in[0,\tau_\varepsilon]}$ will be a diffusion associated to the generator $\widetilde{\mathscr{L}}$ defined in (2.12) and $\tau_\varepsilon \in (0,+\infty]$ will be the exiting time from $\mathcal{F}^{\alpha,\varepsilon}$. We extend the trajectory $(\widetilde{D}_t)_{t\in[0,\tau_\varepsilon]}$ by taking $\widetilde{D}_t = \widetilde{D}_{\tau_\varepsilon}$ for any $t > \tau_\varepsilon$. It amounts to imposing that $\widetilde{\mathscr{L}}$ vanishes outside

$\mathcal{F}^{\alpha,\varepsilon}$. It is possible to define in the same way $(\widetilde{D}_t)_{t\in[0,\tau)}$ on $\mathcal{D}^{2+\alpha}$ (which coincides with $\cup_{\varepsilon>0}\mathcal{F}^{\alpha,\varepsilon}$), where τ is the exiting time from $\mathcal{D}^{2+\alpha}$. But it will be more convenient for us to work with a process with an infinite lifetime (to be able to apply Proposition D.3 in Appendix D) and whose set of values has a boundary which is well-separated from the skeleton.

Let $\beta \in \{0, \alpha\}$. For $D_0 \in \mathcal{D}^{2+\beta}$ and $\delta > 0$ small enough, a δ-neighborhood of D_0 is defined as follow:

$$\mathcal{V}_\delta^{2+\beta}(D_0) := \left\{\mathrm{int}(\exp_{\partial D_0}(f)), f \in C^{2+\beta}(\partial D_0), \|f\|_{C^{2+\beta}(\partial D_0)} < \delta\right\},$$

where for $f \in C^{2+\beta}(\partial D_0)$

$$\exp_{\partial D_0}(f) := \left\{\exp_x(f(x)N^{D_0}(x)), x \in \partial D_0\right\}$$

($\exp$ being the exponential map in M), and $\mathrm{int}(\exp_{\partial D_0}(f))$ is the interior of the hypersurface $\exp_{\partial D_0}(f)$, oriented by the orientation of D_0. Let $\eta(\partial D_0) > 0$ be the radius of the maximal tubular neighborhood of ∂D_0. Notice that $\delta < \eta(\partial D_0)$ garantees that all elements of $\mathcal{V}_\delta^{2+\beta}(D_0)$ are regular deformations of D_0. Also notice that all elements D of $\mathcal{F}^{\alpha,\varepsilon}$ have $\eta(\partial D) \geqslant \varepsilon$.

We identify two domains $D_1, D_2 \in \mathcal{V}_\delta^{2+\beta}(D_0)$ with the functions $f_1, f_2 \in C^{2+\beta}(\partial D_0)$ such that $D_1 = \mathrm{int}\{\exp_{\partial D_0}(f_1)\}$ and $D_2 = \mathrm{int}\{\exp_{\partial D_0}(f_2)\}$ and we define a local distance

$$d_{\beta,D_0}(D_1, D_2) := \|f_1 - f_2\|_{C^{2+\beta}(\partial D_0)}. \tag{2.2}$$

Assumption 2.2.

- The function

$$\begin{aligned} f : M \times \mathcal{F}^{\alpha,\varepsilon} &\to \mathbb{R} \\ (x, D) &\mapsto f(x, D) = f^D(x) \end{aligned}$$

is a $C^{2+\alpha}$ function in the two variables (the differential in D is in the sense of Fréchet with respect to the above local Banach structure) satisfying

$$\left\|\nabla f^D\right\|_\infty \leqslant 1, \tag{2.3}$$

and the functions f^D coincide to the signed distance to the boundary $\rho^+_{\partial D}$ (positive inside D) in a neighbourhood of ∂D. The functions f^D have bounded Hessian, uniformly in $D \in \mathcal{F}^{\alpha,\varepsilon}$. Furthermore, we assume that the coefficients of the α-Hölderianity of $\mathrm{Hess} f^D$ are uniformly bounded over $\mathcal{F}^{\alpha,\varepsilon}$.

- There exists a positive integer m and a C^1 map

$$\begin{aligned} \sigma_c : M \times \mathcal{F}^{\alpha,\varepsilon} &\to \Gamma(TM \otimes (\mathbb{R}^m)^*) \\ (x, D) &\mapsto \sigma_c(x, D) = \sigma_c^D(x) \in L(\mathbb{R}^m, T_xM) \end{aligned}$$

where $\Gamma(TM \otimes (\mathbb{R}^m)^*)$ is the set of sections over M of $TM \otimes (\mathbb{R}^m)^*$ and $L(\mathbb{R}^m, T_xM)$ is the set of linear maps from $\mathbb{R}^m$ to T_xM, such that the linear map

$$\begin{aligned} \sigma^D(x) : \mathbb{R} \times \mathbb{R}^m &\to T_xM \\ (w_0, w) &\mapsto w_0 \nabla f^D(x) + \sigma_c^D(x)(w) \end{aligned} \tag{2.4}$$

satisfies

$$\forall x \in \bar{D},\ \ \sigma^D(\sigma^D)^*(x) = \mathrm{Id}_{T_xM}. \tag{2.5}$$

□

Remark 2.3. The first condition of Assumption 2.2 implies that

$$\begin{aligned} \nabla f^D|_{\partial D} &= (\nabla \rho^+_{\partial D})|_{\partial D} (= N^D) \quad \text{and} \\ \Delta f^D|_{\partial D} &= (\Delta \rho^+_{\partial D})|_{\partial D} (= -h^D). \end{aligned} \tag{2.6}$$

where h^D stands for the mean curvature on ∂D. It also implies that the functions f^D are uniformly Lipschitz and have uniformly bounded Laplacian. Also, for fixed $x \in \partial D$, varying D successively along a field K normal to the boundary ∂D and along N^D for the second derivative:

$$\begin{aligned} \langle \nabla f(x,\cdot), K\rangle(x) &= -\langle N^D(x), K(x)\rangle \quad \text{and} \\ \nabla df(x,\cdot)\left(N^D, N^D\right) &= 0 \end{aligned} \tag{2.7}$$

where $\nabla df(x,\cdot)$ is the Hessian of f in the second variable.

The second condition of Assumption 2.2 implies that for all $u \in T_xM$,

$$\|u\|^2 = \langle u, \nabla f^D(x)\rangle^2 + \sum_{i=1}^m \langle u, \sigma^D_{\mathrm{c}}(x)(e_i)\rangle^2 \tag{2.8}$$

for $e_1, \dots, e_m$ an orthonormal basis of $\mathbb{R}^m$. In particular, if $x \in \partial D$, taking $u = \nabla f^D(x) = N^D(x)$, we get since $\|N^D(x)\| = 1$:

$$0 = \langle \nabla f^D(x), \sigma(x)(e_i)\rangle, \quad i = 1, \dots m. \tag{2.9}$$

Proposition 2.4. *Assumption 2.2 can always be realized, with any $\alpha \in (0,1)$ and $\varepsilon > 0$.*

Proof. We begin with remarking that for $D \in \mathcal{F}^{\alpha,\varepsilon}$, $\rho(\partial D, S(D)) \geqslant \varepsilon$. In particular, the distance to ∂D is $C^{2+\alpha}$ on $D_\varepsilon := \{x \in D,\ \rho(x, \partial D) < \varepsilon\}$. Let h_ε be a smooth nondecreasing function from $[0,\infty)$ to $\mathbb{R}_+$ such that $h_\varepsilon(r) = r$ for $r \in [0, \varepsilon/2]$, $h_\varepsilon(r) = (3/4)\varepsilon$ for $r \geqslant \varepsilon$ and $\|h'_\varepsilon\|_\infty \leqslant 1$. Then $f^D := h_\varepsilon \circ \rho^+_{\partial D}$ satisfies all the requirements of the first condition of Assumption 2.2. Then for constructing σ^D_c we proceed as in [2], Proposition 3.2 taking $\sigma_1 = \nabla f^D$. The wanted regularity in D is easily checked. □

Let W_t and W^m_t two independent Brownian motions with values respectively in $\mathbb{R}$ and $\mathbb{R}^m$.

The equation we are interested in writes in Itô form for all $y \in \partial D_t$:

$$\left\{ \begin{array}{rcl} dX_t &=& \left(\nabla f^{D_t}(X_t)\, dW_t + \sigma^{D_t}_{\mathrm{c}}(X_t)\, dW^m_t\right) \\ d\partial D_t(y) &=& N^{D_t}(y)\left(dW_t + \left(\frac12 h^{D_t}(y) + \Delta f^{D_t}(X_t)\right)\, dt\right) \end{array} \right. \tag{2.10}$$

started at a relatively compact domain D_0 with $C^{2+\alpha}$ boundary and X_0 such that $\mathscr{L}(X_0) = \mathscr{U}(D_0)$, where $\mathscr{U}(D_0)$ is the uniform probability measure on D_0. In fact, as in Definition 2.1, the evolution equation (2.10) is implicitly considered only up to the exit time τ_ε of $\mathcal{F}^{\alpha,\varepsilon}$ for some fixed $\alpha \in (0,1)$, $\varepsilon > 0$, after which the process is assumed not to move.

In (2.10), the processes $(D_t)_{t\geqslant 0}$ and $(X_t)_{t\geqslant 0}$ are fully interacting, since the evolution of one of them depends on the other one. In particular, they are not Markovian by themselves in general.

Another subset-valued process $(\widetilde{D}_t)_{t\geqslant 0}$ will be interesting for our purposes. It is solution to the evolution equation

$$d\partial \widetilde{D}_t(y) \;=\; N^{\widetilde{D}_t}(y)\left(d\widetilde{W}_t + \left(\frac12 h^{\widetilde{D}_t}(y) - \frac{\mu^{\partial \widetilde{D}_t}(\partial \widetilde{D}_t)}{\mu(\widetilde{D}_t)}\right)\, dt\right), \quad \forall y \in \partial \widetilde{D}_t \tag{2.11}$$

where $\widetilde{W}_t$ is a real-valued Brownian motion.

Notice that the equation for $\widetilde{D}_t$ does no longer depend of X_t, so if the solution is unique, $(\widetilde{D}_t)_{t\geqslant 0}$ will be Markovian. It is Equation (51) in [6]. Theorem 39 of [6] proves local existence of a solution.

Theorem 2.5. *Fix $\alpha \in (0,1)$ and $\varepsilon > 0$. Then* (2.11) *admits a unique global solution. In particular the process $(\widetilde{D}_t)_{t\geqslant 0}$ is Markovian.*

Proof. The proof is a consequence of Theorem 9 in [6]. It can be found in Appendix C. □

To describe the generator $\widetilde{\mathscr{L}}$ of $(\widetilde{D}_t)_{t\geqslant 0}$ we must introduce the following notations. For any smooth function k on M, consider the mapping F_k on $\mathcal{D}^{2+\alpha}$ by

$$\forall\ D \in \mathcal{D}^{2+\alpha}, \qquad F_k(D) \quad := \quad \int_D k\, d\mu$$

For any $k, g \in \mathcal{C}^\infty(M)$ and any $D \in \mathcal{D}^{2+\alpha}$, define

$$\widetilde{\mathscr{L}}[F_k](D) \quad := \quad \underline{\mu}^{\partial D}(k)\frac{\underline{\mu}^{\partial D}(\partial D)}{\mu(D)} - \frac{1}{2}\underline{\mu}^{\partial D}(\langle \nabla k, N^D\rangle) \tag{2.12}$$

$$\Gamma_{\widetilde{\mathscr{L}}}[F_k, F_g](D) \quad := \quad \int_{\partial D} k\, d\underline{\mu} \int_{\partial D} g\, d\underline{\mu} \tag{2.13}$$

Next consider $\mathfrak{A}$ the algebra consisting of the functionals of the form $\mathfrak{F} := \mathfrak{f}(F_{k_1}, ..., F_{k_n})$, where $n \in \mathbb{Z}_+$, $k_1, ..., k_n \in \mathcal{C}^\infty(M)$ and $\mathfrak{f} : \mathcal{R} \to \mathbb{R}$ is a $\mathcal{C}^\infty$ mapping, with $\mathcal{R}$ an open subset of $\mathbb{R}^n$ containing the image of $\mathcal{D}^{2+\alpha}$ by $(F_{k_1}, ..., F_{k_n})$. For such a functional $\mathfrak{F}$, define

$$\begin{aligned}\widetilde{\mathscr{L}}[\mathfrak{F}] \quad := \quad & \sum_{l=1}^{n} \partial_j \mathfrak{f}(F_{k_1}, ..., F_{k_n})\widetilde{\mathscr{L}}[F_{k_l}] \\ & + \sum_{j,l\in[\![1,n]\!]} \partial_{j,l}\mathfrak{f}(F_{k_1}, ..., F_{k_n})\Gamma_{\widetilde{\mathscr{L}}}[F_{k_j}, F_{k_l}]\end{aligned} \tag{2.14}$$

To two elements of $\mathfrak{A}$, $\mathfrak{F} := \mathfrak{f}(F_{k_1}, ..., F_{k_n})$ and $\mathfrak{G} := \mathfrak{g}(F_{g_1}, ..., F_{g_m})$, we also associate

$$\Gamma_{\widetilde{\mathscr{L}}}[\mathfrak{F}, \mathfrak{G}] \quad := \sum_{l\in[\![n]\!], j\in[\![m]\!]} \partial_l \mathfrak{f}(F_{k_1}, ..., F_{k_n})\partial_j \mathfrak{g}(F_{g_1}, ..., F_{g_m})\Gamma_{\widetilde{\mathscr{L}}}[F_{k_l}, F_{g_j}] \tag{2.15}$$

Remark 2.6. To see that the above definitions are non-ambiguous, since a priori they could depend on the writing of $\mathfrak{F} \in \mathfrak{A}$ under the form $\mathfrak{f}(F_{k_1}, ..., F_{k_n})$ and similarly for $\mathfrak{G}$, see Remark 2 of [6]. More generally, the forms of (2.14) and (2.15) are consequences of the diffusion feature of $\widetilde{\mathscr{L}}$, for more on the subject, see e.g. the book of Bakry, Gentil and Ledoux [3].

Remark 2.7. In the above considerations, $\widetilde{\mathscr{L}}$ was defined on $\mathcal{D}^{2+\alpha}$, but from now on, $\widetilde{\mathscr{L}}$ will stand for the restriction of this generator to $\mathcal{F}^{\alpha,\varepsilon}$ and will be zero on $\mathcal{D}^{2+\alpha}\backslash\mathcal{F}^{\alpha,\varepsilon}$, in accordance with Definition 2.1. Similarly, all stochastic differential equations will be valid only up to the stopping time τ_ε.

The interest of Assumption 2.2 comes from the following result:

Theorem 2.8. *Let $(x, D) \mapsto f^D(x)$ and $(x, D) \mapsto \sigma_c^D(x)$ satisfy Assumption 2.2. Then equation* (2.10) *has a solution $(X_t, D_t)_{t\geqslant 0}$ started at $D_0 \in \mathscr{F}^{\alpha,\varepsilon}$, $X_0 \sim \mathscr{U}(D_0)$. Moreover the processes $(X_t)_{t\geqslant 0}$ and $(D_t)_{t\geqslant 0}$ are τ_ε-intertwined.*

Proof. We prove here the existence of solution to equation (2.10). The intertwining will be a consequence of Proposition 2.11 below.

We begin to prove the existence of a diffusion with modified drift, and then we will get the result by change of probability. The modified equation writes

$$\left\{\begin{array}{rcl} d\partial D_t(y) &=& N^{D_t}(y)\left(d\widehat{W}_t + \left(\frac{1}{2}h^{D_t}(y) - \frac{\mu^{\partial D_t}(\partial D_t)}{\mu(D_t)}\right)\,dt\right);\\ dX_t &=& \left(\nabla f^{D_t}(X_t)\left[d\widehat{W}_t - \left(\frac{\mu^{\partial D_t}(\partial D_t)}{\mu(D_t)} + \Delta f^{D_t}(X_t)\right)\,dt\right]\right.\\ && \left. +\sigma_{\mathrm{c}}^{D_t}(X_t)\,dW_t^m\right) \end{array}\right. \tag{2.16}$$

for $\widehat{W}_t$ and W_t^m independent Brownian motions. Notice that the first equation is the same as (2.11). Thus due to Assumption 2.5, $(D_t)_{t\geqslant 0}$ is a diffusion process with generator $\widetilde{\mathscr{L}}$. Then given D_t, the equation for X_t

$$\begin{aligned} dX_t = &\left(\nabla f^{D_t}(X_t)\left[d\widehat{W}_t - \left(\frac{\mu^{\partial D_t}(\partial D_t)}{\mu(D_t)} + \Delta f^{D_t}(X_t)\right)\,dt\right]\right.\\ &\left. + \sigma_{\mathrm{c}}^{D_t}(X_t)\,dW_t^m\right) \end{aligned} \tag{2.17}$$

can also be solved, since the coefficients in front of $d\widehat{W}_t$ and dW_t^m are Lipschitz, $\sigma^D(\sigma^D)^*(x) = \mathrm{Id}_{T_xM}$ and Δf^D is bounded and uniformly Hölder continuous (due to Assumption 2.2). Notice that X_t remains in D_t, since when $X_t \in \partial D_t$, we have, using (2.9) which yields on boundary $\langle N^{D_t}(X_t), \sigma_{\mathrm{c}}^{D_t}(X_t)dW_t^m\rangle = 0$,

$$\begin{aligned} &d(\rho^+_{\partial D_t}(X_t))\\ &= \langle \nabla\rho^+_{\partial D_t}, dX_t\rangle - \frac{1}{2}h^{D_t}(X_t)\,dt - \langle d\partial D_t(X_t), N^{D_t}(X_t)\rangle\\ &= \langle N^{D_t}(X_t), dX_t\rangle - \frac{1}{2}h^{D_t}(X_t)\,dt - \langle d\partial D_t(X_t), N^{D_t}(X_t)\rangle = 0. \end{aligned} \tag{2.18}$$

where we used (2.16) and (2.6). We also have no covariation since the martingale part of $d\partial D_t$ acts on the normal flow only, and any normal flow

$$r \mapsto D(r) := \{x \in M,\ \rho^+(x) \geqslant r\}$$

satisfies $\rho^+_{\partial D(r)}(x) = \rho^+_{\partial D(0)}(x) - r$ for $x \in D(0)$ and $|r|$ small, (see Appendix A).

Once we have a solution to (2.16), make by Girsanov theorem a change of probability such that (W_t, W_t^m) is a Brownian motion where

$$W_t := \widehat{W}_t - \int_0^t \left(\frac{\mu^{\partial D_s}(\partial D_s)}{\mu(D_s)} + \Delta f^{D_s}(X_s)\right)\,ds. \tag{2.19}$$

We get a solution to (2.10) in the new probability. □

Proposition 2.9. *Let D_t satisfy*

$$d\partial D_t(y) = N^{D_t}(y)\left(dW_t + \left(\frac{1}{2}h^{D_t}(y) + b_t\right)\,dt\right), \quad \forall y \in \partial D_t \tag{2.20}$$

for some Brownian motion W_t and some adapted locally bounded real-valued process b_t. Let $\mu_t = \mu^{D_t}$ be the Lebesgue measure on D_t and $\bar\mu_t = \bar\mu^{D_t} = \mathscr{U}(D_t) = \dfrac{\mu^{D_t}}{\mu(D_t)}$. Denote *by* $\underline{\mu}_t = \underline{\mu}^{\partial D_t}$ *the Lebesgue measure on* ∂D_t *and* $\bar{\underline{\mu}}_t = \bar{\underline{\mu}}^{\partial D_t} = \dfrac{\underline{\mu}^{\partial D_t}}{\mu(D_t)}$. *Let* k *be a smooth function of* M. *Then*

$$d\mu_t(k) = -\underline{\mu}_t(k)\, dW_t - \frac{1}{2}\left(2b_t\underline{\mu}_t(k) + \underline{\mu}_t\left(\langle dk, N^{D_t}\rangle\right)\right)\, dt \tag{2.21}$$

and

$$\begin{aligned} d\bar{\mu}_t(k) &= \left(-\bar{\underline{\mu}}_t(k) + \bar{\mu}_t(k)\bar{\underline{\mu}}_t(\partial D_t)\right)\, dW_t - \frac{1}{2}\bar{\underline{\mu}}_t\left(\langle dk, N^{D_t}\rangle\right)\, dt \\ &\quad + \left(\bar{\underline{\mu}}_t(\partial D_t) + b_t\right)\left(-\bar{\underline{\mu}}_t(k) + \bar{\mu}_t(k)\bar{\underline{\mu}}_t(\partial D_t)\right)\, dt \end{aligned} \tag{2.22}$$

In particular, if $b_t = -\bar{\underline{\mu}}_t(\partial D_t)$ *we get*

$$d\bar{\mu}_t(k) = \left(-\bar{\underline{\mu}}_t(k) + \bar{\mu}_t(k)\bar{\underline{\mu}}_t(\partial D_t)\right)\, dW_t - \frac{1}{2}\bar{\underline{\mu}}_t\left(\langle dk, N^{D_t}\rangle\right)\, dt. \tag{2.23}$$

Proof. For f a smooth function on M and $r \in \mathbb{R}$ sufficiently close to 0 so that $\partial D(r)$ (defined in (A.3) and (A.4)) is a smooth manifold without boundary, let

$$F(r,k) = \int_{D(r)} k\, d\mu. \tag{2.24}$$

We have

$$F(r,k) = \int_{\partial D}\left(\int_r^{\tau(y)} k\left(\psi(s)(y)\right) e^{-\int_0^s h^{D_t}(\psi(u)(y))\, du}\, ds\right)\, \underline{\mu}(dy) \tag{2.25}$$

with $\tau(y)$ the hitting time of $S(D)$ by the inward normal flow started at y (defined in (A.1)) and $\psi(s)(y) = \psi(0,s)(y) = \exp_y(sN_y)$ defined in (A.5). With this formulation we can differentiate with respect to r, to obtain

$$F'(r,k) = -\int_{\partial D} k(\psi(r,y)) e^{-\int_0^r h^{D_t}(\psi(s)(y)\, ds}\, \underline{\mu}(dy). \tag{2.26}$$

Differentiating again we get

$$F''(r,k) = -\int_{\partial D}\left(\langle dk, \partial_r\psi(r,y)\rangle - (kh)(\psi(r,y))\right) e^{-\int_0^r h^{D_t}(\psi(s)(y))\, ds}\, \underline{\mu}(dy). \tag{2.27}$$

In particular,

$$F'(0,k) = -\underline{\mu}(k) \quad \text{and} \quad F''(0,k) = \underline{\mu}(kh - \langle dk, N\rangle). \tag{2.28}$$

This allows us to compute

$$d(F(W_t,k)) = F'(W_t,k)\, dW_t + \frac{1}{2}F''(W_t,k)\, dt \tag{2.29}$$

and then, since dW_t and $\langle d\partial D_t, N^{D_t}\rangle(\cdot)$ differ only by a finite variation process

$$d\mu_t(k) = \int_{\partial D_t} -k(y)\langle d\partial D_t(y), N^{D_t}(y)\rangle + \frac{1}{2}\left(kh^{D_t} - \langle dk, N^{D_t}\rangle\right)(y)\, \underline{\mu}_t(dy). \tag{2.30}$$

This yields

$$d\mu_t(k) = \int_{\partial D_t} k(y)\left(-dW_t - b_t\, dt\right) - \frac{1}{2}\langle dk, N^{D_t}\rangle(y)\, \underline{\mu}_t(dy)\, dt, \tag{2.31}$$

which gives (2.21). In particular, taking $f \equiv 1$ we obtain

$$d\mu(D_t) = \underline{\mu}_t(\partial D_t)\left(-dW_t - b_t\, dt\right). \tag{2.32}$$

Now we can compute

$$
\begin{aligned}
&d\bar\mu_t(k)\\
&= d\left(\frac{\mu_t(k)}{\mu(D_t)}\right)\\
&= \frac{1}{\mu(D_t)}d\mu_t(k) - \frac{\mu_t(k)}{\mu(D_t)^2}d\mu(D_t) + \frac{\mu_t(k)}{\mu(D_t)^3}d\langle \mu(D_\cdot)\rangle_t - \frac{1}{\mu(D_t)^2}d\langle \mu_\cdot(k), \mu(D_\cdot)\rangle_t\\
&= \frac{1}{\mu(D_t)}d\mu_t(k) - \frac{\mu_t(k)}{\mu(D_t)^2}d\mu(D_t) + \frac{\mu_t(k)}{\mu(D_t)^3}\underline{\mu}(\partial D_t)^2 dt - \frac{1}{\mu(D_t)^2}\underline{\mu}_t(k)\underline{\mu}_t(\partial D_t)dt\\
&= -\bar{\underline{\mu}}_t(k)\left(dW_t + b_t\,dt\right) - \frac{1}{2}\bar{\underline{\mu}}_t(\langle dk, N^{D_t}\rangle)\,dt + \bar\mu_t(k)\bar{\underline{\mu}}_t(\partial D_t)(dW_t + b_t\,dt)\\
&+ \bar\mu_t(k)\bar{\underline{\mu}}_t(\partial D_t)^2\,dt - \bar{\underline{\mu}}_t(k)\bar{\underline{\mu}}_t(\partial D_t)\,dt.
\end{aligned}
$$

This yields (2.22). □

Denote τ_ε the exiting time of $(D_t)_{t\geqslant 0}$ from $\mathcal{F}^{\alpha,\varepsilon}$. As in Definition 2.1, we stop $(X_t, D_t)_{t\geqslant 0}$ at τ_ε.

Proposition 2.10. *Any solution of equation* (2.10) *stopped at* τ_ε *is a Markov process solution to a martingale problem associated to a generator* $\mathscr{L}$ *acting in the following way: for any* g, k *smooth functions on* M *and*

$$
F_k(D) := \int_D k\,d\mu, \tag{2.33}
$$

we have for $(x, D) \in M \times \mathcal{F}^{\alpha,\varepsilon}$,

(2.34)

$$
\begin{aligned}
\mathscr{L}(gF_k)(x, D) &= -g(x)\Delta f^D(x)\underline{\mu}^{\partial D}(k) - \frac{1}{2}g(x)\underline{\mu}^{\partial D}(\langle \nabla k, N^D\rangle) + \frac{1}{2}F_k(D)\Delta g(x)\\
&\quad - \underline{\mu}^{\partial D}(k)\langle \nabla g, \nabla f^D\rangle(x).
\end{aligned}
$$

Proof. From (2.10) and (2.21) with $b_t = \Delta f^{D_t}(X_t)$ we have

$$
dF_k(D_t) = -\,\underline{\mu}^{\partial D_t}(k)\left(dW_t + \Delta f^{D_t}(X_t)\,dt\right) - \frac{1}{2}\underline{\mu}^{\partial D_t}\left(\langle \nabla k, N^{D_t}\rangle\right)\,dt. \tag{2.35}
$$

This implies that

$$
\mathscr{L}(F_k)(x, D) = -\underline{\mu}^{\partial D}(k)\Delta f^D(x) - \frac{1}{2}\underline{\mu}^{\partial D}\left(\langle \nabla k, N^D\rangle\right), \tag{2.36}
$$

and the covariation of $g(X_t)$ and $F_k(D_t)$ is $\Gamma_{\mathscr{L}}[g, F_k](X_t, D_t)\,dt$ with

$$
\Gamma_{\mathscr{L}}[g, F_k](x, D) = -\underline{\mu}^{\partial D}(k)\langle \nabla g, \nabla f^D\rangle(x). \tag{2.37}
$$

Consequently, using

$$
\mathscr{L}(gF_k)(x, D) = g(x)\mathscr{L}(F_k)(x, D) + F_k(D)\frac{1}{2}\Delta g(x) + \Gamma_{\mathscr{L}}[g, F_k](x, D) \tag{2.38}
$$

we get (2.34). □

It is possible to extend the description of $\mathscr{L}$ to more general functions on $M \times \mathcal{F}^{\alpha,\varepsilon}$ (it vanishes on its complementary set), by replacing F_k in (2.34) by a mapping $\mathfrak{F}$ from $\mathfrak{A}$, as presented before Theorem 2.8.

Let $(\mathscr{P}_t)_{t\geqslant 0}$ be the Markovian semi-group associated to the processes $(X_t, D_t)_{t\geqslant 0}$ solution to (2.10) stopped at τ_ε. This semi-group is associated to $\mathscr{L}$ in the weak sense of martingale problems, as described in Appendix D.

Let $(\widetilde{D}_t)_{t\geqslant 0}$ be a diffusion process with generator $\widetilde{\mathscr{L}}$ stopped outside $\mathcal{F}^{\alpha,\varepsilon}$, started at $\widetilde{D}_0 = D_0$ (due to Theorem 2.5, this process can be obtained as a solution to the evolution equation (2.11)), $\widetilde{\nu}_t$ its law at time t and

$$\nu_t(dD, dx) = \widetilde{\nu}_t(dD)\mathscr{U}(D)(dx). \tag{2.39}$$

Proposition 2.11. *We have for all smooth functions g, k on M:*

$$\partial_t \nu_t(gF_k) = \nu_t(\mathscr{L}(gF_k)). \tag{2.40}$$

As a consequence, if (D_0, X_0) has law ν_0 then for all $t \geqslant 0$, the solution (D_t, X_t) to equation (2.10) *has law ν_t, implying that $(X_t)_{t\geqslant 0}$ and $(D_t)_{t\geqslant 0}$ are τ_ε-intertwined. Moreover D_t is a diffusion with generator $\widetilde{\mathscr{L}}$.*

Proof. Integrating (2.34) in x with respect to the uniform law $\bar{\mu}^D := \mathscr{U}(D)$ in D yields
(2.41)

$$-\bar{\mu}^D\left(g\Delta f^D\right)\underline{\mu}^{\partial D}(k) - \frac{1}{2}\bar{\mu}^D(g)\underline{\mu}^{\partial D}(\langle\nabla k, N^D\rangle) + \frac{1}{2}F_k(D)\bar{\mu}^D(\Delta g) - \underline{\mu}^{\partial D}(k)\bar{\mu}^D(\langle\nabla g, \nabla f^D\rangle).$$

By Stokes theorem,

$$\bar{\mu}^D\left(g\Delta f^D + \langle\nabla g, \nabla f^D\rangle\right) = \underline{\bar{\mu}}^{\partial D}\left(g\langle\nabla f^D, -N^D\rangle\right) = -\underline{\bar{\mu}}^{\partial D}(g), \tag{2.42}$$

so the expression (2.41) writes

$$H(D) := \underline{\mu}^{\partial D}(k)\underline{\bar{\mu}}^{\partial D}(g) - \frac{1}{2}\bar{\mu}^D(g)\underline{\mu}^{\partial D}(\langle\nabla k, N^D\rangle) + \frac{1}{2}F_k(D)\bar{\mu}^D(\Delta g) \tag{2.43}$$

On the other hand

$$\nu_t(gF_k) = \widetilde{\nu}_t[\bar{\mu}^{D_t}[g]F_k] \tag{2.44}$$

which implies that

$$\partial_t \nu_t(gF_k) = \partial_t\widetilde{\nu}_t((\bar{\mu}^{D_t}(g)F_k) = \widetilde{\nu}_t\left(\widetilde{\mathscr{L}}\left(\bar{\mu}^{D_t}(g)F_k\right)\right). \tag{2.45}$$

By (2.23),

$$\widetilde{\mathscr{L}}\left(\bar{\mu}^{D_t}(g)\right) = -\frac{1}{2}\underline{\bar{\mu}}^{\partial D_t}(\langle\nabla g, N^{D_t}\rangle), \tag{2.46}$$

so, taking into account (2.13),

$$\begin{aligned}
&\widetilde{\mathscr{L}}\left(\bar{\mu}^{D_t}(g)F_k\right)\\
&= \bar{\mu}^{D_t}(g)\widetilde{\mathscr{L}}(F_k) + F_k\widetilde{\mathscr{L}}\left(\bar{\mu}^{D_t}(g)\right) + \Gamma_{\widetilde{\mathscr{L}}}\left[\bar{\mu}^{D_t}(g), F_k\right]\\
&= \bar{\mu}^{D_t}(g)\Big\{\underline{\mu}^{\partial D_t}(k)\underline{\bar{\mu}}^{\partial D_t}(\partial D_t) - \frac{1}{2}\underline{\mu}^{\partial D_t}(\langle\nabla k, N^{D_t}\rangle)\Big\} - \frac{1}{2}\mu^{D_t}(k)\underline{\bar{\mu}}^{\partial D_t}(\langle\nabla g, N^{D_t}\rangle)\\
&- \left(-\underline{\bar{\mu}}^{\partial D_t}(g) + \bar{\mu}^{D_t}(g)\underline{\bar{\mu}}^{\partial D_t}(\partial D_t)\right)\underline{\mu}^{\partial D_t}(k)\\
&= -\frac{1}{2}\bar{\mu}^{D_t}(g)\underline{\mu}^{\partial D_t}(\langle\nabla k, N^{D_t}\rangle) - \frac{1}{2}\mu^{D_t}(k)\underline{\bar{\mu}}^{\partial D_t}(\langle\nabla g, N^{D_t}\rangle) + \underline{\bar{\mu}}^{\partial D_t}(g)\underline{\mu}^{\partial D_t}(k).
\end{aligned}$$

But $\bar{\mu}^{D_t}(\Delta g) = -\underline{\bar{\mu}}^{\partial D_t}(\langle\nabla g, N^{D_t}\rangle)$ and $F_k(D_t) = \mu^{D_t}(k)$, so

$$H(D_t) = \widetilde{\mathscr{L}}\left(\bar{\mu}^{D_t}(g)F_k\right), \tag{2.47}$$

which together with (2.45) proves (2.40).

Let us now prove that for any $t \geqslant 0$, $\mathscr{P}_t$ transports ν_0 into ν_t, where $(\mathscr{P}_t)_{t\geqslant 0}$ is the semi-group introduced after the proof of Proposition 2.10. Consider the map

$$G(g, k, t)(s) = \nu_s\left(\mathscr{P}_{t-s}(gF_k)\right), \quad s \in [0, t]. \tag{2.48}$$

We compute

$$\begin{aligned} G(g,k,t)'(s) &= (\partial_s\nu_s)\left(\mathscr{P}_{t-s}(gF_k)\right) - \nu_s\left(\partial_t\mathscr{P}_{t-s}(gF_k)\right) \\ &= \nu_s\left(\mathscr{L}\mathscr{P}_{t-s}(gF_k)\right) - \nu_s\left(\mathscr{L}\mathscr{P}_{t-s}(gF_k)\right) = 0 \end{aligned} \tag{2.49}$$

where we used Proposition D.3 in Appendix D to justify the differentiations (as well as the fact that $\mathscr{L}\mathscr{P}_{t-s}(gF_k) = \mathscr{P}_{t-s}\mathscr{L}(gF_k)$ is bounded to be able to use differentiation under the integral ν_s). So we get $G(g,k,t)(0) = G(g,k,t)(t)$ which rewrites as

$$\nu_0\mathscr{P}_t(gF_k) = \nu_t(gF_k), \tag{2.50}$$

More generally, by similar arguments, we can replace in this formula F_k by any mapping $\mathfrak{F}$ from $\mathfrak{A}$. This in turn implies that $\nu_0\mathscr{P}_t = \nu_t$.

To finish, by iteration, we see that if $X_0 \sim \bar{\mu}^{D_0}$ then $(D_t)_{t\geqslant 0}$ has the same finite time marginals as $(\widetilde{D}_t)_{t\geqslant 0}$, proving that (D_t) is a diffusion with generator $\widetilde{\mathscr{L}}$. □

3. Intertwined dual processes: a generalized Pitman theorem

In this section we will consider the case where f^D is the distance to boundary. It is not covered by Section 2 since distance to boundary is not smooth, it is singular on the skeleton of D. We will make an approximation of it, and then go to the limit in law.

Let $\widetilde{W}_t$ be a real-valued Brownian motion and $\widetilde{D}_t$ be the solution of (2.11) started at $\widetilde{D}_0$, with driving Brownian motion $\widetilde{W}_t$.

Assumption 3.1. Fix $\alpha \in (0,1)$ and $\varepsilon > 0$. Let $\widetilde{D}_0 \in \mathcal{F}^{\alpha,\varepsilon}$. There exists a closed bounded subset $\widetilde{\mathcal{F}}^{\alpha,\varepsilon}$ of $\mathcal{F}^{\alpha,\varepsilon}$ in which the process $(\widetilde{D}_t)_{t\geqslant 0}$ a.s. takes its values, such that the map $D \mapsto S(D)$ is uniformly Lipschitz from $\widetilde{\mathcal{F}}^{\alpha,\varepsilon}$ with the C^2 metric to $\mathcal{K}(M)$, the set of compact subsets of M endowed with the Hausdorff metric. Moreover all skeletons $S(D)$ of elements $D \in \widetilde{\mathcal{F}}^{\alpha,\varepsilon}$ have uniformly bounded $(d-1)$-Hausdorff measure, and their regular part have uniformly bounded sectional curvature.

Conjecture 3.2. *We conjecture that Assumption 3.1 is always realized, for any $\alpha \in (0,1)$, $\varepsilon > 0$, $\widetilde{D}_0 \in \mathcal{F}^{\alpha,\varepsilon}$.*

All examples together with the study of the motion of the skeleton in Appendix B make us believe that Conjecture 3.2 is true. However a better knowledge of skeletons is necessary to solve it.

Let us begin with some preparatory results. To describe the approximation of $\rho(x,\partial D)$ we are interested in, let us introduce some notations.

• Let $(x,D) \mapsto \ell_\varepsilon(x,D) := (h_\varepsilon \circ \rho_{\partial D})(x)$ where $h_\varepsilon \equiv 1$ in $[0,\varepsilon/2]$, $h_\varepsilon \equiv 0$ in $[3\varepsilon/4,\infty)$ and h_ε is smooth and nonincreasing in $[0,\infty)$. When D is fixed by the context, we will denote $\ell_\varepsilon(x) := \ell_\varepsilon(x,D)$.

• For any $\delta \in (0,\varepsilon)$, let $\varphi_\delta : \mathbb{R}_+ \to \mathbb{R}$ be a nonnegative function with support in $[0,\delta]$, such that the mapping $\mathbb{R}^d \ni u \mapsto \varphi_d(|u|)$is smooth and $\int_{\mathbb{R}^d}\varphi_d(|u|)\,du = 1$ (in the sequel, $|\cdot|$ will stand for the usual Euclidean norm or for the Riemannian norm on any tangent space of M, depending on the context) .

• Let g_δ be a smooth, 1-Lipschitz and odd function defined on $\mathbb{R}$, with $g_\delta(r) = r$ on $[0,\varepsilon/4]$, $0 \leqslant g_\delta(r) \leqslant r$ for any $r \geqslant 0$, and $g_\delta(r) = c_\delta r$ on $[3\varepsilon/8,\infty)$, for an appropriate constant $c_\delta \leqslant 1$ very close to 1 that will be defined below in (3.2). We write $\rho_\delta(x,\partial D) := g_\delta(\rho(x,\partial D))$.

The approximation of $\rho(x, \partial D)$ we choose is

(3.1)
$$f_\delta(x, D) = \ell_\varepsilon(x, D)\rho_\delta(x, \partial D) + (1 - \ell_\varepsilon(x, D)) \int_{T_xM} \varphi_\delta(|v|)\rho_\delta(\exp_x(v), \partial D)\, dv$$

(where dv stands for the Lebesgue measure on T_xM).

Since we will need to differentiate f_δ, in particular with respect to the first variable (the corresponding gradient will be denoted ∇_1), it is more convenient to replace the last term of $f_\delta(x, D)$ by an integral over the fixed domain $\mathbb{R}^d$. More precisely, for any linear isometry (an orthonormal frame) $\imath_x : \mathbb{R}^d \to T_xM$, we have

$$\int_{T_xM} \varphi_\delta(|v|)\rho_\delta(\exp_x(v), \partial D)\, dv \quad = \quad \int_{\mathbb{R}^d} \varphi_\delta(|u|)\rho_\delta(\exp_x(\imath_x(u)), \partial D)\, du$$

In this purpose, for any $x_0, x \in M$, let be given a linear isometry (an orthonormal frame) $\imath_{x_0,x} : \mathbb{R}^d \to T_xM$, smooth with respect to x_0, x, and such that $\nabla_x \imath_{x_0,x}|_{x=x_0} = 0$. Define

$$e(\delta) \quad := \quad \sup\{|\!|\!|d_x[\exp_x(\imath_{x_0,x}(u))|_{x=x_0}]|\!|\!|,\ x_0 \in B(o, R),\ u \in B(0, \delta) \subset \mathbb{R}^d\}$$

where $|\!|\!| \cdot |\!|\!|$ is the operator norm, when $T_{x_0}M$ is endowed with its Euclidean structure. Recall that $R = 1/\varepsilon$ and that ε is fixed as in Assumption 3.1. The previously mentioned constant c_δ is given by

$$c_\delta \quad := \quad e^{-1}(\delta)\,(1 - \delta\|\nabla_1\ell_\varepsilon\|_\infty) \tag{3.2}$$

Notice that c_δ does not depend on D and is as close as we want to 1.

More precisely, we have

Lemma 3.3. *There exists two constants $C_1', C_1'' > 0$, depending only on ε, such that for $\delta > 0$ sufficiently small,*

$$\begin{aligned} 0 \leqslant e(\delta) - 1 &\leqslant C_1'\delta \\ |c_\delta - 1| &\leqslant C_1''\delta \end{aligned}$$

Proof. The first inequalities are a consequence of $\nabla_x \imath_{x_0,x}|_{x=x_0} = 0$ and of the properties of the exponential mapping. The second bound follows, since $\|\nabla_1\ell_\varepsilon\|_\infty = \|h_\varepsilon'\|_\infty$ is independent of D (and of order $1/\varepsilon$). □

From the second bound, we can and will assume that the function g_δ is furthermore chosen so that $g_\delta(r)$ converges uniformly to r on compact sets of $\mathbb{R}_+$, as well as the corresponding derivatives up to order 2 as $\delta \searrow 0$. In addition, we choose $\delta > 0$ sufficiently small so that the map $(x, y) \mapsto \exp_x^{-1}(y)$ is well-defined and smooth in the δ-neighborhood the diagonal of $B(o, R) \times B(o, R)$. Then, for any $x_0 \in M$, we can rewrite (3.1) under the forms

(3.3)
$$\begin{aligned} f_\delta(x, D) &= \ell_\varepsilon(x, D)\rho_\delta(x, \partial D) + (1 - \ell_\varepsilon(x, D)) \int_{\mathbb{R}^d} \varphi_\delta(|u|)\rho_\delta(\exp_{x_0,x}(\imath_{x_0,x}(u)), \partial D)\, du \\ &= \ell_\varepsilon(x, D)\rho_\delta(x, \partial D) \\ &\quad + (1 - \ell_\varepsilon(x, D)) \int_M \varphi_\delta(|(\exp_x \circ \imath_{x_0,x})^{-1}(y)|)\rho_\delta(y, \partial D)\, J(\exp_x \circ \imath_{x_0,x})^{-1}(y) dy, \end{aligned}$$

where $J(\exp_x \circ \imath_{x_0,x})^{-1}$ is the absolute value of the Jacobian of $(\exp_x \circ \imath_{x_0,x})^{-1}$.

The interest of all these preparations is:

Proposition 3.4. *For all $\delta > 0$, the function $(x, D) \mapsto f_\delta(x, D) := f_\delta^D(x)$ has the following properties*

- *f_δ satisfies the conditions of Assumption 2.2;*
- *there exists $C_1 > 0$ such that $\forall D \in \widetilde{\mathcal{F}}^{\alpha,\varepsilon}$ and $x \in D$, we have*

$$|f_\delta(x, D) - \rho(x, \partial D)| \leqslant C_1\delta; \tag{3.4}$$

- *the gradient and the Hessian of f_δ with respect to the second variable D satisfy $\forall D \in \widetilde{\mathcal{F}}^{\alpha,\varepsilon}$, $\forall x \in D\backslash S(D)$, for all vector fields K normal to ∂D:*

$$\langle \nabla_2 f_\delta(x, D), K\rangle \leqslant C_4\|K\|_\infty \quad \text{and} \quad \|\nabla_2 d_2 f_\delta(x, D)\,(N_{\partial D}, N_{\partial D})\| \leqslant C_4 \tag{3.5}$$

for a C_4 not depending on x, D, δ.

Proof. We first prove $\|\nabla_1 f_\delta(x, D)\| \leqslant 1$. Fix $x_0 \in B(o, R)$ and differentiate with respect to x at x_0, we have

$$\begin{aligned}\nabla_1 f_\delta(x, D) =& \ell_\varepsilon(x, D)\nabla_1\rho_\delta(x, \partial D)\\ &+ (1 - \ell_\varepsilon(x, D))\int_{\mathbb{R}^d}\varphi_\delta(|u|)\nabla_1\rho_\delta((\exp_x \circ \imath_{x_0,x})(u), \partial D)\, du\\ &+ \nabla_1\ell_\varepsilon(x, D)\int_{\mathbb{R}^d}\varphi_\delta(|u|)\left(\rho_\delta(x, \partial D) - \rho_\delta((\exp_x \circ \imath_{x_0,x})(u), \partial D)\right)\, du.\end{aligned} \tag{3.6}$$

If $\rho(x, \partial D) \leqslant \varepsilon/2$ then $\ell_\varepsilon(x, D) = 1$, $\nabla\ell_\varepsilon(x, D) = 0$ and

$$\|\nabla_1 f_\delta(x, D)\| \leqslant \ell_\varepsilon(x, D)\|\nabla_1\rho_\delta(x, \partial D)\| \leqslant 1.$$

If $\rho(x, \partial D) \geqslant \varepsilon/2$ then for $\delta \leqslant \varepsilon/8$, we have, for $u \in \mathbb{R}^d$ with $|u| \leqslant \delta$, $\rho(\exp_x \circ \imath_{x_0,x})(u), \partial D) \geqslant 3\varepsilon/8$. It follows

$$\begin{aligned}\|\nabla_1 f_\delta(x, D)\| \leqslant& \ell_\varepsilon(x)e^{-1}(\delta)\,(1 - \delta\|\nabla_1\ell_\varepsilon\|_\infty)\\ &+ (1 - \ell_\varepsilon(x))\int_{\mathbb{R}^d}\varphi_\delta(|u|)c_d\|\nabla_x(\exp_x \circ \imath_{x_0,x})(u)\|\, du\\ &+ \|\nabla_1\ell_\varepsilon(x)\|_\infty\int_{\mathbb{R}^d}\varphi_\delta(|u|)\delta\, du\\ \leqslant& 1, \quad \text{using } \nabla_x\imath_{x_0,x} = 0.\end{aligned}$$

It is easily checked that the function f_δ satisfies the other properties of Assumption 2.2. Let us check that it also satisfies (3.4).

We have

$$f_\delta(x, D) - \rho_\delta(x, \partial D) = (1 - \ell_\varepsilon(x, D))\int_{\mathbb{R}^d}\varphi_\delta(|u|)\left(\rho_\delta(\exp_x(\imath_{x,x}(u)), \partial D) - \rho_\delta(x, \partial D)\right)\, du \tag{3.7}$$

which implies

$$|f_\delta(x, D) - \rho_\delta(x, \partial D)| \leqslant \delta.$$

On the other hand

$$|\rho(x, \partial D) - \rho_\delta(x, \partial D)| \leqslant \left(1 - e^{-1}(\delta)\,(1 - c_\delta)\right)2R \leqslant C_1''''\delta$$

for some constant $C_1'''' > 0$ (depending on ε). This yields (3.4) with $C_1 := 1 + C_1''''$.

For proving (3.5), we take a vector field $K(y) = k(y)N(y)$, $y \in \partial D$ and compute

$$\langle \nabla_2\rho(x, \partial D), K\rangle = \langle -N(P(x)), K(P(x))\rangle = -k(P(x)) \tag{3.8}$$

where $P(x)$ is the projection of x onto ∂D, and

$$\nabla_2 d_2\rho(x, \partial D)\,(N_{\partial D}, N_{\partial D}) = 0. \tag{3.9}$$

Remarking that $\|\nabla_2 \ell_\varepsilon(x, D)\|$ is bounded by $\|h'_\varepsilon\|_\infty$, we get (3.5) via a straightforward computation. □

Theorem 3.5. *Fix $D_0 = \tilde{D}_0 \in \widetilde{\mathcal{F}}^{\alpha,\varepsilon}$ and let $X_0 \sim \mathscr{U}(D_0)$. Under Assumption 3.1, there exists a pair $(X_t, D_t)_{t\geqslant 0}$ of τ_ε intertwined processes in the sense of Definition 1.1, such that the process $(D_t)_{t\geqslant 0}$ satisfies*

(3.10)

$$d\partial D_t(y) = N^{D_t}(y)\left(\left\langle dX_t, N^{D_t}(X_t)\right\rangle + \left(\frac{1}{2}h^{D_t}(y) - h^{D_t}(X_t)\right)\,dt - 2\sin(\theta^{S_t}(X_t))\,dL_t^{S_t}(X)\right)$$

Here $\theta^{S_t}(x) = \pi/2 - \varphi^{S_t}(x)$, $\varphi^{S_t}(x)$ being the angle between the orthogonal line to S_t at x and any of the two minimal geodesics from ∂D_t to $x \in S_t$. In other words $\theta^{S_t}(x)$ is the smallest angle between S_t and the geodesics. The process L^{S_t} is the local time of X_t at $S_t := S(D_t)$:

$$L_t^{S_t}(X) = \lim_{\beta\searrow 0}\frac{1}{2\beta}\int_0^t 1_{\{X_s\in S_s^\beta\}}\,ds, \tag{3.11}$$

S_s^β being the thickening of the regular part of S_s in normal direction, of thickness β in both directions.

Remark 3.6.
- Compared to Section 2 with f^D replaced by distance to boundary $\rho_{\partial D}$, we have outside the skeleton S^D

$$\nabla\rho_{\partial D}(x) = N^D(x) \quad\text{and}\quad \Delta\rho_{\partial D}(x) = -h^D(x) \tag{3.12}$$

and we will see that on the moving skeleton $S_t = S^{D_t}$:

$$\text{“}\Delta\rho_{\partial D_t}(X_t)\,dt\text{”} = -2\sin(\theta^{S_t}(X_t))\,dL_t^{S_t}(X). \tag{3.13}$$

Proof. Under Assumption 3.1, Proposition 3.4 allows us to construct for each $\delta > 0$, intertwined processes $(X_t^\delta, D_t^\delta)_{t\geqslant 0}$ started at $(X_0^\delta, D_0^\delta) = (X_0, D_0)$, associated with the functions f_δ^D, stopped at τ_ε^δ, the exit time from $\widetilde{\mathcal{F}}^{\alpha,\varepsilon}$. We have from Equation (2.10)

$$d\partial D_t^\delta(y) = N^{D_t^\delta}(y)\left(dW_t^\delta + \left(\frac{1}{2}h^{D_t^\delta}(y) + \Delta f^{D_t^\delta}(X_t^\delta)\right)\,dt\right) \tag{3.14}$$

for some Brownian motion W_t^δ. On the other hand, from Proposition 2.11 and (2.1),

$$(\widetilde{D}_t^\delta)_{t\geqslant 0} := (D_t^\delta)_{t\geqslant 0} \tag{3.15}$$

satisfies equation (2.11):

$$d\partial D_t^\delta(y) = N^{D_t^\delta}(y)\left(d\widetilde{W}_t^\delta + \left(\frac{1}{2}h^{D_t^\delta}(y) - \frac{\underline{\mu}^{\partial D_t^\delta}(\partial D_t^\delta)}{\mu(D_t^\delta)}\right)\,dt\right) \tag{3.16}$$

where $\widetilde{W}_t^\delta$ is the $\mathscr{F}_t^{D^\delta}-$Brownian motion

$$d\widetilde{W}_t^\delta = dW_t^\delta + \Delta f^{D_t^\delta}(X_t)\,dt + \frac{\underline{\mu}^{\partial D_t^\delta}(\partial D_t^\delta)}{\mu(D_t^\delta)}\,dt. \tag{3.17}$$

A remarkable fact about all $(X_t^\delta, D_t^\delta)_{t\geqslant 0}$ is that their marginals are constant in law. Notice that also $((D_t^\delta)_{t\geqslant 0}, \tau_\varepsilon^\delta)$ is constant in law since τ_ε^δ is a functional of $(D_t^\delta)_{t\geqslant 0}$ independent of δ. As a consequence, the family

$$\Big((X_t^\delta, D_t^\delta, W_t^\delta, \widetilde{W}_t^\delta, W_t^{\delta,m})_{t\geqslant 0}, \tau_\varepsilon^\delta\Big) \tag{3.18}$$

is tight (in (3.18) the Brownian motions W_t^δ and $W_t^{\delta,m}$ are the ones defined by equation (2.10)). Denote by

$$\Big((X_t, D_t, W_t, \widetilde{W}_t, W_t^{m})_{t\geqslant 0}, \tau_\varepsilon\Big) \tag{3.19}$$

a limiting point. Let us prove the intertwining. Using Proposition 2.11, for any smooth functions g and k on M,

$$\begin{aligned}
\mathbb{E}[g(X_t^\delta)F_k(D_t^\delta)] &= \mathbb{E}[\mathbb{E}[g(X_t^\delta)F_k(D_t^\delta)|\mathscr{F}_t^{D^\delta}]]\\
&= \mathbb{E}[\mathscr{U}(D_t^\delta)(g)F_k(D_t^\delta)]\\
&= \mathbb{E}\left[\frac{F_g(D_t^\delta)}{F_1(D_t^\delta)}F_k(D_t^\delta)\right]
\end{aligned}$$

and passing to the limit yields the intertwining.

This property of $(D_t^\delta, \widetilde{W}_t^\delta)_{t\geqslant 0}$ being constant in law passes to the limit, and we have

$$d\partial D_t(y) = N^{D_t}(y)\left(d\widetilde{W}_t + \left(\frac{1}{2}h^{D_t}(y) - \frac{\underline{\mu}^{\partial D_t}(\partial D_t)}{\mu(D_t)}\right)\,dt\right). \tag{3.20}$$

We need to work with real-valued processes: we have from (2.32), for all $\delta > 0$,

$$\int_0^t \frac{d\mu(D_s^\delta)}{\underline{\mu}(\partial D_s^\delta)} = -W_t^\delta - \int_0^t \Delta_1 f_\delta(X_s^\delta, D_s^\delta)\,ds. \tag{3.21}$$

This together with (3.17) yields

$$d\partial D_t^\delta(y) = N^{D_t^\delta}(y)\left(-\frac{d\mu(D_s^\delta)}{\underline{\mu}(\partial D_s^\delta)} + \frac{1}{2}h^{D_t^\delta}(y)\,dt\right) \tag{3.22}$$

Again by constantness in law:

$$d\partial D_t(y) = N^{D_t}(y)\left(-\frac{d\mu(D_s)}{\underline{\mu}(\partial D_s)} + \frac{1}{2}h^{D_t}(y)\,dt\right). \tag{3.23}$$

So to prove our result we only need to prove that

$$\int_0^t \frac{d\mu(D_s)}{\underline{\mu}(\partial D_s)} = -W_t + \int_0^t h^{D_s}(X_s)\,ds + \int_0^t 2\sin\left(\theta^{S_s}(X_s)\right)\,dL_s^{S_s}(X) \tag{3.24}$$

and that

$$W_t = \int_0^t \langle N^{D_s}(X_s), dX_s\rangle. \tag{3.25}$$

Let us prove (3.25). In all this paragraph we consider M as isometrically embedded in some Euclidean space. In particular we are allowed to integrate vectorial quantities. We use the fact that $dX_t^\delta \otimes dW_t^\delta$ converges in law to $dX_t \otimes dW_t$ (where $\otimes$ stands for bracket of semimartingales). But $dX_t^\delta \otimes dW_t^\delta$ is equal to $\nabla_1 f_\delta(X_t^\delta, D_t^\delta)\,dt$. Then by Lemma H.1 applied to $\nabla_1 f_\delta(X_t^\delta, D_t^\delta)$ (which is uniformly bounded) and $U = \{(x, D),\ x \notin S(D)\}$ defined in (H.3) we see that the integral of $\nabla_1 f_\delta(X_t^\delta, D_t^\delta)\,dt$ converges to the one of

$N^{D_t}(X_t)\,dt$. But almost surely $N^{D_t}(X_t)$ has norm 1 dt-a.e., implying that $dW_t = \langle N^{D_t}(X_t), dX_t\rangle$.

Let us now establish (3.24). It will be a consequence of the convergence of $(f_\delta(X_t^\delta, D_t^\delta))_{t\geqslant 0}$ to $(\rho(X_t, \partial D_t)_{t\geqslant 0}$.

Write the Itô formula for $f_\delta(X_t^\delta, D_t^\delta)$:

(3.26)

$$\begin{aligned}
d\left(f_\delta(X_t^\delta, D_t^\delta)\right) =& \langle \nabla_1 f_\delta(X_t^\delta, D_t^\delta), dX_t^\delta\rangle + \frac{1}{2}\Delta_1 f_\delta(X_t^\delta, D_t^\delta)\,dt\\
&+\langle \nabla_2 f_\delta(X_t^\delta, D_t^\delta), d\partial D_t^\delta\rangle + \frac{1}{2}\nabla_2 d_2 f_\delta(X_t^\delta, D_t^\delta)(d\partial D_t^\delta, d\partial D_t^\delta)\,dt\\
&+\langle \nabla_{21} f_\delta(X_t^\delta, D_t^\delta), d\partial D_t^\delta \otimes dX_t^\delta\rangle.
\end{aligned}$$

From Proposition 3.4, possibly by extracting a subsequence,

$$\left(f_\delta(X_t^\delta, D_t^\delta)\right)_{t\geqslant 0} \xrightarrow{\mathscr{L}} (\rho(X_t, \partial D_t))_{t\geqslant 0}\,. \tag{3.27}$$

From (3.7) we get for $i = 1, 2$,

$$\begin{aligned}
&\nabla_i f_\delta(x, D) - \nabla_i \rho_\delta(x, \partial D)\\
&= -\nabla_i \ell_\varepsilon(x, D)\int_{\mathbb{R}^d} \varphi_\delta(|u|)\left(\rho_\delta(\exp_x(\imath_{x,x}(u)), \partial D) - \rho_\delta(x, \partial D)\right)\,du\\
&+ (1-\ell_\varepsilon(x, D))\int_{\mathbb{R}^d} \varphi_\delta(|u|)\left(\nabla_i \rho_\delta(\exp_x(\imath_{x,x}(u)), \partial D) - \nabla_i\rho_\delta(x, \partial D)\right)\,du
\end{aligned} \tag{3.28}$$

(where we use the same procedure as in the proof of Proposition 3.4: ∇_1 is not applied to the first x of $\imath_{x,x}$). From this we see that $\nabla_1 f_\delta(\cdot, D)$ converges, locally uniformly outside $S(D)$, to $\nabla_1\rho(\cdot, \partial D)$ with respect to the distance d_0 of Appendix H. We obtain, with Lemma H.1, possibly by again extracting a subsequence, that

$$\left(\int_0^t \langle \nabla_1 f_\delta(X_s^\delta, D_s^\delta), dX_s^\delta\rangle\right)_{t\geqslant 0} \xrightarrow{\mathscr{L}} \left(\int_0^t \langle \nabla_1 \rho(X_s, \partial D_s), dX_s\rangle\right)_{t\geqslant 0}. \tag{3.29}$$

More precisely, we have a sequence of martingales converging in law to a martingale M_t which is a Brownian motion by Theorem 3 in [21]. For identifying the limiting martingale we use the convergence of $\langle \nabla_1 f_\delta(X_s^\delta, D_s^\delta), dX_s^\delta\rangle \otimes dX_s^\delta$ to $dM_s \otimes dX_s$ obtained again by Theorem 3 in [21] (here again we use an isometric embedding of M). But Lemma H.1 proves that the limit is equal to $\nabla_1\rho(X_s, \partial D_s)\,ds$, yielding (3.29).

Next we prove that

$$\left(\int_0^t \langle \nabla_2 f_\delta(X_s^\delta, D_s^\delta), d\partial D_s^\delta\rangle\right)_{t\geqslant 0} \xrightarrow{\mathscr{L}} \left(\int_0^t \langle \nabla_2 \rho(X_s, \partial D_s), d\partial D_s\rangle\right)_{t\geqslant 0}. \tag{3.30}$$

The argument is similar except that as we see with (3.14), the drift part of $d\partial D_s^\delta$ is not well controlled as X_t^δ approaches the skeleton. So one cannot proceed exactly the same way. But fortunately, for x outside a $3\varepsilon/4$-neighbourhood of ∂D and outside $S(D)$, we have

$$\begin{aligned}
&\langle \nabla_2 f_\delta(x, D), N|_{\partial D}\rangle\\
&= c_\delta \int_{\mathbb{R}^d} \varphi_\delta(|u|)\langle -N\left(P(\exp_x(\imath_{x,x}(u))\right), N\left(P(\exp_x(\imath_{x,x}(u)))\right)\rangle\,du = -c_\delta
\end{aligned} \tag{3.31}$$

where c_δ is defined in (3.2). This together with (3.22) suggests to write

$$\int_0^t \langle \nabla_2 f_\delta(X_s^\delta, D_s^\delta), d\partial D_s^\delta\rangle = \left(\int_0^t \langle \nabla_2 f_\delta(X_s^\delta, D_s^\delta), d\partial D_s^\delta\rangle + c_\delta \int_0^t \frac{d\mu(D_s^\delta)}{\underline{\mu}(\partial D_s^\delta)}\right) - c_\delta \int_0^t \frac{d\mu(D_s^\delta)}{\underline{\mu}(\partial D_s^\delta)}.$$

The second line clearly converges. The right hand side in the first line can be written

$$\int_0^t \tilde{\ell}_\varepsilon(x,D) \left\langle \nabla_2 f_\delta(X_s^\delta, D_s^\delta) + c_\delta N|_{\partial D}, d\partial D_s^\delta \right\rangle \tag{3.32}$$

with $(x,D) \mapsto \tilde{\ell}_\varepsilon(x,D) := (\tilde{h}_\varepsilon \circ \rho_{\partial D})(x)$ where $\tilde{h}_\varepsilon \equiv 1$ in $[0, 3\varepsilon/4]$, $\tilde{h}_\varepsilon \equiv 0$ in $[\varepsilon, \infty)$ and $\tilde{h}_\varepsilon$ is smooth and nonincreasing in $[0,\infty)$.

With this last integral we can proceed as for (3.29)

Similarly we obtain the two following convergences for the second derivatives.

$$\begin{aligned} &\left(\int_0^t \nabla_2 d_2 f_\delta(X_s^\delta, D_s^\delta)(d\partial D_s^\delta, d\partial D_s^\delta)\right)_{t\geqslant 0} \\ &\xrightarrow{\mathscr{L}} \left(\int_0^t \nabla_2 d_2 \rho(X_s, \partial D_s)\left(N(P^{\partial D_s}(X_s), N(P^{\partial D_s}(X_s)\right) \, ds\right)_{t\geqslant 0} \equiv 0 \end{aligned} \tag{3.33}$$

where $P^{\partial D_s}(X_s)$ is the orthogonal projection of X_s on ∂D_s (which is defined ds-almost everywhere),

$$\begin{aligned} &\left(\int_0^t \langle \nabla_2 d_1 f_\delta(X_s^\delta, D_s^\delta), d\partial D_t^\delta \otimes dX_t^\delta\rangle)\right)_{t\geqslant 0} \\ &\xrightarrow{\mathscr{L}} \left(\int_0^t \langle \nabla_2 d_1 \rho(X_s, \partial D_s), d\partial D_s \otimes dX_s\rangle\right)_{t\geqslant 0} \equiv 0 \end{aligned} \tag{3.34}$$

since $d_1\rho(X_s, \partial D_s) = -\langle N^{D_s}(X_s), \cdot\rangle$ which implies that the covariant derivative in the second variable with respect to N^{D_s} is equal to 0. On the other hand, by Itô-Tanaka formula, see Proposition F.1 in Appendix F, using that $\rho(x, \partial D)$ is almost everywhere the minimum of two smooth functions, we have

$$\begin{aligned} d\left(\rho(X_t, \partial D_t)\right) =& \langle \nabla_1 \rho(X_t, \partial D_t), dX_t\rangle - \frac{1}{2} h^{D_t}(X_t)\, dt + \langle \nabla_2 \rho(X_t, \partial D_t), d\partial D_t\rangle \\ &+ 0 + 0 - \sin\left(\theta^{S_t}(X_t)\right) \, dL_t^{S_t}(X). \end{aligned} \tag{3.35}$$

Using (3.26), (3.27), (3.29), (3.30), (3.33), (3.34), (3.35) we obtain that

$$\left(\int_0^t \Delta_1 f_\delta(X_s^\delta, D_s^\delta)\, ds\right)_{t\geqslant 0} \xrightarrow{\mathscr{L}} \left(\int_0^t -h^{D_s}(X_s)\, ds - \int_0^t 2\sin\left(\theta^{S_s}(X_s)\right) \, dL_s^{S_s}(X)\right)_{t\geqslant 0}. \tag{3.36}$$

It remains to pass in the limit as δ goes to zero in (3.21), to deduce (3.24). □

Remark 3.7. From (3.35), it can be deduced that

$$d\left(\rho(X_t, \partial D_t)\right) = \frac{1}{2}\left(h^{D_t}(X_t) - h^{D_t}\left(P^{\partial D_t}(X_t)\right)\right) \, dt + \sin\left(\theta^{S_t}(X_t)\right) \, dL_t^{S_t}(X). \tag{3.37}$$

Indeed, (3.25) implies that

$$\langle \nabla_1 \rho(X_t, \partial D_t), dX_t \rangle \quad = \quad dW_t$$

and due to (3.30), we have

$$\begin{aligned} &\langle \nabla_2 \rho(X_t, \partial D_t), d\partial D_t \rangle \\ &= \lim_{\delta \to 0} \langle \nabla_2 \rho(X_t^\delta, \partial D_t^\delta), d\partial D_t^\delta \rangle \\ &= \lim_{\delta \to 0} -dW_t^\delta - \left(\Delta_1 f_\delta (P^{\partial D_t^\delta}(X_t^\delta), D_t^\delta) + \frac{1}{2} h^{D_t^\delta}(P^{\partial D_t^\delta}(X_t^\delta)) \right) dt \end{aligned}$$

where we used (3.21) in conjunction with (3.22).

Taking into account (3.36), we identify the last limit with

$$-dW_t + \left(h^{D_t}(x_t) + 2\sin\left(\theta^{S_t}(X_t)\right) \, dL_t^{S_t}(X) - \frac{1}{2} h(P^{\partial D_t}(X_t)) \right) dt$$

4. Intertwined dual processes: decoupling and reflection on boundary

In this section we consider another canonical and extremal situation, the case where f^D vanishes almost everywhere. More precisely, it is the limiting situation where f^D is constant outside a ε-neighbourhood of the boundary. This situation is completely opposite to the one of Section 3 where the coupling is maximal.

Theorem 4.1. *There exists a pair $(X_t, D_t)_{t\geqslant 0}$ of τ_ε-intertwined processes in the sense of Definition 1.1 satisfying*

$$d\partial D_t(y) \mathbb{1}_{\{X_t \notin \partial D_t\}} = N^{D_t}(y) \left(dW_t + \frac{1}{2} h^{D_t}(y) dt - dL_t^{\partial D_t}(X) \right) \tag{4.1}$$

where X_t is a M-valued Brownian motion started at uniform law in D_0, W_t is a real-valued Brownian motion independent of X_t, $L_t^{\partial D_t}(X)$ is the local time of X_t on the moving boundary ∂D_t.

Remark 4.2. Equation (4.1) can be considered as a limiting case of (2.10). Here Assumption 3.1 is not needed since the morphological skeleton of D does not play a role, and the map $D \mapsto \partial D$ is already sufficiently regular.

Proof. The proof is quite similar to the one of Theorem 3.5, but with another family of functions f_δ^D, namely $f_\delta^D := h_\delta \circ \rho_{\partial D}$ where h_δ is defined in the proof of Proposition 2.4: h_δ is a smooth nondecreasing function from $[0, \infty)$ to $\mathbb{R}_+$ such that $h_\delta(r) = r$ for $r \in [0, \delta/2]$, $h_\delta(r) = (3/4)\delta$ for $r \geqslant \delta$ and $\|h_\delta'\|_\infty \leqslant 1$. But here, as ε is fixed, we will let $\delta \searrow 0$. Again we construct for each $\delta > 0$, an intertwined processes $(X_t^\delta, D_t^\delta)_{t\geqslant 0}$ stopped at τ_ε^δ. Again all $(X_t^\delta, D_t^\delta)_{t\geqslant 0}$ are tight, and a limiting process $(X_t, D_t)_{t\geqslant 0}$ stopped at τ_ε provides an intertwining. The proof of (4.1) goes along the same lines as the one of (3.10). □

We end this section with another canonical construction, where the functions f_δ^D approximate $-\rho_{\partial D}$.

Theorem 4.3. *Under assumption 3.4, there exists an intertwining $(X_t, D_t)_{t\geqslant 0}$ stopped at τ_ε, satisfying*

$$
\begin{aligned}
d\partial D_t(y) = & N^{D_t}(y)\Bigg(-\big\langle dX_t, N^{D_t}(X_t)\big\rangle + \left(\frac{1}{2}h^{D_t}(y) + h^{D_t}(X_t)\right)\, dt \\
& + 2\sin(\theta^{S_t}(X_t))\, dL_t^{S_t}(X) - 2dL_t^{\partial D_t}(X)\Bigg)
\end{aligned}
\tag{4.2}
$$

Proof. It is completely similar to the ones of Theorems 3.5 and 4.1. □

5. Some fundamental examples

5.1. **Real Brownian motion and three-dimensional Bessel process.** We come back to the case where $M = \mathbb{R}$. Assume that the Brownian motion X starts from 0 (to respect rigorously the above framework, X should start from the uniform distribution on $D_0 := [-\epsilon, \epsilon]$ and next we should let ϵ go to 0_+). Due to the invariance by symmetry of (3.10), for any $t > 0$, D_t remains a symmetric interval, let us write it $[-R_t, R_t]$. In this simple setting, we have $N^{D_t}(\cdot) = -\mathrm{sign}(\cdot)$ on $\mathbb{R}\backslash\{0\}$, $h^{D_t} = 0$ and $S_t = \{0\}$, for any $t > 0$. Thus (3.10) writes

$$dR_t = \mathrm{sign}(X_t)dX_t + 2dL_t \tag{5.1}$$

where $L := (L_t)_{t\geqslant 0}$ is the local time of X at 0. Namely we get that

$$
\begin{aligned}
\forall\, t \geqslant 0, \qquad R_t &= \int_0^t \mathrm{sign}(X_s)\, dX_s + 2L_t \\
&= |X_t| + L_t
\end{aligned}
$$

by Tanaka's formula. It is well-known that $R := (R_t)_{t\geqslant 0}$ is a Bessel process of dimension 3 (cf. e.g. Corollary 3.8 of Chapter 6 of Revuz and Yor [17]). In particular, we get that with the notation introduced in (A.4),

$$\forall\, t \geqslant 0, \qquad \rho^+_{\partial D_t}(X_t) \;=\; \min(X_t + R_t, R_t - X_t)$$

But except at time $t = 0$, this quantity is always positive: a.s. X_t never touch the boundary of D_t for $t > 0$. Indeed, if for some $t > 0$ we have $|X_t| = R_t$, we deduce that $L_t = 0$, namely a contradiction, since $X_0 = 0$.

In particular, we see that the intertwining coupling we have constructed is different from the one proposed by Pitman [16], which is a.s. touching (the upper) boundary repeatedly. Instead we end up with the intertwining dual constructed in [14] via stochastic flows. It is mentioned there how to deduce the classical Pitman's dual, via Lévy's theorem.

Here is an alternative approach. While Equation (5.1) is obtained from approximating $x \mapsto |r - x|$ outside an ε-neighbourhood of 0 when $D = [-r, r]$ by smooth functions f^D satisfying Assumption 2.2, we are able to recover Pitman theorem by rather approximating $x \mapsto -x$ in $D = [-r, r]$ outside the only ε-neighbourhood of $-r$. In the limit of (2.10) as ϵ goes to zero, on the one hand we have

$$\mathbb{1}_{\{X_t \neq R_t\}} dR_t = dX_t, \tag{5.2}$$

on the other hand we have $X_t + R_t \geqslant 0$, so that $X_t + R_t$ is the solution to the Skorohod problem associated to $2X_t$. We get

$$R_t + X_t = 2X_t - 2\min_{0\leqslant s\leqslant t} X_s. \tag{5.3}$$

which is equivalent to

$$R_t = X_t - 2 \min_{0 \leqslant s \leqslant t} X_s. \tag{5.4}$$

The answer to the question: what would be a symmetric construction with local time at the two ends of $[-R_t, R_t]$ is given by Theorem 4.3. We obtained intertwined processes with

$$R_t = -\int_0^t \operatorname{sign}(X_s)\, dX_s - 2L_t^0(X) + 2L_t^0(R - X) + 2L_t^0(R + X). \tag{5.5}$$

5.2. **Brownian motion and disks in rotationally symmetric manifolds.** This is the most simple example since the skeleton is never hit by the Brownian motion. Consider a complete d-dimensional manifold with $d \geqslant 2$, rotationally symmetric around a point $o \in M$. Denote by (r, Θ) polar coordinates with $r(x) = \rho(o, x)$ and

$$ds^2 = dr^2 + f^2(r)\, d\Theta^2 \tag{5.6}$$

the metric in polar coordinates. Then the radial Laplacian is

$$\Delta_r = \frac{\partial^2}{(\partial r)^2} + b(r) \frac{\partial}{\partial r} \quad \text{with} \quad b = (d-1)(\ln f)'. \tag{5.7}$$

We will investigate set-valued processes $D_t = B(o, R_t)$ where $B(o, r)$ is the open geodesic ball centered at o, with radius r. The skeleton of $B(o, R_t)$ is the point o.

Let X_t be a Brownian motion in M satisfying $X_0 \sim \mathscr{U}(D_0)$ for some $D_0 = B(o, r_0)$. Denote by $\rho_t := r(X_t)$ the radial part of X_t. Then

$$d\rho_t = d\beta_t + \frac{1}{2} b(\rho_t)\, dt, \qquad \rho_0 \sim \mathscr{U}^f((0, r_0)) \tag{5.8}$$

where $(\beta_t)_{t \geqslant 0}$ is a real Brownian motion and

$$\mathscr{U}^f(dr) := \frac{f(r)}{\int_0^{r_0} f(s)\, ds}\, dr. \tag{5.9}$$

The evolution equation (3.10) for D_t shows by symmetry that for all $t \geqslant 0$, $D_t = B(0, R_t)$ for some real-valued process R_t. Moreover it writes

$$\begin{aligned} d\rho_t &= d\beta_t + \frac{1}{2} b(\rho_t)\, dt \\ dR_t &= d\beta_t + \left[-\frac{1}{2} b(R_t) + b(\rho_t) \right] dt. \end{aligned} \tag{5.10}$$

Proposition 5.1. *The system of equations* (5.10) *has a solution up to explosion time of* R_t

$$\tau^D := \inf\{t \geqslant 0, R_t \notin (0, \infty)\}, \tag{5.11}$$

which satisfies for all $t < \tau^D$,

$$0 < \rho_t < R_t. \tag{5.12}$$

The corresponding set-valued process $D_t = B(o, R_t)$ *is solution to equation* (3.10)*, and in particular, for all* $\mathscr{F}^D$*-stopping time* τ,

$$\mathscr{L}(X_\tau | \mathscr{F}_\tau^D) = \mathscr{U}(D_\tau) \quad \text{as well as} \quad \mathscr{L}(\rho_\tau | \mathscr{F}_\tau^D) = \mathscr{U}^f((0, R_\tau)). \tag{5.13}$$

Proof. We only have to check (5.12). By (5.10),

$$d(R_t - \rho_t) = \frac{1}{2}\left[b(\rho_t) - b(R_t)\right] dt, \tag{5.14}$$

which vanishes on $\{R_t = \rho_t\}$, and since b is smooth, if $\rho_0 < R_0$, then $\rho_t < \mathbb{R}_t$ for all times. □

5.3. **Brownian motion and annulus in 2-dimensional rotationally symmetric manifolds.** Let M be a complete 2-dimensional Riemannian manifold, rotationally symmetric around a point $o \in M$. Denote by (r, θ) polar coordinates with $r(x) = \rho(o, x)$ and

$$ds^2 = dr^2 + f^2(r)\, d\theta^2 \tag{5.15}$$

the metric in polar coordinates. Then the radial Laplacian is

$$\Delta_r = \frac{\partial^2}{(\partial r)^2} + b(r)\frac{\partial}{\partial r} \quad \text{with} \quad b = (\ln f)'. \tag{5.16}$$

If $0 \leqslant r^- \leqslant r^+$, let

$$A(r^-, r^+) := \{x \in M,\ r^- < r(x) < r^+\} \quad \text{if} \quad r^- < r^+, \quad A(r^-, r^-) := \varnothing, \tag{5.17}$$

the open annulus delimited by the radius r^- and r^+.

In the following we will investigate set-valued processes $D_t = A(R_t^-, R_t^+)$. The skeleton of $A(R_t^-, R_t^+)$ is the circle

$$S_t = C(o, R_t^0) \quad \text{with} \quad R_t^0 := \frac{1}{2}(R_t^- + R_t^+). \tag{5.18}$$

Let X_t be a Brownian motion in M satisfying $X_0 \sim \mathscr{U}(D_0)$ for some $D_0 = A(r_0^-, r_0^+)$. Denote by $\rho_t := r(X_t)$ the radial part of X_t. Then

$$d\rho_t = d\beta_t + \frac{1}{2}b(\rho_t)\, dt, \qquad \rho_0 \sim \mathscr{U}^f((r_0^-, r_0^+)) \tag{5.19}$$

where β_t is a real Brownian motion and

$$\mathscr{U}^f((r_0^-, r_0^+))(dr) := \frac{f(r)}{\int_{r_0^-}^{r_0^+} f(s)\, ds}\, dr. \tag{5.20}$$

The evolution equation (3.10) for D_t shows by symmetry that for all $t \geqslant 0$, $D_t = A(R_t^-, R_t^+)$ for some real-valued processes $R_t^- \leqslant R_t^+$. Moreover it writes

$$\begin{aligned} d\rho_t &= \operatorname{sign}(\rho_t - R_t^0)\, dW_t + \frac{1}{2}b(\rho_t)\, dt \\ dR_t^+ &= dW_t + \left[-\frac{1}{2}b(R_t^+) + \operatorname{sign}(\rho_t - R_t^0)b(\rho_t)\right] dt + 2L_t^{R_t^0}(\rho) \\ dR_t^- &= -dW_t + \left[-\frac{1}{2}b(R_t^-) - \operatorname{sign}(\rho_t - R_t^0)b(\rho_t)\right] dt - 2L_t^{R_t^0}(\rho) \\ R_t^0 &= \frac{1}{2}\left(R_t^- + R_t^+\right) \end{aligned} \tag{5.21}$$

and these equations imply

$$dR_t^0 = -\frac{1}{4}\left[b(R_t^+) + b(R_t^-)\right] dt. \tag{5.22}$$

Proposition 5.2. *The system of equations* (5.21) *has a solution up to explosion time*

$$\tau^D := \inf\{t \geqslant 0, (R_t^-, R_t^+) \notin (0,\infty)^2\}, \tag{5.23}$$

which satisfies for all $t < \tau^D$*,*

$$R_t^- \leqslant \rho_t \leqslant R_t^+. \tag{5.24}$$

The corresponding set-valued process $D_t = A(R_t^-, R_t^+)$ *is solution to equation* (3.10)*, and in particular, for all* $\mathscr{F}^D$*-stopping time* τ*,*

$$\mathscr{L}(X_\tau | \mathscr{F}_\tau^D) = \mathscr{U}(D_\tau) \quad \textit{as well as} \quad \mathscr{L}(\rho_\tau | \mathscr{F}_\tau^D) = \mathscr{U}^f((R_\tau^-, R_\tau^+)). \tag{5.25}$$

Proof. Fix $\varepsilon > 0$ and $\alpha \in (0,1)$. We will first solve the system of equations until the exit time τ_ε and then let $\varepsilon \searrow 0$. Let us construct functions $f_\delta^D(x)$ which satisfies equation (3.1). It will be easier here because there is no need of functions ℓ_ε and g_δ.

For $\delta \in (0,\varepsilon)$, let $\varphi_\delta : \mathbb{R} \to \mathbb{R}$ be the function with support equal to $[-\delta/2, \delta/2]$, satisfying for $-\delta/2 < r < \delta/2$:

$$\varphi_\delta(r) := \frac{1}{c(\delta)} \exp\left(-\frac{1}{\left(\frac{\delta}{2}\right)^2 - r^2}\right) \quad \text{with} \quad c(\delta) := \int_{-\delta/2}^{\delta/2} \exp\left(-\frac{1}{\left(\frac{\delta}{2}\right)^2 - s^2}\right) ds, \tag{5.26}$$

and let

$$\begin{aligned} \operatorname{sign}_\delta &: \mathbb{R} \to \mathbb{R} \\ r &\mapsto -1 + 2\int_{-\infty}^r \varphi_\delta(s)\, ds. \end{aligned} \tag{5.27}$$

The functions φ_δ and $\operatorname{sign}_\delta$ are both smooth and Lipschitz, and they respectively approximate δ_0 and sign. For $0 < r^- < r^+$ satisfying $r^+ - r^- \geqslant 2\varepsilon$, defining $r^0 := \frac{1}{2}(r^- + r^+)$, for $x \in A(r^-, r^+)$ let

$$f^{A(r^-,r^+)}(x) = f(x, r^-, r^+) = g(r(x)) \quad \text{with} \quad g(r) = g(r, r^-, r^+) = \int_{r^-}^r -\operatorname{sign}_\delta(s - r^0)\, ds. \tag{5.28}$$

Clearly $f(x, r^-, r^+)$ is 1-Lipschitz in the first variable. A computation shows that

$$\partial_{r^+} g(r, r^-, r^+) = \int_{-\varepsilon}^{r^0} \varphi_\delta(v)\, dv \quad \text{and} \quad \partial_{r^-} g(r, r^-, r^+) = -\int_{r-r^0}^{\varepsilon} \varphi_\delta(v)\, dv \tag{5.29}$$

showing that g and f are 1-Lipschitz. Then the vector $N := N_{\partial A(r^-,r^+)}$ is equal to $-\mathbb{1}_{\{r(x)=r^+\}}\partial_{r^+} + \mathbb{1}_{\{r(x)=r^-\}}\partial_{r^-}$ so that

$$\langle \nabla f, N \rangle \equiv 1 \quad \text{and} \quad \nabla df(N,N) \equiv 0. \tag{5.30}$$

This yields an elementary proof of the properties of Proposition 3.4. We can use Theorem 3.5 to solve equation (5.21) until the stopping time τ_ε.

We are left to prove that $\tau_\varepsilon \nearrow \tau^D$ a.s. as $\varepsilon \searrow 0$. This is a direct consequence of the fact that the volume of $A(R_t^-, R_t^+)$ is a time changed Bessel process of dimension 3 (by [6] Theorem 5), proving that $A(R_t^-, R_t^+)$ cannot collapse onto its skeleton. □

Remark 5.3. After the hitting time of 0 by R_t^-, the processes can continue to evolve under the regime of Section 5.2.

We recover from Proposition 5.2 a result from [13] stating that $([R_t^-, R_t^+])_{t\geqslant 0}$ is an intertwining dual process for the real diffusion $(\rho_t)_{t\geqslant 0}$. In particular, we deduce that if $(\rho_t)_{t\geqslant 0}$ is positive recurrent and if $+\infty$ is an entrance boundary, then $([R_t^-, R_t^+])_{t\geqslant 0}$ reaches $[0, +\infty]$ in finite time and this finite time is a strong stationary time for $(\rho_t)_{t\geqslant 0}$, see [13] for more details.

5.4. **Brownian motion and symmetric convex sets in $\mathbb{R}^2$.** In this section we take $M = \mathbb{R}^2$ endowed with the Euclidean metric. Consider a smooth strictly convex bounded set $D_0 \subset M$ with smooth boundary, symmetric with respect to the horizontal and vertical axes. Also assume that its skeleton is an horizontal interval $S_0' = [-x_0, x_0] \times \{0\}$. An example of such a set is the interior of an ellipse, the skeleton being the interval between the two foci. Assume that X_t is a Brownian motion in $\mathbb{R}^2$ satisfying $X_0 \sim \mathscr{U}(D_0)$. Let us investigate the evolution of (X_t, D_t). Notice that it is the first example where we really have to deal with infinite dimensional processes. By conservation of the convexity by the normal and mean curvature flows, D_t will stay convex. It will also stay symmetric. Propositions 5.4, 5.5 and 5.7 below will be proved in a forthcoming paper:

Proposition 5.4. *The skeleton of D_t always takes the form $[-x_t, x_t] \times \{0\}$.*

Proof. This will be proved in [1] □

Denote by $(\imath, \jmath)$ the canonical basis of $\mathbb{R}^2$, and $X_t = (X_t^{(1)}, X_t^{(2)})$. In this notation, the vector $N^{D_t}(X_t)$ of Equation (3.10) writes

$$N^{D_t}(X_t) = -\mathrm{sign}(X_t^{(1)})\cos(\theta^{S_t}(X_t))\imath - \mathrm{sign}(X_t^{(2)})\sin(\theta^{S_t}(X_t))\jmath \tag{5.31}$$

where $\theta^{S_t}(x)$ is naturally extended to D_t by being constant on lines normal to the boundary (see [1]). Notice that $x \mapsto \theta^{S_t}(x)$ is locally Lipschitz on D_t and is equal to 0 on $D_t \cap ((-\infty, -x_t] \times \{0\} \cup [x_t, \infty) \times \{0\})$. Also notice that the function h^{D_t} is locally Lipschitz on $D_t \backslash \{(-x_t, 0), (x_t, 0)\}$. With these notations, equation (3.10) writes

$$\begin{aligned} d\partial D_t(y) = &- N^{D_t}(y)\Bigg(\mathrm{sign}(X_t^{(1)})\cos(\theta^{S_t}(X_t))\,dX_t^{(1)} + \sin(\theta^{S_t}(X_t))\mathrm{sign}(X_t^{(2)})\,dX_t^{(2)} \\ &+ \left(\frac{1}{2}h^{D_t}(y) - h^{D_t}(X_t)\right)\,dt - 2\sin(\theta^{S_t}(X_t))dL_t(X^{(2)})\Bigg). \end{aligned} \tag{5.32}$$

Let us investigate the motion of the skeleton $\tilde{S}_t$ of the solution $\tilde{D}_t$ of equation (2.11) (garanteed by Theorem 2.5).

Proposition 5.5. *The process $\left(\tilde{D}_t\right)_{t\geqslant 0}$ takes its values in a closed subset $\widetilde{\mathcal{F}}^{\alpha,\varepsilon}$ of $\mathcal{F}^{\alpha,\varepsilon}$, symmetric with respect to the horizontal and vertical axes, such that on $\widetilde{\mathcal{F}}^{\alpha,\varepsilon}$, the map $D \mapsto h^D|_{\partial D}$ is continuous from $\widetilde{\mathcal{F}}^{\alpha,\varepsilon}$ (with the $C^{2+\alpha}$ metric) to $C^2(\partial D)$. Its skeleton $\tilde{S}_t$ satisfies $\tilde{S}_t = [-\tilde{x}_t, \tilde{x}_t] \times \{0\}$ for some process $\tilde{x}_t$.*

Proof. This will be proved in [1] □

In the next result we prove that the skeleton has finite variation and is monotonly decreasing.

Proposition 5.6. *The right endpoint $(\tilde{x}_t, 0)$ of the skeleton $\tilde{S}_t$ satisfies*

$$\frac{d\tilde{x}_t}{dt} = \frac{\rho^2((\tilde{x}_t, 0), \tilde{y}_t)}{2}(h^{\tilde{D}_t})''(\tilde{y}_t), \tag{5.33}$$

$\tilde{y}_t$ being the point of $\partial\tilde{D}_t$ in the horizontal line with the greatest abscissa, and the second derivative being calculated with curvilign coordinates on $\partial\tilde{D}_t$. Notice that $(h^{\tilde{D}_t})''(\tilde{y}_t) \leqslant 0$, proving that the process $S(\tilde{D}_t)$ is monotonly decreasing.

Proof. Let us investigate the motion of a point in $\tilde{S}_t$ close to $(\tilde{x}_t, 0)$. This point has two closest points in $\partial\tilde{D}_t$, which we call $\widetilde{y}_{1,t}$ and $\widetilde{y}_{2,t}$, the first one having positive second coordinate. We will use Theorem B.1 and (B.28). Call $\hat{x}_t$ the point in the skeleton corresponding to $\widetilde{y}_{1,t}$ and $\widetilde{y}_{2,t}$. We have $N_1(\hat{x}_t) = -\cos\theta(\hat{x}_t)\imath - \sin\theta(\hat{x}_t)\jmath$, $N_2(\hat{x}_t) = -\cos\theta(\hat{x}_t)\imath + \sin\theta(\hat{x}_t)\jmath$, $N_1^S(\hat{x}_t) = -\jmath$. Denote $T(\widetilde{y}_{1,t})$ the tangent vector to $\partial\tilde{D}_t$ at $\widetilde{y}_{1,t}$, corresponding to increasing of θ: $T(\widetilde{y}_{1,t}) = -\sin\theta(\hat{x}_t)\imath + \cos\theta(\hat{x}_t)\jmath$. Write $h'(\widetilde{y}_{1,t})$ the curvilign derivative of $h(\widetilde{y}_{1,t})$ in the direction of $T(\widetilde{y}_{1,t})$. Then the vector $J_1^\perp(1)$ of (B.28) is equal to $-\dfrac{1}{2}\rho_S(\widetilde{y}_{1,t})h'(\widetilde{y}_{1,t})T(\widetilde{y}_{1,t})$. So we get from (B.28):

$$\begin{aligned}\frac{d}{dt}\hat{x}_t &= \frac{1}{2}\rho_S(\widetilde{y}_{1,t})h'(\widetilde{y}_{1,t})\left(\sin\theta(\hat{x}_t) + \frac{\cos^2\theta(\hat{x}_t)}{\sin\theta(\hat{x}_t)}\right)\imath \\ &= \frac{\rho_S(\widetilde{y}_{1,t})h'(\widetilde{y}_{1,t})}{2\sin\theta(\hat{x}_t)}\imath \\ &= \frac{\rho_S^2(\widetilde{y}_{1,t})h'(\widetilde{y}_{1,t})}{2\widetilde{y}_{1,t}^{(2)}}\imath \quad \text{with} \quad \widetilde{y}_{1,t} = (\widetilde{y}_{1,t}^{(1)}, \widetilde{y}_{1,t}^{(2)}).\end{aligned} \tag{5.34}$$

In the limit, as $\widetilde{y}_{1,t}^{(2)}$ goes to zero, we obtain the motion of $\tilde{x}_t$ and using the symmetry of the convex set, we have $h'(\widetilde{y}_t) = 0$ so that we can replace $\dfrac{h'(\widetilde{y}_{1,t})}{\widetilde{y}_{1,t}^{(2)}}$ by $h''(\widetilde{y}_t)$. This yields (5.33). □

In particular a Brownian motion X_t will never meet the ends of $\tilde{S}_t$.

A solution to (5.32) can be found with the help of Theorem 3.5. The family of functions $f_\delta(x, D)$ defined in (3.1) takes the form:

$$\begin{aligned}f_\delta(x, D) &= \ell_\varepsilon(x)\rho_\delta(x, \partial D) + (1 - \ell_\varepsilon(x))\int_{\mathbb{R}^2}\varphi_\delta(|x-y|)\rho_\delta(y, \partial D)\,dy \\ &= \ell_\varepsilon(x)\rho_\delta(x, \partial D) + (1 - \ell_\varepsilon(x))\int_{\mathbb{R}^2}\varphi_\delta(|y|)\rho_\delta(x-y, \partial D)\,dy.\end{aligned} \tag{5.35}$$

Proposition 5.7. *Equation* (5.32) *provides an intertwining with infinite lifetime*

Proof. This will be proved in [1] □

Appendix A. An integration by parts on domains with boundary

Let M be a d-dimensional Riemannian manifold and $D \subset M$ a compact and connected domain with smooth boundary ∂D. For $y \in \partial D$, let $N(y)$ be the inward normal vector. Denote by S' the inward (morphological) skeleton of D: S' is the set of points in D such that the distance to ∂D is not smooth with non vanishing gradient around them. Denote

$$\tau(y) = \inf\{t > 0,\ \exp_y(tN(y)) \in S'\}. \tag{A.1}$$

Let S be the set of regular points of S', which we can describe as follows: if $x \in S$, then there exists a unique couple (y_1, y_2) of distinct points from ∂D such that

$$x = \exp_{y_1}(\tau(y_1)N(y_1)) = \exp_{y_2}(\tau(y_2)N(y_2)). \tag{A.2}$$

We have $\tau(y_1) = \tau(y_2)$, and for $i = 1, 2$, the differential at $(\tau(y_i), y_i)$ of the map $\mathbb{R}_+ \times \partial D \ni (t, y) \mapsto \exp_y(tN(y))$ is nondegenerate. The set S is a codimension 1 submanifold of M and $S'\backslash S$ has Hausdorff dimension smaller than or equal to $d-2$. It is the union of the focal set which is the set of points $x = \exp_y(\tau(y)N(y))$ such that $(t, y') \mapsto \exp_{y'}(tN(y'))$ is degenerate at $(\tau(y), y)$, and the union of the sets defined like S but withstrictly more than two points y_1, y_2, y_3,... For $r \geqslant 0$, let

$$D(r) = \{z \in D\backslash S',\ \rho_{\partial D}(z) \geqslant r\}. \tag{A.3}$$

where ρ is the Riemannian distance. The set $D(r)$ is a (possibly empty) manifold with smooth boundary $\partial D(r)$ on which one can define an inward normal $N(y)$ and an orientation by parallel transporting oriented basis of ∂D along normal geodesics. So we have for all $y \in D\backslash S'$: $N(y) = \nabla \rho_{\partial D}(y)$.

We will also need the sets $D(r)$ for all $r \in \mathbb{R}$. We will let for $r < 0$

$$D(r) = \{z \in M,\ \rho^+_{\partial D}(z) \geqslant r\} \tag{A.4}$$

where $\rho^+_{\partial D}$ is the signed distance to ∂D, positive inside D, negative outside D.

Define for $s, t \in \mathbb{R}$

$$\begin{aligned} \psi(s,t) : \partial D(s) &\to \partial D(t) \\ y &\mapsto \exp_y\left((t-s)N(y)\right) \end{aligned} \tag{A.5}$$

and $\psi(t) = \psi(0, t)$. We will indifferently write $\psi(t)(x) = \psi(t, x)$. The function $\psi(s, t)$ is not defined for all points of $\partial D(s)$ because we ask $\psi(s, t)(y) \in \partial D(t)$, nor is $N(\cdot)$. However for $|s|$ and $|t|$ small it is a map, defined for all $y \in \partial D(s)$, and is is also a diffeomorphism with inverse $\psi(t, s)$.

We have for $0 \leqslant s \leqslant t$, $\psi(t) = \psi(s, t) \circ \psi(s)$, which implies

$$\det T\psi(t) = \det T\psi(s,t) \times \det T\psi(s). \tag{A.6}$$

Notice that thanks to the orientation of the sets $\partial D(r)$ we get an orientation of $D\backslash S'$ by adding N as first vector to oriented basis, consequently $\det T\psi$ is well defined and always positive. It is well-known that

$$\left.\frac{d}{dt}\right|_{t=s} \det T\psi(s,t)(y) = -h(y) \tag{A.7}$$

where $h(y)$ is the inward mean curvature of $\partial D(s)$ (the minus sign of the r.h.s. of (A.7) insures that h is non-negative on $\partial D(s)$ when $D(s)$ is convex). This together with (A.6) yields

$$\left.\frac{d}{dt}\right|_{t=s} \det T\psi(t)(y) = -h\left(\psi(s)(y)\right) \det T\psi(s)(y) \tag{A.8}$$

and consequently, using $\psi(0) = \mathrm{id}$ and $\det T\psi(0) \equiv 1$,

$$\det T\psi(t)(y) = \exp\left(\int_0^t -h\left(\psi(s)(y)\right)\, ds\right). \tag{A.9}$$

Denote by μ the volume measure of D and by $\underline{\mu}$ the volume measures of the manifolds $\partial D(s)$ and of S. Then

$$\mu(D) = \int_0^\infty \underline{\mu}\left(\partial D(r)\right)\, dr. \tag{A.10}$$

But for $r \geqslant 0$

$$\underline{\mu}\left(\partial D(r)\right) = \int_{\partial D} \det T\psi(r)(y)\, \underline{\mu}(dy) \tag{A.11}$$

with convention $\det T\psi(r)(y) = 0$ if $r \geqslant \tau(y)$. We get

$$\underline{\mu}\left(\partial D(r)\right) = \int_{\partial D} \exp\left(-\int_0^r h(\psi(s)(y))\,ds\right) 1_{\{r<\tau(y)\}}\,\underline{\mu}(dy) \tag{A.12}$$

which yields with (A.10)

$$\mu\left(D\right) = \int_{\partial D} \left(\int_0^{\tau(y)} \exp\left(-\int_0^r h(\psi(s,y))\,ds\right)\,dr\right)\underline{\mu}(dy). \tag{A.13}$$

More generally, for a measurable function $g : D \to \mathbb{R}$ bounded below,

$$\int_D g\,d\mu = \int_{\partial D} \left(\int_0^{\tau(y)} g\left(\psi(r,y)\right) \exp\left(-\int_0^r h(\psi(s,y))\,ds\right)\,dr\right)\underline{\mu}(dy). \tag{A.14}$$

Applying this formula to the function gh which we assume to be bounded below or integrable, we get by integration by parts

$$\begin{aligned}\int_D gh\,d\mu &= \int_{\partial D} \left(\int_0^{\tau(y)} -g\left(\psi(r,y)\right) \frac{d}{dr}\exp\left(-\int_0^r h(\psi(s,y))\,ds\right)\,dr\right)\underline{\mu}(dy)\\ &= \int_{\partial D}\left[-g\left(\psi(r,y)\right)\exp\left(-\int_0^r h(\psi(s,y))\,ds\right)\right]_0^{\tau(y)}\underline{\mu}(dy)\\ &\quad+\int_{\partial D}\left(\int_0^{\tau(y)}\langle dg,N\rangle\left(\psi(r,y)\right)\exp\left(-\int_0^r h(\psi(s,y))\,ds\right)\,dr\right)\underline{\mu}(dy)\\ &= \int_{\partial D} g(y)\,\underline{\mu}(dy) - \int_{\partial D} g(\psi(\tau(y),y))e^{-\int_0^{\tau(y)} h(\psi(u,y))\,du}\,\underline{\mu}(dy)\\ &\quad+\int_D \langle dg,N\rangle\,d\mu.\end{aligned}$$

Define the map

$$\begin{aligned}\varphi : \partial D &\to S'\\ y &\mapsto \psi(\tau(y),y).\end{aligned} \tag{A.15}$$

For $z = \psi(\tau(y_i),y_i) \in S$ $(i = 1,2)$ define $\theta(z) \in (0,\pi/2]$ the angle between $N(\psi(\tau(y_i)-,y_i))$ and S. In the sequel we assume that $\theta(z) \neq \pi/2$ (the case $\theta(z) = \pi/2$ is simpler to deal with and Proposition A.1 is always valid). Notice that this angle does not depend on i, this is a consequence of $z \in S$ staying at the same distance to y_1 and y_2 by infinitesimal variation. For later use, let also $\theta(z) = 0$ when $z \in S'\backslash S$. Let us prove that for $z = \psi(\tau(y_i),y_i)) \in S$,

$$\det T\psi(\tau(y_i),y_i) = \sin\theta(\varphi(y_i))\det T\varphi(y_i), \qquad i = 1,2. \tag{A.16}$$

Set $y = y_1$. Let $e_1 = N(y)$, $e_1^S = N(\psi(\tau(y)-,y))$, $N^S(z)$ the normal to S at z such that $\langle N^S(z), e_1^S\rangle > 0$, let $e'' = (e_3,\ldots,e_d)$ be a family of orthonormal normalized vectors in $T_y\partial D$ such that letting $e_2 = \dfrac{\nabla\tau(y)}{\|\nabla\tau(y)\|}$ (we have $\nabla\tau(y) \neq 0$, since $\theta(z) \neq \pi/2$), $e' := (e_2,e'')$ is an orthonormal basis of $T_y\partial D$, let $(e^S)'' = (e_3^S,\ldots,e_d^S)$ be an orthonormal basis of $T_y\varphi(\mathrm{Vect}(e''))$, let e_2^S such that $(e^S)' := (e_2^S,\ldots,e_d^S)$ is an orthonormal basis of T_zS. Finally let $e_2^\theta \in T_zM$ be such that $\langle e_2^\theta, N(z)\rangle < 0$ (e_2^θ and $N^S(z)$ are not orthogonal, since $\theta(z) \neq \pi/2$) and $(e_1^S, e_2^\theta, (e^S)'')$ is an orthonormal basis of T_zM. Figure 1 shows the configuration of $e_1^S, N^S(z), e_2^S$ and e_2^θ on an example of dimension 2. In the sequel we will denote for instance $T\varphi(e') = \begin{pmatrix} T\varphi(e_2) \\ \vdots \\ T\varphi(e_d) \end{pmatrix}$, so that $\langle T\varphi(e'), (e^S)' \rangle$ will be the matrix of all scalar products. We have

$$\begin{aligned}
&\langle T\varphi(e'), (e^S)' \rangle \\
&= \langle d\tau, e' \rangle \langle \partial_t \psi(\tau(y), y), (e^S)' \rangle + \langle T\psi(e'), (e^S)' \rangle \\
&= \begin{pmatrix} \langle d\tau, e_2 \rangle \langle \partial_t \psi, e_2^S \rangle + \langle T\psi(e_2), e_2^S \rangle & \langle T\psi(e_2), (e^S)'' \rangle \\ \langle d\tau, e'' \rangle \langle \partial_t \psi, e_2^S \rangle + \langle T\psi(e''), e_2^S \rangle & \langle T\psi(e''), (e^S)'' \rangle \end{pmatrix}.
\end{aligned}$$

Let us simplify and make more explicit this expression. We have $\langle d\tau, e'' \rangle = 0$. Also $e_2^\theta \perp (e^S)''$ and $e_2^S \perp (e^S)''$ so $e_2^S \in \mathrm{Vect}(e_1^S, e_2^\theta)$ and more precisely

$$e_2^S = \cos(\theta(z)) e_1^S + \sin(\theta(z)) e_2^\theta. \tag{A.17}$$

On the other hand $T\psi(e') \perp e_1^S$ which implies

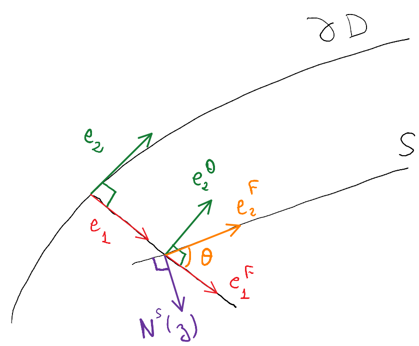

FIGURE 1. The vectors $e_1^S, N^S(z), e_2^S$ and e_2^θ

$$\langle T\psi(e'), e_2^S \rangle = \sin(\theta(z)) \langle T\psi(e'), e_2^\theta \rangle. \tag{A.18}$$

Also $\langle \partial_t \psi, e_2^S \rangle = \cos(\theta(z))$. We arrive at

$$\begin{aligned}
&\det \langle T\varphi(e'), (e^S)' \rangle \\
&= \sin\theta(z) \det \begin{pmatrix} \langle T\psi(e_2), e_2^\theta \rangle & \langle T\psi(e''), e_2^\theta \rangle \\ \langle T\psi(e_2), (e^S)'' \rangle & \langle T\psi(e''), (e^S)'' \rangle \end{pmatrix} \\
&+ \cos\theta(z) \det \begin{pmatrix} \langle d\tau, e_2 \rangle & 0 \\ \langle T\psi(e_2), (e^S)'' \rangle & \langle T\psi(e''), (e^S)'' \rangle \end{pmatrix} \\
&= \sin\theta(z) \det T\psi + \cos\theta(z) \langle d\tau, e_2 \rangle \det \langle T\psi(e''), (e^S)'' \rangle.
\end{aligned} \tag{A.19}$$

For the last equation we used the fact that $\det T\psi = \det \langle T\psi(e'), (e_2^\theta, (e^S)'') \rangle$, since e' and $(e_2^\theta, (e^S)'')$ are orthonormal bases. Note that by definition, $\langle T\psi(e''), e_2^\theta \rangle = 0$, so we also get $\det T\psi = \det\langle T\psi(e''), (e^S)''\rangle \times \langle T\psi(e_2), e_2^\theta\rangle$. On the other hand, we have

$$\langle d\tau, e_2\rangle = \langle T\psi(e_2), e_2^\theta\rangle \cot\theta(z). \tag{A.20}$$

Indeed, note that

$$\begin{aligned} 0 &= \left\langle T\varphi(e_2), N^S\right\rangle \\ &= \left\langle d\tau, e_2\right\rangle\left\langle e_1^S, N^S\right\rangle + \left\langle T\psi(e_2), N^S\right\rangle \\ &= \left\langle d\tau, e_2\right\rangle \sin(\theta(z)) - \cos(\theta(z))\left\langle T\psi(e_2), e_2^\theta\right\rangle \end{aligned}$$

where the last term is obtained by taking into account that $T\psi(e_2)$ is parallel to e_2^θ. This is the change of length of the geodesic needed to stay in S. We obtain

$$\begin{aligned} \det T\varphi &= \sin\theta(z)\det T\psi + \cos\theta(z)\cot\theta(z)\det T\psi \\ &= \frac{\sin^2\theta(z) + \cos^2\theta(z)}{\sin\theta(z)}\det T\psi. \end{aligned}$$

This yields (A.16).

We arrived at

$$\begin{aligned} \int_D gh\, d\mu = \int_{\partial D} g(y)\, \underline{\mu}(dy) - \int_{\partial D} g(\psi(\tau(y), y)) \det T\psi(\tau(y), y)\, \underline{\mu}(dy) \\ + \int_D \langle dg, N\rangle\, d\mu. \end{aligned} \tag{A.21}$$

this yields with (A.16)

$$\begin{aligned} \int_D gh\, d\mu = \int_{\partial D} g(y)\, \underline{\mu}(dy) - \int_{\partial D} g(\varphi(y)) \sin\theta(\varphi(y)) \det T\varphi(y)\, \underline{\mu}(dy) \\ + \int_D \langle dg, N\rangle\, d\mu. \end{aligned} \tag{A.22}$$

Using the change of variable $y \mapsto \varphi(y)$ and the fact that all $z \in S$ is equal to $\varphi(y_i)$, $i = 1, 2$, we obtain the key formula

Proposition A.1. *With the above notations, for any smooth function g defined on D such that gh is integrable or bounded below, we have:*

$$\int_D gh\, d\mu = \int_{\partial D} g(y)\, \underline{\mu}(dy) - 2\int_S g(z)\sin\theta(z)\, \underline{\mu}(dz) + \int_D \langle dg, N\rangle\, d\mu. \tag{A.23}$$

Appendix B. Moving sets

In this section we describe how to move a domain with smooth boundary by deformation of its boundary. We will investigate the deformation of its skeleton The deformation we will consider will have a general absolutely continuous finite variation part, together with a very specific martingale part and singular finite variation part. First we introduce some notation.

For a domain D with smooth boundary ∂D, $s \in \mathbb{R}$, define

$$\begin{aligned} \psi^D(s) = \psi^D(0, s) : \partial D \to \partial D(s) \\ y \mapsto \psi^D(s)(y) = \psi^D(s, y) = \exp_y\left(sN^D(y)\right). \end{aligned} \tag{B.1}$$

Here $N^D = N$ is the inward normal defined in Section A. Consider a moving domain $t \mapsto D_t$. Be careful not to confound $D(t)$ with D_t, since in general they are quite different subsets. We first assume that the deformation is sufficiently regular so that for all $0 \leqslant s \leqslant t$, we can write D_t as

$$D_t = \left\{ \psi^{D_s}([Z_t^{D_s}(y), \tau_{D_s}(y)], y), \quad y \in \partial D_s \right\}. \tag{B.2}$$

In particular, we must have $S'_s \subset D_t$. Notice that in the special case where the real valued function $t \mapsto Z_t^{D_s}(y)$ does not depend on y, for any $0 \leqslant s \leqslant t$, then we have

$$D_t = D_s(Z_t^{D_s}) = D_0(Z_t^{D_0}), \qquad Z_t^{D_0} = Z_t^{D_s} + Z_s^{D_0} \tag{B.3}$$

where $D(r)$ is defined in (A.3), replacing distance to ∂D by signed distance with positive sign inside D and negative sign outside. In this situation, the skeleton is not moving, at least as long as ∂D_t remains smooth (i.e. until ∂D_t hits S'_0 or is too far outside D_0), and $t \mapsto Z_t^{D_0}$ can be allowed to be a semimartingale with singular continuous drift.

When $t \mapsto Z_t^{D_s}(y)$ depends on y the situation is a little bit more complicated. Starting from $(t,y) \mapsto Z_t^{D_0}(y)$ which is assumed to be defined on $[0,\varepsilon) \times \partial D_0$, the sets D_t are defined for $0 \leqslant t < \varepsilon$, as well as the $Z_t^{D_s}(y)$, $0 \leqslant s \leqslant t$, $y \in D_s$. In fact, if $(y,t) \mapsto Z_t^{D_0}(y)$ is C^1, then one can reconstruct all $Z_t^{D_s}(y)$ with the only knowledge of $\dot Z_t^{D_t}(z)$, $z \in \partial D_t$. Let us do it for $s = 0$: the map $(t,y) \mapsto \psi^{D_0}(t,y)$ from $(-\alpha,\alpha) \times \partial D_0$ to M is a diffeomorphism on its range, for $\alpha > 0$ sufficiently small. Let us denote $z \mapsto (\tau_0(z), \varphi_0(z))$ its inverse. Then a variation $z + N^{D_t}(z) dZ_t^{D_t}$ corresponds to a variation $(\tau_0(z), \varphi_0(z)) + (d\tau_0, T\varphi_0) N^{D_t}(z) dZ_t^{D_t}$ of the coordinates in $(-\alpha,\alpha) \times \partial D_0$. But this is not convenient at all, since it is not intrinsic. Moreover, when passing to stochastic processes and Stratonovich equations, it will involve second derivatives of $z \mapsto (\tau_0(z), \varphi_0(z))$. So we prefer to leave the reference to D_0 and to always stay at the level of the moving D_t.

For all $y \in \partial D_0$ we define a stochastic process $t \mapsto Y_t(y)$ representing the motion of D_t satisfying $Y_0(y) = y$ and the Itô equation in manifold with respect to the Levi Civita connection ∇

(B.4)

$$dY_t(y) = d^\nabla Y_t(y) = \partial_1 \psi^{D_t}(\cdot, Y_t(y))(dZ_t^{D_t}(Y_t(y))) = N^{D_t}(Y_t(y)) dZ_t^{D_t}(Y_t(y)).$$

Recall that fomally $d^\nabla Y_t(y)$ is a vector which writes in local coordinates $(y^1, \ldots, y^d)$ with the Christoffel symbols $\Gamma^i_{j,k}$:

$$d^\nabla Y_t(y) = \left(dY_t^i(y) + \frac{1}{2} \Gamma^i_{j,k}(Y_t(y))\, d\langle Y_t^j(y), Y_t^k(y) \rangle \right) D_i(Y_t(y)) \tag{B.5}$$

where $D_i(Y_t(y))$ is the vector $\dfrac{\partial}{\partial y^i}$ taken at point $Y_t(y)$. We will always assume that the martingale part dm_t of $dZ_t^{D_t}(y)$ does not depend on y. In this situation, the Itô equation is equivalent to the Stratonovich one: indeed, using (B.3) the Itô to Stratonovich convertion term is

$$\frac{1}{2} \nabla_{N^{D_t}(Y_t(y)) dm_t} N^{D_t}(\cdot) dm_t = \frac{1}{2} \nabla_{N^{D_t}(Y_t(y))} N^{D_t}(\cdot) d\langle m, m \rangle_t = 0$$

since $N^{D_t}(Y_t(y))$ is the speed at time $a = 0$ of the geodesic $a \mapsto \psi^{D_t}(a)(Y_t(y))$.

More precisely, we will let $dZ_t^{D_t}(y)$ be of the form

$$dZ_t^{D_t}(y) = H^{D_t}(Y_t(y))\, dt + dz_t \tag{B.6}$$

where H^{D_t} is a smooth function on ∂D_t (which later on will be chosen to be $h^{D_t}/2$, where h^{D_t} is the mean curvature of ∂D_t) and $(z_t)_{t \geqslant 0}$ is a real valued continuous semimartingale. We assume that Equation (B.4) has a strong solution up to some positive stopping time.

Moreover, since $dY_t(y)$ represents the motion of ∂D_t and for small time the map $y' \mapsto Y_t(y')$ is a diffeomorphism from ∂D_0 to ∂D_t, writing $Y_t(y') = y$, equation (B.4) rewrites as

$$d\partial D_t(y) := dY_t(y') = N^{D_t}(y)\left(H^{D_t}(y)\,dt + dz_t\right). \tag{B.7}$$

Let us now investigate the motion of the skeleton S_t under this motion of D_t. First we remark that by local inversion theorem, at regular points of the skeleton, the variation in Stratonovich sense is linear and the sum of all variations of the concerned point at the boundary. As we already remarked, the motion dz_t does not change S_t, so this together with the linearity just mentioned implies that we have a finite variation of the skeleton.

Recall the situation of (A.2) in Section A. We consider a domain D, $x \in S$, y_1, y_2 the two elements of ∂D such that $\exp_{y_1}(\tau(y_1)N(y_1)) = \exp_{y_2}(\tau(y_2)N(y_2))$, with $\tau(y_1) = \tau(y_2)$. For $i = 1, 2$, we will consider a variation of the minimal geodesic from y_i to x, represented by a Jacobi field J_i satisfying $J_i(0) \in T_{y_i}M$, $J_1(1) = J_2(1) \in T_xM$,

$$J_i(0) = \lambda_i N(y_i) + J_i^\perp(0), \quad J_i'(0) = \lambda_i' N(y_i) + (J_i^\perp)'(0), \tag{B.8}$$

with $J_i^\perp$ orthogonal to $N(y_i)$. The motion of S corresponding to the motion of y_1 and y_2 will be represented by $J_1(1)$. Since S has a boundary, the observation of the orthogonal part to S of $J_1(1)$ is not sufficient.

Let γ_i be the projection on M of J_i. It is the geodesic in time 1 from y_i to x (as usual in the computations of Jacobi fields, the speed is not normalized). Denote $N_i(x) = \dot\gamma_i(1)/\|\dot\gamma_i(1)\|$. Recall that the angle between $N_i(x)$ and T_xS is $\theta(x) \in (0, \pi/2]$. We will also let

$$N_1^S(x) = \frac{1}{2\sin\theta(x)}(N_1(x) - N_2(x)). \tag{B.9}$$

Figure 2 shows the configuration of the points x, y_1, y_2 and the vectors $N_1(x)$, $N_2(x)$, $N_1^S(x)$. The vector $N_1^S(x)$ is is the normal vector to S at point x, in the same side as

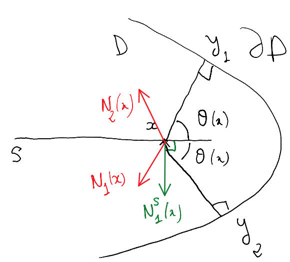

FIGURE 2. The points x, y_1, y_2 and the vectors $N_1(x), N_2(x), N_1^S(x)$

$N_1(x)$. We will consider variations of geodesics with same final value:

$$J_1(1) = J_2(1) = \lambda N_1^S(x) + J_1^T(1) \tag{B.10}$$

for some $\lambda \in \mathbb{R}$, where $J_1^T(1) \in T_xS$. Writing $\lambda N_1^S(x) = \frac{\lambda}{2\sin\theta(x)}(N_1(x) - N_2(x))$ we have

$$\begin{aligned} \langle J_1(1), N_1(x)\rangle &= \frac{\lambda}{2\sin\theta(x)}\left(1 - \cos(2\theta(x)\right) + \langle J_1^T(1), N_1(x)\rangle \\ &= \lambda \sin\theta(x) + \langle J_1^T(1), N_1(x)\rangle \end{aligned} \tag{B.11}$$

and

$$\begin{aligned} \langle J_1(1), N_2(x)\rangle &= -\frac{\lambda}{2\sin\theta(x)}\left(1 - \cos(2\theta(x)\right) + \langle J_1^T(1), N_2(x)\rangle \\ &= -\lambda \sin\theta(x) + \langle J_1^T(1), N_2(x)\rangle \end{aligned} \tag{B.12}$$

On the other hand we require that the variation of length of the two geodesics are the same. This writes as

$$\langle J_1(1), N_1(x)\rangle - \langle J_1(0), N(y_1)\rangle = \langle J_2(1), N_2(x)\rangle - \langle J_2(0), N(y_2)\rangle \tag{B.13}$$

or

$$\lambda\sin\theta(x) + \langle J_1^T(1), N_1(x)\rangle - \lambda_1 = -\lambda\sin\theta(x) + \langle J_1^T(1), N_2(x)\rangle - \lambda_2, \tag{B.14}$$

which finally, with $\langle J_1^T(1), N_1(x) - N_2(x)\rangle = 0$, yields $\lambda = \dfrac{\lambda_1 - \lambda_2}{2\sin\theta(x)}$, so the normal variation of S is given by

$$\langle J_1(1), N_1^S(x)\rangle N_1^S(x) = \frac{\lambda_1 - \lambda_2}{2\sin\theta(x)} N_1^S(x). \tag{B.15}$$

Next we will compute the tangential displacement $J^T(1)$ of x in S. As we will see later, we will only need a Jacobi field J_1 such that $J_1^\perp(0)$ and $(J_1^\perp)'(0)$ are known and

$$J_1(0) = \lambda_1 N(y_1), \text{ i.e. } \quad J_1^\perp(0) = 0. \tag{B.16}$$

So we know $J_1^\perp(1)$: and

$$J_1^\perp(1) = J\left(1, 0, (J_1^\perp)'(0)\right) \tag{B.17}$$

where $J(1,u,v)$ is the value at time 1 of the Jacobi field J with $J(0) = u$ and $J'(0) = v$. From

$$\begin{aligned} J_1(1) &= J_1^T(1) + \langle J_1(1), N_1^S(x)\rangle N_1^S(x) \\ J_1(1) &= J_1^\perp(1) + \langle J_1(1), N_1(x)\rangle N_1(x) \end{aligned} \tag{B.18}$$

we get

$$J_1^T(1) = J_1^\perp(1) + \langle J_1(1), N_1(x)\rangle N_1(x) - \langle J_1(1), N_1^S(x)\rangle N_1^S(x). \tag{B.19}$$

On the other hand we have

$$\begin{aligned} \langle J_1(1), N_2(x)\rangle &= \langle J_1^\perp(1), N_2(x)\rangle + \langle J_1(1), N_1(x)\rangle\langle N_1(x), N_2(x)\rangle \\ \langle J_1(1), N_2(x)\rangle &= \langle J_1(1), N_1(x)\rangle - (\lambda_1 - \lambda_2) \end{aligned} \tag{B.20}$$

where the second equation is a direct consequence of (B.15). Substracting the second equation to the first one yields

$$(1 - \cos(2\theta(x)))\langle J_1(1), N_1(x)\rangle = \langle J_1^\perp(1), N_2(x)\rangle + \lambda_1 - \lambda_2. \tag{B.21}$$

Replacing $\langle J_1(1), N_1(x)\rangle$ in (B.19) and after simplification, using (B.9) and (B.15), we finally obtain the horizontal displacement

(B.22)
$$(J_1^T)(1) = J_1^\perp(1)+\frac{1}{4\sin^2\theta(x)}\left(2\langle J_1^\perp(1), N_2(x)\rangle N_1(x) + (\lambda_1-\lambda_2)(N_1(x)+N_2(x))\right).$$

We are now in position to write the motion of the skeleton S_t when the motion of the boundary is given by (B.7). For $x \in S_t$ with corresponding points y_1 and y_2 in ∂D_t,

$$dS_t^\perp(x) = \frac{1}{2\sin\theta^{S_t}(x)}\left(H^{D_t}(y_1) - H^{D_t}(y_2)\right) N_1^{S_t}(x)\,dt \tag{B.23}$$

which has finite variation. Observe that, as already mentioned, the term dz_t disappears.

Here we wrote $dS_t^\perp(x)$ for the normal variation of the regular skeleton. But as we already remarked, since S_t is not a closed manifold, it can expand via the motion of its boundary. So we have to investigate the horizontal motion $dS^T(x)$.

Notice that $J_1^\perp)'(0)$ is the perpendicular part of the time derivative of the speed at y_1 of the geodesic in time 1 from y_1 to x. So from equation (B.7) we deduce the rotation

$$(J_1^\perp)'(0)\,dt = \rho_S(y_1)\nabla_t N^{D_t}(y_1) = -\rho_S(y_1)\nabla H^{D_t}(y_1)\,dt. \tag{B.24}$$

(in the r.h.s. the gradient corresponds to the tangential gradient on ∂D_t, recall that H^{D_t} is only defined on this hypersurface).

We conclude that the horizontal displacement of x is $J_1^T(1)\,dt$

$$\begin{aligned} J_1^T(1)\,dt = J_1^\perp(1)\,dt + \frac{1}{4\sin^2\theta^{S_t}(x)}\Bigg(& 2\langle J_1^\perp(1), N_2^{D_t}(x)\rangle N_1^{D_t}(x) \\ &+ (H^{D_t}(y_1) - H^{D_t}(y_2))(N_1^{D_t}(x)+N_2^{D_t}(x))\Bigg)\,dt \end{aligned} \tag{B.25}$$

where $J_1^\perp(1) = J(1,0,-\rho_S(y_1)\nabla H^{D_t}(y_1))$. Again the processus z_t does not play a role.

To summarize, we have the following result for the evolution of S_t:

Theorem B.1. *When D_t evolves as* (B.7)

$$d\partial D_t(y) = N^{D_t}(y)(H^{D_t}(y)\,dt + dz_t), \tag{B.26}$$

the regular skeleton S_t has the normal evolution (B.23)

$$dS_t^\perp(x) = \frac{H^{D_t}(y_1) - H^{D_t}(y_2)}{4\sin^2\theta^{S_t}(x)}\left(N_1^{D_t}(x) - N_2^{D_t}(x)\right)\,dt \tag{B.27}$$

and the tangential evolution (B.25) *which can be rewritten as*

$$\begin{aligned} &dS_t^T(x) \\ &= p_S(J_1^\perp(1))\,dt \\ &+ \left(-\frac{\langle J_1^\perp(1), N_1^S(x)\rangle}{2\sin\theta^{S_t}(x)} + \frac{H^{D_t}(y_1)-H^{D_t}(y_2)}{4\sin^2\theta^{S_t}(x)}\right)(N_1^{D_t}(x)+N_2^{D_t}(x))\,dt \end{aligned} \tag{B.28}$$

where p_S denotes the orthogonal projection on TS, $J_1^\perp(1) = J(1,0,-\rho_S(y_1)\nabla H^{D_t}(y_1))$, and y_1, y_2 are defined in Figure 2.

Remark B.2. The points y_1 and y_2 do not play the same role in Theorem B.1. As formula (B.27) is symmetric in y_1 and y_2, formula (B.28) is not. The reason is that if we assume the motion of y_1 to be normal to the boundary ∂D_t and to have speed given by (B.26), the motion of y_2 has no reason to be normal to the boundary: $J_2^\perp(0)$ does not vanish.

APPENDIX C. DOSS-SUSSMAN REPRESENTATION OF ITÔ'S EQUATION (2.11)

In this section we adapt the results of [6] to our notations. Let the stochastic mean curvature flow be a solution of :

$$\forall\, t \in [0,\tau),\, \forall\, x \in C_t, \qquad d\partial D_t(y) \;=\; \left(dW_t + \frac{1}{2}h^{D_t}(y)dt\right) N^{D_t}(y) \tag{C.1}$$

where $C_t := \partial D_t$, starting at D_0.

Let ∂G_t be a solution of

$$\begin{cases} & G_0 = D_0 \\ \forall\, t \in [0,\widetilde{\epsilon}),\, \forall\, x \in \partial G_t, & \partial_t x = \frac{1}{2}\alpha_{\partial G_t, W_t}(x) N^{G_t}(x) \end{cases} \tag{C.2}$$

for some $\widetilde{\epsilon} > 0$ small enough, where α is defined by

$$\forall\, r > 0,\, \forall\, D \in \mathcal{D}_r,\, \forall\, x \in C, \qquad \alpha_{C,r}(x) \;:=\; \frac{1}{2}h^{\Psi(C,r)}(\psi_{C,r}(x)) \tag{C.3}$$

and $\Psi(C,r)$ is the normal flow starting at C at time r.

Similarly to the proof of Theorem 9 from [6], we show that $D_t = \Psi(G_t, -W_t)$ is a solution of the stopped martingale problem associated to the generator $(\mathcal{D}, \widetilde{\mathcal{L}})$ where for $f \in C^\infty(M)$ and $\mathbb{F}_f(D) = \int_D f\, d\mu$,

$$\widetilde{\mathcal{L}}\mathbb{F}_f(D) := \int_{\partial D} \langle \nabla f, \nu \rangle\, d\underline{\mu} = \mathbb{F}_{\Delta f}(D).$$

Recall that the equation (C.2), is in fact a quasiparabolic equation with coefficients that depend on trajectory of the Brownian motion (the meaning is trajectory by trajectory). Similarly to Section 4.1 from [6], we show that the solution of (C.2) have a regularity $C^{1+\frac{\alpha}{2}, 2+\alpha}$, for all $\alpha < 1$.

Proposition C.1. *Let ∂G_t be a solution of* (C.2)*. Then $\partial D_t = \Psi(\partial G_t, -W_t)$ is a solution of* (C.1) *in the Itô sense.*

Proof. Let $x \in \Psi(\partial G_t, -W_t)$, we have :

$$\begin{aligned} & d\Psi(\partial G_t, -W_t)(x) = \\ & = T_1\Psi_{(\partial G_t, -W_t)}(\frac{d}{dt}\partial G_t)(\Psi^{-1}(\partial G_t, -W_t)(x)\, dt \\ & - N^{\Psi(\partial G_t, W_t)}(x) dW_t \\ & = \left(-dW_t + \frac{1}{2}h^{\Psi(\partial G_t, -W_t)}(x)dt\right) N^{\Psi(\partial G_t, -W_t)}(x), \end{aligned} \tag{C.4}$$

where in the first equality we use the Itô formula, and the fact that $t \mapsto \partial G_t$ is $C^{1+\frac{\alpha}{2}}$, $\frac{d^2}{d^2 r}\Psi(x,r) = 0$, and in the second equality we used Lemma 13 in [6], i.e. ∂D_t is a solution in the Itô form :

$$\begin{cases} d\partial D_t(x) &= (-dW_t + \frac{1}{2}h^{\partial D_t}(x)dt)\nu_{\partial D_t}(x) \\ x &\in \partial D_t. \end{cases} \tag{C.5}$$

□

Proposition C.2. *Conversely, if ∂D_t is a solution of* (C.5) *then $\partial G_t = \Psi(\partial D_t, -W_t)$ is a solution of* (C.2).

Proof. Let $x \in \partial\Psi(\partial D_t, -W_t)$

$$\begin{aligned} & d\Psi(\partial D_t, -W_t)(x) \\ & = T_1\Psi_{(\partial D_t,-W_t)}(\circ d\partial D_t)(x) + N^{\Psi(\partial D_t,W_t)}(x)dB_t \\ & = -T_1\Psi_{(\partial D_t,-W_t)}((-dW_t + \frac{1}{2}h^{\partial D_t}dt)N^{\partial D_t})(x) \\ & - N^{\Psi(\partial D_t,W_t)}(x)dW_t \\ & = \left(\frac{1}{2}h^{\partial D_t}(\Psi^{-1}(\partial D_t, W_t)(x))N^{\partial G_t}(x)dt\right) \\ & = \frac{1}{2}N^{\Psi(\partial G_t,-W_t)}(\Psi(\partial G_t, -W_t)(x))N^{\partial G_t}(x)dt \end{aligned} \tag{C.6}$$

where we use that the Stratonovich differential is equal to the Itô's one, i.e. $\circ d\partial D_t(x) = d\partial D_t$. So ∂G_t is a solution of (C.2). □

By the uniqueness of the solution of (C.2) and the fact that it is adapted to the filtration of B we deduce that the solution of (C.5) is unique and is a strong solution. Similarly we have the uniqueness of the solution of

$$d\partial D_t(x) = \left(dW_t + \frac{1}{2}h^{\partial D_t}(x)dt - \frac{\mu(\partial D_t)}{\mu(D_t)}dt\right)N^{\partial D_t}(x).$$

Moreover, since we could also make a change of time in the Itô equation, Equation (2.11) has a unique strong solution.

APPENDIX D. WEAK SEMI-GROUP THEORY IN THE MARTINGALE PROBLEM SENSE

This theory has been developed in several books, see for instance Stroock and Varadhan [19] or Ethier and Kurtz [8]. Here we present a minimal version suitable for our purposes.

Let V be a measurable state space and consider Ω a set of trajectories from $\mathbb{R}_+$ to V. The canonical coordinates on Ω are denoted by the X_t, for $t \geqslant 0$: for $\omega \in \Omega$, $X_t(\omega)$ is the position at time t of ω. The set Ω is endowed with the sigma-field generated by the X_t, for $t \geqslant 0$. Our first assumption is that the mapping

$$\Omega \times \mathbb{R}_+ \ni (\omega, t) \quad \mapsto \quad X_t(\omega) \in V$$

is measurable, which usually means that "Ω is not too big".

For $t \geqslant 0$, we define

$$\mathcal{F}_t \quad := \quad \sigma(X_s \,:\, s \in [0, t])$$

For $t \geqslant 0$, we will also need the time shift Θ_t associating to any $\omega \in \Omega$ the trajectory $\Theta_t(\omega)$ defined by

$$\forall\, s \geqslant 0, \qquad X_s(\Theta_t(\omega)) \quad = \quad X_{s+t}(\omega)$$

We assume that $\Theta_t(\Omega) \subset \Omega$.

A given family $\mathbb{P} := (\mathbb{P}_x)_{x\in V}$ of probability measures on Ω is said to be **Markovian** if for any $x \in V$ and any $t \geqslant 0$, the image by Θ_t of $\mathbb{P}_x$ conditioned by $\mathcal{F}_t$ is $\mathbb{P}_{X_t}$. In particular, it is assumed that $\mathbb{P}$ has the regularity of a Markov kernel from V to Ω.

From now on, we suppose that a Markovian family $\mathbb{P}$ is given. Let $\mathcal{B}$ be the space of bounded and measurable functions defined on V. The **semi-group** $P := (P_t)_{t\geqslant 0}$ associated to $\mathbb{P}$ is the family of operators acting on $\mathcal{B}$ via

$$\forall\, t \geqslant 0,\, \forall\, f \in \mathcal{B},\, \forall\, x \in V, \qquad P_t[f](x) \quad := \quad \mathbb{E}_x[f(X_t)]$$

The Markovianity of $\mathbb{P}$ implies at once the semi-group property

$$\forall\, s,t \geqslant 0, \qquad P_tP_s \;=\; P_{t+s}$$

and in particular the elements of P commute.

A subclass of "regular" functions that will be important for our purposes is $\mathcal{R}$ defined as

$$\mathcal{R} \;:=\; \left\{ f\in\mathcal{B}\,:\,\forall\, x\in V,\ \lim_{t\to 0_+} P_t[f](x) \;=\; f(x)\right\}$$

Exceptionally in the above limit, we assumed that $t \geqslant 0$ (i.e. not only that $t > 0$), so that by definition, for any $f \in \mathcal{R}$ and $x \in V$, $P_0[f](x) = f(x)$.

Let us observe that $\mathcal{R}$ is left stable by the semi-group:

Lemma D.1. *For any $t \geqslant 0$, we have $P_t[\mathcal{R}] \subset \mathcal{R}$. Thus for any given $f \in \mathcal{R}$ and $x \in V$, the mapping*

$$\mathbb{R}_+ \ni t \;\mapsto\; P_t[f](x)$$

is right continuous.

Proof. Indeed, fix $t \geqslant 0$ and $f \in \mathcal{R}$, we have for any $x \in V$ and $s \geqslant 0$,

$$\begin{aligned} P_s[P_t[f]](x) &= P_t[P_s[f]](x) \\ &= \mathbb{E}_x[P_s[f](X_t)]] \end{aligned}$$

We have for any $s \geqslant 0$, $\|P_s[f]\|_\infty \leqslant \|f\|_\infty$ (where $\|\cdot\|_\infty$ stands for the supremum norm on $\mathcal{B}$) and since $f \in \mathcal{R}$, we get everywhere

$$\lim_{s\to 0_+} P_s[f](X_t) \;=\; f(X_t)$$

Dominated convergence implies that

$$\begin{aligned} \lim_{s\to 0_+} \mathbb{E}_x[P_s[f](X_t)]] &= \mathbb{E}_x[f(X_t)] \\ &= P_t[f] \end{aligned}$$

as desired. □

The **generator** L associated to P is the operator

$$L\,:\,\mathcal{D}(L) \;\to\; \mathcal{R}$$

defined in the following way: the space $\mathcal{D}(L)$ is the set of functions $f \in \mathcal{R}$ for which there exists a function $g \in \mathcal{R}$ such that the process $M^{f,g} := (M^{f,g}_t)_{t\geqslant 0}$ defined by

$$\forall\, t \geqslant 0, \qquad M^{f,g}_t \;:=\; f(X_t) - f(X_0) - \int_0^t g(X_s)\,ds$$

is a martingale under $\mathbb{P}_x$, for all $x \in V$.

Let us remark that g is then uniquely determined. Indeed, we have for any $x \in V$ and $t \geqslant 0$,

$$\mathbb{E}_x[f(X_t)] - \mathbb{E}[f(X_0)] - \mathbb{E}\left[\int_0^t g(X_s)\,ds\right] \;=\; 0$$

Using Fubini's lemma (applicable due to our measurability requirement on Ω) and taking into account the definition of P, we get

$$P_t[f](x) - P_0[f](x) - \int_0^t P_s[g](x)\,ds \;=\; 0$$

namely, recalling that we required that $g \in \mathcal{R}$,

$$\begin{aligned} g &= P_0[g] \\ &= \lim_{t\to 0_+} \frac{1}{t}\int_0^t P_s[g](x)\,ds \\ &= \lim_{t\to 0_+} \frac{P_t[f](x)-f(x)}{t} \end{aligned} \tag{D.1}$$

(we came back to the usual convention that $t > 0$ in the above limit) and as a by-product, we are assured of the existence of the latter limit.

We define $L[f] := g$ and $M^f := M^{f,g}$.

The differentiation property (D.1) can be extended into

Lemma D.2. *For any $f \in \mathcal{D}(L)$, $x \in V$ and $t \geqslant 0$, we have*

$$\partial_t P_t[f](x) = P_t[L[f]](x) \tag{D.2}$$

Proof. For any $f \in \mathcal{D}(L)$, $x \in V$ and $t, s \geqslant 0$, we have

$$\begin{aligned} \mathbb{E}_x\left[M^f_{t+s} - M^f_t\right] &= \mathbb{E}_x\left[\mathbb{E}_x\left[M^f_{t+s} - M^f_t|\mathcal{F}_t\right]\right] \\ &= 0 \end{aligned}$$

We compute that

$$M^f_{t+s} - M^f_t = f(X_{t+s}) - f(X_t) - \int_t^{t+s} L[f](X_u)\,du$$

so that

$$\mathbb{E}_x\left[M^f_{t+s} - M^f_t\right] = P_{t+s}[f](x) - P_t[f](x) - \int_0^s P_{t+u}[L[f]](x)\,du$$

Since $L[f] \in \mathcal{R}$, the mapping $[0, s] \ni u \mapsto P_{t+u}[L[f]](x)$ is right continuous, according to Lemma D.1, and the same argument as in (D.1) enables to conclude to (D.2). □

We can now come to the main goal of this appendix:

Proposition D.3. *For any $t \geqslant 0$, $\mathcal{D}(L)$ is stable by P_t and on $\mathcal{D}(L)$ we have $LP_t = P_tL$.*

Proof. Fix $f \in \mathcal{D}(L)$ and $x \in V$, the assertion of the lemma amounts to checking that the process $N := (N_s)_{s\geqslant 0}$ defined by

$$(N_s)_{s\geqslant 0} := \left(P_t[f](X_s) - P_t[f](X_0) - \int_0^s P_t[L[f]](X_u)\,du\right)_{s\geqslant 0}$$

is a martingale under $\mathbb{P}_x$. Consider $s' \geqslant s \geqslant 0$, we have to prove that

$$\mathbb{E}_x[N_{s'} - N_s|\mathcal{F}_s] = 0 \tag{D.3}$$

The l.h.s. is equal to

$$\begin{aligned} &\mathbb{E}_x\left[P_t[f](X_{s'}) - P_t[f](X_s) - \int_s^{s'} P_t[L[f]](X_u)\,du\Big|\mathcal{F}_s\right] \\ &= \mathbb{E}_x\left[P_t[f](X_{s'-s}\circ\Theta_s) - P_t[f](X_0\circ\Theta_s) - \int_0^{s'-s} P_t[L[f]](X_u\circ\Theta_s)\,du\Big|\mathcal{F}_s\right] \\ &= \mathbb{E}_y\left[P_t[f](X_{s'-s}) - P_t[f](X_0) - \int_0^{s'-s} P_t[L[f]](X_u)\,du\right] \end{aligned}$$

where $y = X_s$. By Fubini's lemma, the previous r.h.s. can be written

$$\mathbb{E}_y\left[P_t[f](X_{s'-s})\right] - \mathbb{E}_y\left[P_t[f](X_0)\right] - \int_0^{s'-s} \mathbb{E}_y[P_t[L[f]](X_u)]\,du$$
$$= \quad P_{t+s'-s}[f](y) - P_t[f](y) - \int_0^{s'-s} P_{t+u}[L[f]](y)\,du$$

Taking into account (D.2), the last integral is equal to

$$\int_0^{s'-s} \partial_u P_{t+u}[f](y)\,du \quad = \quad P_{t+s'-s}[f](y) - P_t[f](y)$$

which ends the proof of (D.3). □

The advantage of the above approach is that it is quite sable by optional stopping, as it is the case for martingales. Let us succinctly give a simple example in the spirit of Section 2.

Assume that in the above framework, V is a metric space, endowed with its Borelian measurable structure, and that Ω is the set of continuous trajectories $\mathcal{C}(\mathbb{R}_+, V)$. Furthermore, we suppose that P is **Fellerian**, in the sense that it preserves $\mathcal{C}_{\mathrm{b}}(V)$, the set of bounded and continuous real functions on V.

Let be given $A \subset V$ a closed set. We consider τ the hitting time of A:

$$\tau \quad := \quad \inf\{t \geqslant 0 \,:\, X_t \in A\} \,\in\, \mathbb{R}_+ \sqcup \{+\infty\}$$

Define the "new" process $\widetilde{X} := (\widetilde{X}_t)_{t\geqslant 0}$ via

$$\forall\, t \geqslant 0, \qquad \widetilde{X}_t \quad := \quad X_{t\wedge\tau}$$

and for $x \in V$, let $\widetilde{\mathbb{P}}_x$ be the image of $\mathbb{P}_x$ by $\widetilde{X}$, it is still a probability measure on $\mathcal{C}(\mathbb{R}_+, V)$. All notions corresponding to $\widetilde{\mathbb{P}} := (\widetilde{\mathbb{P}}_x)_{x\in V}$, which is still a Markovian family, receive a tilde. It appears without difficulty that $\widetilde{\mathcal{R}}$ is the set of functions $\widetilde{f} \in \mathcal{B}$ such that there exists $f \in \mathcal{R}$ with $\widetilde{f}$ coinciding with f on $V\backslash A$. The domain $\mathcal{D}(\widetilde{L})$ is the set of $\widetilde{f} \in \widetilde{\mathcal{R}}$ such that there exists $f \in \mathcal{D}(L)$ with $\widetilde{f}$ coinciding with f on $V\backslash A$. In addition, we have

$$\forall\, x \in V, \qquad \widetilde{L}[\widetilde{f}](x) \quad = \quad \begin{cases} L[f](x) & \text{, when } x \notin A \\ 0 & \text{, when } x \in A \end{cases}$$

This expression does not depend on the choice of f, due to the fact that $\mathbb{P}$ is a diffusion, i.e. that $\Omega = \mathcal{C}(\mathbb{R}_+, V)$, which implies that L is a local operator (see for instance Theorem 7.29 of Schilling and Partzsch [18], they are working with Euclidean spaces, but the result can be extended to metric spaces).

According to (D.2) and Proposition D.3, we get

$$\forall\, \widetilde{f} \in \mathcal{D}(\widetilde{L}),\, \forall\, x \in V,\, \forall\, t \geqslant 0 \qquad \partial_t \widetilde{P}_t[\widetilde{f}](x) \quad = \quad \widetilde{P}_t[\widetilde{L}[\widetilde{f}]](x) \;=\; \widetilde{L}[\widetilde{P}_t[\widetilde{f}]](x)$$

Such relations are not so obvious if we had chosen to work in a Banach setting (cf. e.g. the book of Yosida [20]), considering for instance semi-groups acting on the space $\mathcal{C}_{\mathrm{b}}(V)$ (endowed with the supremum norm), since in general $\widetilde{L}$ would not naturally take values in $\mathcal{C}_{\mathrm{b}}(V)$.

APPENDIX E. A MEASURE THEORY RESULT

This appendix is not used in this paper. However it could offer an alternative to Lemma H.1 if we were able to establish that for $i = 1, 2$ and

$$\left((X_t^{i,\delta}, D_t^\delta, W_t^{i,\delta}, \widetilde{W}_t^\delta, W_t^{i,\delta,m})_{t\geqslant 0}, \tau_\varepsilon^\delta\right) \tag{E.1}$$

as in (3.18) such that

$$\left(X_t^{i,\delta}, W_t^{i,\delta}, W_t^{i,\delta,m}\right)_{t\geqslant 0}, \quad i = 1, 2,$$

are conditionally independent given $\left((D_t^\delta, \widetilde{W}_t^\delta)_{t\geqslant 0}, \tau_\varepsilon^\delta\right)$, then the conditional independence remains true in the limit. Conditioning with respect to $\left((D_t^\delta, \widetilde{W}_t^\delta)_{t\geqslant 0}, \tau_\varepsilon^\delta\right)$ would allow to work with finite dimensional processes and Lemma 4 in [21] would be sufficient for the convergences required by Theorem 3.5, instead of resorting to Lemma H.1.

Consider V and W two Polish spaces. Let μ be a probability measure on V and $(K_n)_{n\in\mathbb{N}\sqcup\{\infty\}}$ a family of Markov kernels from V to W. For any $n \in \mathbb{N} \sqcup \{\infty\}$, define the probability measure m_n on $V \times W \times W$ via

$$\forall\ (x, y, z) \in V \times W \times W, \qquad m_n(dx, dy, dz) \quad := \quad \mu(dx)K_n(x, dy)K_n(x, dz)$$

We have the following result:

Proposition E.1. *Assume that the sequence $(m_n)_{n\in\mathbb{N}}$ weakly converges toward m_∞ on $V \times W \times W$. Then there exists a subsequence $(n_l)_{l\in\mathbb{N}}$ such that μ-a.s. in $x \in V$, the sequence $K_{n_l}(x, \cdot)$ weakly converges toward $K_\infty(x, \cdot)$ on W.*

Proof. Fix $f \in \mathcal{C}_b(V)$ and $g \in \mathcal{C}_b(W)$, where $\mathcal{C}_b(V)$ (respectively $\mathcal{C}_b(W)$) stands for the space of bounded continuous functions on V (resp. W).

Consider the function $h := f \otimes g \otimes \mathbb{1}_W$, where $\mathbb{1}_W$ is the mapping on W only taking the value 1. We have $h \in \mathcal{C}_b(V \times W^2)$, so we have

$$\lim_{n\to\infty} m_n[h] \quad = \quad m_\infty[h]$$

namely

$$\lim_{n\to\infty} \int f K_n[g]\, d\mu \quad = \quad \int f K_\infty[g]\, d\mu \tag{E.2}$$

Let us extend this convergence to any $f \in \mathbb{L}^2(\mu)$.

Indeed, given $\epsilon > 0$, we can find $\widetilde{f} \in \mathcal{C}_b(V)$ such that

$$\left\| f - \widetilde{f} \right\|_{\mathbb{L}^2(\mu)} \quad \leqslant \quad \epsilon$$

For any $n \in \mathbb{N} \sqcup \{\infty\}$, we have

$$\begin{aligned}
\left| \int f K_n[g]\, d\mu - \int \widetilde{f} K_n[g]\, d\mu \right| \quad &\leqslant \quad \int \left| f - \widetilde{f} \right| |K_n[g]|\ d\mu \\
&\leqslant \quad \|K_n[g]\|_\infty \int \left| f - \widetilde{f} \right|\ d\mu \\
&\leqslant \quad \|g\|_\infty \left\| f - \widetilde{f} \right\|_{\mathbb{L}^2(\mu)} \\
&\leqslant \quad \|g\|_\infty\, \epsilon
\end{aligned}$$

where $\|\cdot\|_\infty$ stands for the supremum norm and where we used the Cauchy-Schwarz inequality.

It follows that

$$\begin{aligned}\limsup_{n\to\infty}\int fK_n[g]\,d\mu &\leqslant \epsilon+\lim_{n\to\infty}\int \widetilde{f}K_n[g]\,d\mu\\ &\leqslant \epsilon+\int \widetilde{f}K_\infty[g]\,d\mu\\ &\leqslant 2\epsilon+\int fK_\infty[g]\,d\mu\end{aligned}$$

Similarly, we deduce that

$$\limsup_{n\to\infty}\int fK_n[g]\,d\mu \geqslant \int fK_\infty[g]\,d\mu-2\epsilon$$

Since $\epsilon>0$ can be chosen arbitrary small, we get the validity of (E.2), for any $f\in\mathbb{L}^2$, when $g\in\mathcal{C}_{\mathrm{b}}(W)$ is fixed, namely the weak convergence of $(K_n[g])_{n\in\mathbb{N}}$ toward $K_\infty[g]$ in $\mathbb{L}^2(\mu)$.

To transform this weak convergence into a strong convergence, it is sufficient to check the convergence of the corresponding $\mathbb{L}^2(\mu)$ norms, i.e. that

$$\lim_{n\to\infty}\int(K_n[g])^2\,d\mu = \int(K_\infty[g])^2\,d\mu$$

This is also a consequence of the weak convergence of $(m_n)_{n\in\mathbb{N}}$ toward m, by considering the mapping $\mathbb{1}_V\otimes g\otimes g$ on $V\times W\times W$.

Thus we have shown that for any given $g\in\mathcal{C}_{\mathrm{b}}(W)$, the sequence $(K_n[g])_{n\in\mathbb{N}}$ converges toward $K_\infty[g]$ in $\mathbb{L}^2(\mu)$. As a consequence, there exists a subsequence $(n_l)_{l\in\mathbb{N}}$ such that $(K_{n_l}[g])_{l\in\mathbb{N}}$ converges μ-a.s. toward $K_\infty[g]$.

A priori, this subsequence $(n_l)_{l\in\mathbb{N}}$ may depend on g, so we resort to a diagonal procedure to avoid this difficulty. More precisely, let $(g_k)_{k\in\mathbb{N}}$ be a sequence of functions from $\mathcal{C}_{\mathrm{b}}(W)$ characterizing the weak convergence on W. According to the above arguments, for any $k\in\mathbb{N}$, there exists a subsequence $(n_l^{(k)})_{l\in\mathbb{N}}$ such that $(K_{n_l^{(k)}}[g_k])_{l\in\mathbb{N}}$ converges μ-a.s. toward $K_\infty[g_k]$. Furthermore, these subsequences can be constructed iteratively: first we find $(n_l^{(1)})_{l\in\mathbb{N}}$, next $(n_l^{(2)})_{l\in\mathbb{N}}$ is obtained as a subsequence of $(n_l^{(1)})_{l\in\mathbb{N}}$, and so on, for any $k\in\mathbb{N}$, $(n_l^{(k+1)})_{l\in\mathbb{N}}$ is a subsequence of $(n_l^{(k)})_{l\in\mathbb{N}}$. Define the subsequence $(n_l)_{l\in\mathbb{N}}$ via

$$\forall\, l\in\mathbb{N},\qquad n_l := n_l^{(l)}$$

The sequence $(n_l)_{l\in\mathbb{N}}$ is a subsequence of all the subsequences $(n_l^{(k)})_{l\in\mathbb{N}}$, and so for any $k\in\mathbb{N}$, $(K_{n_l}[g_k])_{l\in\mathbb{N}}$ converges μ-a.s. toward $K_\infty[g_k]$. Taking the union of the underlying μ-negligible sets, we get that μ-a.s., for any $k\in\mathbb{N}$, $(K_{n_l}[g_k])_{l\in\mathbb{N}}$ converges toward $K_\infty[g_k]$. By choice of the sequence $(g_k)_{k\in\mathbb{N}}$, the desired result follows.

■

APPENDIX F. AN ITÔ-TANAKA FORMULA

Let M be a d-dimensional Riemannian manifold and $D\subset M$ a compact and connected domain with C^2 boundary ∂D, and S be the regular skeleton of D, and $\rho_{\partial D}^{+}$ the signed distance to ∂D, which is positive inside D and negative outside D. The notations will be the same as in Appendix A.

Proposition F.1. *Let X_t a Brownian motion in M. We have the following Itô-Tanaka formula :*

$$d\rho^+_{\partial D}(X_t) = \langle N^D(X_t), dX_t\rangle - \frac{1}{2}h^D(X_t)dt - \sin\left(\theta^S(X_t)\right)\, dL^S_t(X),$$

in the above formula, $N^D(x) = \nabla\rho^+_{\partial D}(x)$ and $-h^D(x) = \Delta\rho^+_{\partial D}(x)$ for $x \notin S$, and define to be 0 elsewhere, $L^S_t(X)$ is the local time defined as in (3.11).

Proof. The formula is a consequence of the Itô formula outside the skeleton. Since the non regular part of the skeleton has Hausdorff dimension smaller than or equal to $d-2$, it is not visited by the Brownian motion. So we only focus on the regular skeleton. For all $x \in S$, the distance to the boundary is the minimum of two C^2 functions f, g defined on some neighborhood U of x in M. The function f (resp. g) is the distance function to a piece of ∂D containing y_1 (resp. y_2) as in (A.2). We have locally,

$$\rho^+_{\partial D} = f \wedge g = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|.$$

Using Itô formula and Tanaka formula we have

$$\begin{aligned} d\rho^+_{\partial D}(X_t) = {}& \frac{1}{2}\Big(\frac{1}{2}\Delta(f+g)(X_t)dt + \langle\nabla(f+g)(X_t), dX_t\rangle\Big) \\ & - \frac{1}{2}\Big(\mathrm{sign}((f-g)(X_t))d((f-g)(X_t)) + dL^{0,+}_t((f-g)(X_\cdot))\Big), \end{aligned}$$

where $L^{0,+}_t((f-g)(X_\cdot)) = \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_0^t \mathbb{1}_{[0,\varepsilon]}((f-g)(X_s))d\langle(f-g)(X),(f-g)(X)\rangle_s$. Since locally $S = \{f-g=0\}$ and $\mu(S) = 0$, we have

$$d\rho^+_{\partial D}(X_t) = \frac{1}{2}\mathbb{1}_{X_t\notin S}\Delta\rho^+_{\partial D}(X_t)dt + \mathbb{1}_{X_t\notin S}\langle\nabla\rho^+_{\partial D}(X_t), dX_t\rangle - \frac{1}{2}dL^{0,+}_t((f-g)(X_\cdot)).$$

After changing the role of f and g we get
(F.1)

$$d\rho^+_{\partial D}(X_t) = \frac{1}{2}\mathbb{1}_{X_t\notin S}\Delta\rho^+_{\partial D}(X_t)dt + \mathbb{1}_{X_t\notin S}\langle\nabla\rho^+_{\partial D}(X_t), dX_t\rangle - \frac{1}{2}dL^{0}_t((f-g)(X_\cdot)),$$

where

$$L^0_t((f-g)(X_\cdot)) = \lim_{\varepsilon\to 0^+}\int_0^t \frac{1}{2\varepsilon}\mathbb{1}_{[-\varepsilon,\varepsilon]}((f-g)(X_s))\|\nabla(f-g)\|^2(X_s)\,ds.$$

In Appendix A it is shown that for $x \in S$, $\|\nabla(f-g)(x)\| = 2\sin\left(\theta^S(x)\right)$.

Using the flow $\frac{d}{dt}\gamma(t) = -\frac{\nabla(f-g)(\gamma(t))}{\|\nabla(f-g)(\gamma(t))\|^2}$ that starts at $y \in U$, we get

$$\{y \in M, \text{ s.t. } |f-g|(y) \leqslant \varepsilon\} \subset \{y \in M, \text{ s.t. } |d_S(y)| \leqslant \frac{\varepsilon}{2\sin\left(\theta^S(\gamma(g(y)))\right)} + o(\varepsilon)\},$$

where d_S is the distance to S. On the other hand, using the minimal geodesic from S to $y \in U$ we get

$$\{y \in M, \text{ s.t. } |d_S(y)| \leqslant \varepsilon\} \subset \{y \in M, \text{ s.t. } |f-g|(y) \leqslant 2\varepsilon\sin\left(\theta^S(P^S(y))\right) + o(\varepsilon)\}.$$

Hence

$$dL^0_t((f-g)(X_\cdot)) = 2\sin\left(\theta^S(X_t)\right)L^S_t(X_\cdot).$$

Together with (F.1), this yield the Proposition. □

APPENDIX G. UNIQUENESS IN LAW OF $\tilde{\mathcal{L}}$ DIFFUSION

Let us consider the following generator $\widehat{\mathscr{L}}$ of a stochastic modified mean curvature flow. The action of this generator and its carré du champs on elementary observables are defined as follows. For any smooth function k on M, consider the mapping F_k on $\mathcal{D}^{2+\alpha}$ defined by

$$\forall\, D \in \mathcal{D}^{2+\alpha}, \qquad F_k(D) \quad := \quad \int_D k\, d\mu$$

For any $k, g \in \mathcal{C}^\infty(M)$ and any $D \in \mathcal{D}^{2+\alpha}$,

$$\text{(G.1)} \qquad \begin{cases} \widehat{\mathscr{L}}[F_k](D) & := \quad -\frac{1}{2}\underline{\mu}^{\partial D}(\langle \nabla k, N^D\rangle) = F_{\frac{1}{2}\Delta k}(D) \\ \Gamma_{\widehat{\mathscr{L}}}[F_k, F_g](D) & := \quad \int_{\partial D} k\, d\underline{\mu} \int_{\partial D} g\, d\underline{\mu}. \end{cases}$$

Note that $\widehat{\mathscr{L}}$ has the same carré du champs as the carré du champs associated to $\widetilde{\mathscr{L}}$. From now the generator $\widehat{\mathscr{L}}$ is defined as in (2.14).

Proposition G.1. *The martingale problem associated $\widehat{\mathscr{L}}$ is well-posed.*

Proof. We have already shown the existence result in [6], so it remains to prove the uniqueness in law. Let us first consider the two-dimensional Euclidean case, namely $M = \mathbb{R}^2$. For all $\lambda \in \mathbb{R}$ and for any function $k_\lambda \in \mathrm{vect}(e^{\lambda x}, e^{\lambda y})$ we have $\frac{1}{2}\Delta k_\lambda(x, y) = \frac{\lambda^2}{2} k_\lambda(x, y)$. Let $f_\lambda((x, y), D) := k_\lambda(x, y) F_{k_\lambda}(D)$, for $(x, y) \in \mathbb{R}^2$ and $D \in \mathcal{D}^{2+\alpha}$. This function satisfies the following property:

$$\begin{aligned} \widehat{\mathscr{L}} f_\lambda((x, y), D) &= k_\lambda(x, y) \widehat{\mathscr{L}} F_{k_\lambda}(D) \\ &= k_\lambda(x, y) F_{\frac{1}{2}\Delta k_\lambda}(D) \\ &= k_\lambda(x, y) F_{\frac{\lambda^2}{2} k_\lambda}(D) \\ &= \frac{\lambda^2}{2} k_\lambda(x, y) F_{k_\lambda}(D) \\ &= \frac{1}{2}\Delta k_\lambda(x, y) F_{k_\lambda}(D). \\ &= \frac{1}{2}\Delta f_\lambda((x, y), D) \end{aligned}$$

Let $(X_t)_{t\geqslant 0}$ be a $\mathbb{R}^2$-valued Brownian motion that starts at $X_0 = (x_1, x_2) \in \mathbb{R}^2$ and $(\hat{D}_t)_{t\geqslant 0}$ a $\widehat{\mathscr{L}}$ diffusion that starts at D_0 independent of $(X_t)_{t\geqslant 0}$. Even if we stop the diffusion, we can assume that its lifetime is infinite and we add indicators as described in Appendix D. For all $0 \leqslant s \leqslant t$, we have

$$df_\lambda(X_{t-s}, \hat{D}_s) \overset{m}{=} -\frac{1}{2}\Delta f_\lambda(X_{t-s}, \hat{D}_s)ds + \widehat{\mathscr{L}} f_\lambda(X_{t-s}, \hat{D}_s)ds \overset{m}{=} 0.$$

Hence for all $\lambda \in \mathbb{R}$ we have

$$\text{(G.2)} \qquad \mathbb{E}[f_\lambda(X_t, D_0)] = \mathbb{E}[f_\lambda(X_0, \hat{D}_t)].$$

Since the left hand side of the above equation does not depend on the $\widehat{\mathscr{L}}$ diffusion, we get that for any $\widehat{\mathscr{L}}$ diffusion $(\tilde{D}_t)_{t\geqslant 0}$ that starts at D_0 :

$$\mathbb{E}[f_\lambda(X_0, \hat{D}_t)] = \mathbb{E}[f_\lambda(X_0, \tilde{D}_t)],$$

and so

$$\mathbb{E}[F_{k_\lambda}(D_t)] = \mathbb{E}[F_{k_\lambda}(\tilde{D}_t))].$$

In order to apply Theorem 4.2 of [8], we have to show that the above equation characterizes the law of the one-dimensional distribution, i.e. we have to show that (F_{k_λ}) is separating in the space of probability measures on $\mathcal{D}^{2+\alpha}$. This is equivalent to separate domains. Let $A, B \in \mathcal{D}^{2+\alpha}$ such that $F_{k_\lambda}(A) = F_{k_\lambda}(B)$ for all $\lambda \in \mathbb{R}$ and $k_\lambda \in \langle e^{\lambda x}, e^{\lambda y}\rangle$, we have for all λ:

$$\int_A k_\lambda(x,y)d\mu = \int_B k_\lambda(x,y)d\mu.$$

After successive derivations in λ and evaluation at $\lambda = 0$, we get for all $n \in \mathbb{N}$

$$\int_A x^n d\mu = \int_B x^n d\mu,$$

$$\int_A y^n d\mu = \int_B y^n d\mu,$$

The above computations could be done also for $\tilde{k}_{\lambda_1,\lambda_2} = e^{\lambda_1 x + \lambda_2 y}$, since $\frac{1}{2}\Delta \tilde{k}_{\lambda_1,\lambda_2} = \frac{\lambda_1^2+\lambda_2^2}{2}\tilde{k}_{\lambda_1,\lambda_2}$, and after derivations in λ_1, λ_2 and evaluating at $(0,0)$ we get that for all $n, m \in \mathbb{N}$:

$$\int_A x^n y^m d\mu = \int_B x^n y^m d\mu,$$

hence, using the boundary regularity, we get $A = B$.

We could also apply Stone-Weierstrass' theorem to the function algebra generated by the mappings $(x,y) \mapsto e^{\lambda_1 x}$ and $(x,y) \mapsto e^{\lambda_2 y}$.

The proof is the same for all Euclidean spaces.

If M is a compact manifold let

$$f_{\lambda_i}(X,D) := k_{\lambda_i}(X)F_{k_{\lambda_i}}(D),$$

where λ_i is an eigenvalue of $\frac{1}{2}\Delta$ and k_i is the associated eigenfunction (respectively the Neumann eigenvalue). By the same computation as above (G.2) is also valid for the boundary reflecting Brownian motion), to get the conclusion we have to show that $(F_{k_{\lambda_i}})_i$ separates domains. Since $(k_{\lambda_i})_i$ is an orthonormal basis of $L^2(\mu)$ we get that if $A, B \in \mathcal{D}^{2+\alpha}$ be such that for all i,

$$F_{k_{\lambda_i}}(A) = F_{k_{\lambda_i}}(B)$$

i.e $\langle \mathbb{1}_A, k_{\lambda_i}\rangle_{L^2} = \langle \mathbb{1}_B, k_{\lambda_i}\rangle_{L^2}$, then $\mathbb{1}_A \overset{L^2}{=} \mathbb{1}_B$ hence $A = B$.

For the complete manifold M, let Ω_k be an exhaustion of M with a regular boundary such that $D_0 \subset \Omega_k$, and stop the $\widehat{\mathscr{L}}$ diffusion when it hit Ω_k^c and use the above result for the manifold with boundary Ω_k, we get the result by localization.

□

Proposition G.2. *The martingale problem associated to $\mathscr{L}$ is well-posed.*

Proof. Let D_t be a $\mathscr{L}$ diffusion that starts at D_0, defined on $(\Omega, \mathcal{F}^D, \mathbb{Q})$. We first recall that there exist an enlargement of the probability space such that it carries a one dimensional Brownian motion B such that for all $k \in C^\infty(M)$

$$F_k(D_t) = F_k(D_0) + \int_0^t \mathscr{L}[F_k](D_s)\,ds + \int_0^t \sqrt{\Gamma_{\mathscr{L}}[F_k,F_k]}(D_s)\,dB_s \tag{G.3}$$

where $\sqrt{\Gamma_{\mathscr{L}}[F_k,F_k]}(D) := \int_{\partial D} k\,d\sigma$, this is actually Proposition 53 in [6]. Note that this procedure of enlargement (Theorem 1.7 chapter V in [17]) could be done by gluing the same independent Brownian motion for each $(\Omega,\mathcal{F}^D,\mathbb{Q})$. We denote by $(\tilde\Omega,\tilde{\mathcal{F}}^D,\tilde{\mathbb{Q}})$ the enlarged probability space. Since $\mathscr{L}$ is an h-transform of $\widehat{\mathscr{L}}$ namely

$$\mathscr{L}[F_k]=\widehat{\mathscr{L}}[F_k]+\frac{\Gamma_{\widehat{\mathscr{L}}}(F_1,F_k)}{F_1},$$

equation (G.3) becomes in a differential form

$$dF_k(D_t)-\widehat{\mathscr{L}}[F_k](D_t)dt=\Big(\int_{\partial D}k\,d\sigma\Big)\Big(dB_t+\frac{\underline{\mu}^{\partial D_t}(\partial D_t)}{\mu(D_t)}dt\Big). \tag{G.4}$$

Let

$$M_t=e^{-\int_0^t\langle\frac{\underline{\mu}^{\partial D_s}(\partial D_s)}{\mu(D_s)},dB_s\rangle-\frac12\int_0^t\left(\frac{\underline{\mu}^{\partial D_s}(\partial D_s)}{\mu(D_s)}\right)^2ds},$$

$$\mathbb{P}_{|\mathcal{F}_t}=M_t\tilde{\mathbb{Q}}_{|\mathcal{F}_t}.$$

Using Girsanov transform, D_t is solution of the $\widehat{\mathscr{L}}$ martingale problem on the probability space $(\tilde\Omega,\tilde{\mathcal{F}}^D,\mathbb{P})$. Since $\tilde{\mathbb{Q}}=M^{-1}\mathbb{P}$ we get the uniqueness in law of the $\mathscr{L}$ diffusion by Proposition G.1. □

Appendix H. Convergence in law: a key lemma

This Appendix is devoted to the adaptation to some domain-valued sequences of processes, of Lemma 4 in [21], which states stability of some time integrals under convergence in law.

Lemma H.1. *Let $\tilde{\mathscr{F}}:=\tilde{\mathscr{F}}^{\alpha,\varepsilon}$. We endow the set of continuous paths $\mathscr{C}\left([0,\infty),M\times\tilde{\mathscr{F}}\right)$ with the two dissimilarity measures d_β, $\beta\in\{0,\alpha\}$, defined as:*

$$d_\beta\left((x^1,D^1),(x^2,D^2)\right)=\sup_{t\geqslant0}\rho(x^1(t),x^2(t))+\sup_{t\geqslant0}d_{\beta,\tilde{\mathscr{F}}}(D^1(t),D^2(t)), \tag{H.1}$$

where for two domains D and D'

$$d_{\beta,\tilde{\mathscr{F}}}(D,D')=\begin{cases}d_{\beta,D}(D,D')\wedge d_{\beta,D'}(D',D)\wedge\varepsilon & \text{if } H(D,D')<\varepsilon\\ \varepsilon & \text{otherwise.}\end{cases} \tag{H.2}$$

Here $H(D,D')$ is the Hausdorff distance between D and D' and the distance $d_{\beta,D}$ is defined in (2.2).

Let $(X_t^n,D_t^n,\tau_\varepsilon^n)_{t\geqslant0}:=(X_t^{\delta_n},D_t^{\delta_n},\tau_\varepsilon^{\delta_n})_{t\geqslant0}$ a subsequence of (3.18) *converging in law to the limit defined in* (3.19) *for the product of d_α and the Euclidean distance in $\mathbb{R}_+$.*

Let $f_n:(x,D)\mapsto f_n(x,D)$ and $f:(x,D)\mapsto f(x,D)$ be maps on $M\times\tilde{\mathscr{F}}$ with values in some Euclidean space, and U an open set in $M\times\tilde{\mathscr{F}}$ for d_0. Assume that:

(i) *the random variables $\displaystyle\int_0^\infty|f_n(X_s^n,D_s^n)|^p\,ds$ are uniformly bounded in probability for some $p>1$,*
(ii) *in the open set U, the functions f_n converge locally uniformly to f with respect to d_0, and are d_0-continuous,*
(iii) *for a.e. $t\geqslant0$, $(X_t,D_t)\in U$.*

Then $\left(X_t^n,D_t^n,\displaystyle\int_0^t f_n(X_s^n,D_s^n)\,ds\right)_{t\geqslant0}$ converges in law to $\left(X_t,D_t,\displaystyle\int_0^t f(X_s,D_s)\,ds\right)_{t\geqslant0}$ for $(d_\alpha,|\cdot|)$.

Remark H.2. In the applications we will always take

$$U = \left\{(x, D) \in M \times \tilde{\mathscr{F}},\ x \in D\backslash S(D)\right\}, \tag{H.3}$$

which is easily seen to be d_0-open thanks to Assumption 3.1 on $\tilde{\mathscr{F}}$.

Proof. We will follow the proof of Lemma 4 in [21], but with several differences due to infinite dimensional spaces. Set for $n \in \mathbb{N}$, $t \geqslant 0$,

$$A_t^n := \int_0^t f_n(X_s^n, D_s^n)\, ds, \quad A_t := \int_0^t f(X_s, D_s)\, ds. \tag{H.4}$$

Condition (i) implies that the processes A^n are tight. To get the conclusion il is sufficient to show that all the converging subsequences have the same limit. So assume that

$$(X_t^n, D_t^n, A_t^n)_{t\geqslant 0} \xrightarrow{\mathscr{L}} (X_t, D_t, a_t)_{t\geqslant 0}\,. \tag{H.5}$$

and let us prove that $(a_t)_{t\geqslant 0} = (A_t)_{t\geqslant 0}$. By Skorohod theorem we may realize all processes

$$(X_t^n, D_t^n, A_t^n, X_t, D_t, a_t)_{t\geqslant 0} \tag{H.6}$$

on the same probability space $(\Omega, \mathscr{F}, \mathbb{P})$ in such a way that

$$(Z_t^n)_{t\geqslant 0} := (X_t^n, D_t^n, A_t^n)_{t\geqslant 0} \xrightarrow{\text{a.s.}} (X_t, D_t, a_t)_{t\geqslant 0} =: (Z_t)_{t\geqslant 0}. \tag{H.7}$$

This means that $Z_t^n \to Z_t$ a.s. uniformly in $t \geqslant 0$.

Fix $\omega \in \Omega$. Let $t > 0$ be such that $(X_t(\omega), D_t(\omega)) \in U$. For some $\varepsilon' > 0$ we have $(X_s(\omega), D_s(\omega)) \in U$ for all $s \in [t - \varepsilon', t + \varepsilon']$. The set

$$S := \left\{(X_s(\omega), D_s(\omega))\,, \quad s \in [t - \varepsilon', t + \varepsilon']\right\} \tag{H.8}$$

is d_α-compact in $M \times \tilde{\mathscr{F}}$, so it has a d_α-neighbourhood V included in U of the form

$$V = \left\{(x, D) \in M \times \tilde{\mathscr{F}}, \quad d_\alpha\left((x, D), S\right) \leqslant \varepsilon''\right\}. \tag{H.9}$$

for some small enough $\varepsilon'' > 0$. For n sufficiently large, $(X_s^n(\omega), D_s^n(\omega)) \in V$ for all $s \in [t - \varepsilon', t + \varepsilon']$. On the other hand V is bounded for the distance d_α. This implies by Arzela-Ascoli theorem that it is compact for the distance d_0. We have the two following facts, the first one being an assumption on the f_n and f, the second one being a consequence of the d_0-compactness of V

(a) $f_n \to f$ as $n \to \infty$ uniformly in (V, d_0);
(b) f is uniformly continuous in (V, d_0).

Then

$$\begin{aligned}
&\sup_{s\in[t-\varepsilon, t+\varepsilon]} \left|f_n(X_s^n(\omega), D_s^n(\omega)) - f(X_s(\omega), D_s(\omega))\right| \\
&\leqslant \sup_{s\in[t-\varepsilon, t+\varepsilon]} \left|f_n(X_s^n(\omega), D_s^n(\omega)) - f(X_s^n(\omega), D_s^n(\omega))\right| \\
&\quad + \sup_{s\in[t-\varepsilon, t+\varepsilon]} \left|f(X_s^n(\omega), D_s^n(\omega)) - f(X_s(\omega), D_s(\omega))\right|.
\end{aligned}$$

Both terms in the right converge to 0, the first one by (a) and the second one by (b). So we have by (H.7) and the above calculation

$$\left\{\begin{array}{rl}
(A_s^n(\omega))_{s\in[t-\varepsilon,t+\varepsilon]} & \to (a_s(\omega))_{s\in[t-\varepsilon,t+\varepsilon]} \\
((A_s^n(\omega))' = f_n(X_s^n(\omega), D_s^n(\omega)))_{s\in[t-\varepsilon,t+\varepsilon]} & \to (f(X_s(\omega), D_s(\omega)))_{s\in[t-\varepsilon,t+\varepsilon]}
\end{array}\right. \tag{H.10}$$

both uniformly in $s \in [t-\varepsilon, t+\varepsilon]$. This implies that $a_s(\omega)$ is differentiable in $(t-\varepsilon, t+\varepsilon)$ with derivative $f(X_s(\omega), D_s(\omega))$ and in particular at t.

We have that for all $t \geqslant 0$, $(X_t(\omega), D_t(\omega)) \in U$ a.s.. So for all $t \geqslant 0$,

$$\frac{d}{dt} a_t(\omega) = f(X_t(\omega), D_t(\omega)) \quad \text{a.s..} \tag{H.11}$$

This implies that ω a.s.

$$\frac{d}{dt} a_t(\omega) = f(X_t(\omega), D_t(\omega)) \quad \text{for a.e. } t. \tag{H.12}$$

On the other hand we know by [12] Theorem 10 that $(a_t)_{t\geqslant 0}$ is absolutely continuous :

$$a_t(\omega) = \int_0^t \ell_s(\omega)\, ds. \tag{H.13}$$

By Lebesgue theorem, ω a.s., for a.e. $t \geqslant 0$

$$\lim_{\varepsilon \searrow 0} \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} |\ell_s(\omega) - \ell_t(\omega)|\, ds = 0. \tag{H.14}$$

Equalities (H.12) and (H.13) imply that ω a.s.

$$\lim_{\varepsilon \searrow 0} \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} \ell_s(\omega)\, ds = f(X_t(\omega), D_t(\omega)) \quad \text{for a.e. } t. \tag{H.15}$$

On the other hand

$$\left| \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} \ell_s(\omega) - \ell_t(\omega)\, ds \right| \leqslant \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} |\ell_s(\omega) - \ell_t(\omega)|\, ds$$

so (H.14) implies that ω a.s. for a.e. $t \geqslant 0$

$$\lim_{\varepsilon \searrow 0} \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} \ell_s(\omega)\, ds = \ell_t(\omega). \tag{H.16}$$

Consequently, using (H.12) and (H.16), we get ω a.s. for a.e. $t \geqslant 0$

$$\ell_t(\omega) = f(X_t(\omega), D_t(\omega)) \tag{H.17}$$

Integrating we get ω-a.s. for all $t \geqslant 0$

$$a_t(\omega) = A_t(\omega) = \int_0^t f(X_s(\omega), D_s(\omega))\, ds. \tag{H.18}$$

This together with (H.4) proves the lemma. □

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