# On the Helmholtz decomposition for finite Markov processes

Laurent Miclo

**Abstract** Helmholtz decompositions break down any vector field into a sum of a gradient field and a divergence-free vector field. Such a result is extended to finite irreducible and reversible Markov processes, where vector fields correspond to anti-symmetric functions on the oriented edges of the underlying graph.

**Key words:** Helmholtz decompositions, compact Riemannian manifolds, vector fields, gradient fields, Poisson equation, optimal transport, finite Markov processes, finite weighted graphs.

**Mathematics Subject Classification (2020):** 60J27, 60J60, 58J65, 46E27, 37C10, 58B20.

## 1 Introduction

The Helmholtz decomposition on a compact Riemannian manifold writes any vector field as a sum of a gradient field and a vector field whose divergence vanishes. In some sense, this decomposition corresponds to the Riemannian probability distribution $\ell$ and can be extended to any other probability measure admitting a positive and smooth density with respect to $\ell$, with its corresponding skewed divergence, see Theorem 1 below. An interest of this extension is to explain why in the variational formulation of Benamou and Brenier [4] of the Wasserstein distance, only gradient fields are needed.

Our main goal here is to transpose this decomposition to the finite Markov process setting, see Theorem 2 in Section 3, via some geometric definitions

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inspired by Kenyon [8], interpreting 1-forms as anti-symmetric functions on the oriented edges of the underlying graph. The main difference with the continuous situation turns out to be that the two-sided flow of diffeomorphisms generated by a vector field has to be replaced by a one-sided flow on probability measures generated by some non-linear Markov generators. It will enable us to revisit the works of Maas [9] and Erbar and Maas [6] on optimal transport for finite Markov processes. Denote $\mathcal{P}_+(V)$ the set of all probability measures on the finite state space $V$ giving a positive weight to all its points. We will see furthermore that any smooth mapping $\mathcal{P}_+(V) \ni \rho \mapsto L_\rho$, where $L_\rho$ is an irreducible Markov generator admitting $\rho$ as reversible measure, leads to a Riemannian structure on $\mathcal{P}_+(V)$ which is natural in the context of finite optimal transport. But we will check that not all Riemannian structures on $\mathcal{P}_+(V)$ are of this Markovian form, except when $V$ has only two points.

The plan of this note is as follows. In the next section we recall in details the Helmholtz decompositions relative to "nice" probability measures on compact Riemannian manifolds, since we did not find a pedagogical exposition in the literature. This point of view is transferred to finite irreducible and reversible Markov processes in Section 3, which leads to the introduction of Markov-Riemannian metrics. The paper finishes with two appendices. One dealing, on finite state spaces, with a continuity property of the solutions to Poisson equations, which are at the heart of the Helmholtz decompositions. The second appendix computes a Metropolis-Riemann distance on the two point state space.

## 2 The compact Riemannian manifold setting

A reminder of the Helmholtz decomposition of vector fields is presented here, which will serve as a guide for the extension to the finite state space framework.

Consider $M$ a compact and connected Riemannian manifold. Denote $TM = \bigsqcup_{x\in M} T_xM$ the corresponding tangent space, where $T_xM$ is the tangent space above $x \in M$. Let $\Sigma(TM)$ be the set of tangent vector fields on $M$, namely the smooth sections from $M$ to $TM$.

Denote $\mathcal{P}_+(M)$ the set of probability measures $\rho$ on $M$ admitting a smooth and positive density, still denoted $\rho$, with respect to the Riemannian probability measure $\ell$. For any $\rho \in \mathcal{P}_+(M)$, we see $\Sigma(TM)$ as a tangent space above $\rho$ and we endow it with the scalar product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\rho$ given by

$$\forall\ b, b' \in \Sigma(TM), \qquad \langle\!\langle b, b' \rangle\!\rangle_\rho \coloneqq \int_M \langle b, b' \rangle_x \ \rho(dx)$$

(where $\langle \cdot, \cdot \rangle_x$ in the scalar product in the tangent space $T_xM$). This introduces a notion of $\rho$-**orthogonality** on $\Sigma(TM)$.

Given a smooth function $U : M \to \mathbb{R}$, the Riemannian gradient $\nabla U$ of $U$ is an example of vector field. We denote $\mathcal{G}$ their set as $U$ runs through all smooth functions. It appears that $\mathcal{G}$ is a linear sub-space of the space of sections $\Sigma(TM)$.

A vector field $b \in \Sigma(TM)$ is said to leave $\rho \in \mathcal{P}_+(M)$ invariant, if

$$\forall\ t \geqslant 0, \qquad \varphi_t(\rho) = \rho$$

where $(\varphi_t)_{t\in\mathbb{R}}$ is the flow generated by $b$ and where for any $t \geqslant 0$, $\varphi_t(\rho)$ is the push-forward of $\rho$ by $\varphi_t$. Denote by $\mathcal{I}(\rho)$ the set of $b \in \Sigma(TM)$ leaving $\rho$ invariant.

Here is the statement of the Helmholtz $\rho$-decomposition of $\Sigma(TM)$:

**Theorem 1** *For any $\rho \in \mathcal{P}_+(M)$, we have*

$$\Sigma(TM) = \mathcal{G} \oplus \mathcal{I}(\rho)$$

*where the terms of the r.h.s. are $\rho$-orthogonal.*

Before giving proof of this decomposition, let us recall a computational characterization of $\mathcal{I}(\rho)$. For any $\rho \in \mathcal{P}_+(M)$, denote $\mathrm{div}_\rho$ the **$\rho$-skewed divergence** defined by

$$\forall\ b \in \Sigma(TM), \qquad \mathrm{div}_\rho(b) \coloneqq \frac{1}{\rho}\mathrm{div}(\rho b)$$

where $\mathrm{div}(\cdot)$ is the usual Riemannian divergence. The corresponding $\rho$-skewed Laplacian $\triangle_\rho$ is defined similarly

$$\forall\ f \in \mathcal{C}^\infty(M), \qquad \triangle_\rho(f) \coloneqq \mathrm{div}_\rho(\nabla f) = \triangle(f) + \langle \nabla \ln(\rho), \nabla f\rangle$$

where $\triangle$ is the Laplace-Beltrami operator. The second order operator $\triangle_\rho$ has several names, depending in the context in which it is used: the generalized Ornstein-Uhlenbeck diffusion generator in probability theory, the over-damped Langevin operator in analysis and the Witten Laplacian in geometry (especially concerning its extensions to differential forms).

The interest of $\rho$-skewed divergence is the well-known:

**Lemma 1** *For any fixed $\rho \in \mathcal{P}_+(M)$, we have for any $b \in \Sigma(TM)$,*

$$b \in \mathcal{I}(\rho) \Leftrightarrow \mathrm{div}_\rho(b) = 0$$

***Proof*** This is a simple consequence of Stokes' theorem. Indeed, $\rho \in \mathcal{P}_+(M)$ and $b \in \Sigma(TM)$ being fixed, denote $\rho_t \coloneqq \varphi_t(\rho)$ for any $t \geqslant 0$. We have that $\rho \in \mathcal{I}(\rho)$ if and only if for any test function $f \in \mathcal{C}^\infty(M)$,

$$\forall\ t \geqslant 0, \qquad \rho_t[f] = \rho[f]$$

or equivalently

$$\forall\ t \geqslant 0, \qquad \partial_t \rho_t[f] = 0$$

It leads us to compute $\partial_t|_{t=0}\rho_t[f]$, since $\partial_t\rho_t[f]$ can be computed similarly at any time $t \geqslant 0$, replacing $\rho$ by $\rho_t$ (which is well-known to belong to $\mathcal{P}_+(M)$ too).

We have by definition of the flow,

$$\begin{aligned}
\partial_t|_{t=0}\rho_t[f] &= \partial_t|_{t=0} \int f(\varphi_t(x))\, \rho(dx) \\
&= \int \langle b, \nabla f \rangle\, d\rho \\
&= \int \langle \rho b, \nabla f \rangle\, d\ell \\
&= -\int \mathrm{div}(\rho b) f\, d\ell \\
&= -\int \mathrm{div}_\rho(b) f\, d\rho
\end{aligned} \qquad \square$$

Since the density $\rho$ is positive, this expression vanishes for all $f \in \mathcal{C}^\infty(M)$ if and only $\mathrm{div}_\rho(b) = 0$, as desired.

We can now come to the

***Proof (of Theorem 1)*** The elements $\rho \in \mathcal{P}_+(M)$ and $b \in \Sigma(TM)$ being fixed, we consider the Poisson equation in $U$:

$$\begin{cases} \triangle_\rho(U) = \mathrm{div}_\rho(b) \\ \ell[U] = 0 \end{cases} \tag{1}$$

This equation is well-known to admit a unique solution if and only if $\rho[\mathrm{div}_\rho(b)] = 0$. We compute

$$\begin{aligned}
\rho[\mathrm{div}_\rho(b)] &= \int \mathrm{div}_\rho(b)\, d\rho \\
&= \int \mathrm{div}(\rho b)\, d\ell \\
&= 0
\end{aligned}$$

where Stokes' theorem has been invoked again.

Consider $U$ the unique solution of (1) and define $\beta \coloneqq b - \nabla U$. We have

$$\begin{aligned}
\mathrm{div}_\rho(\beta) &= \mathrm{div}_\rho(b - \nabla U) \\
&= \mathrm{div}_\rho(b) - \triangle_\rho(U) \\
&= 0
\end{aligned}$$

by construction of $U$.

It remains to check that for any $\nabla U \in \mathcal{G}$ and any $\beta \in \mathcal{I}(\rho)$ are $\rho$-orthogonal. We compute

$$\begin{aligned}\langle\!\langle \nabla U, \beta \rangle\!\rangle_\rho &= \int \langle \nabla U, \beta \rangle \, d\rho \\ &= \int \langle \nabla U, \rho\beta \rangle \, d\ell \\ &= -\int \operatorname{div}(\rho\beta) U \, d\ell \\ &= -\int \operatorname{div}_\rho(\beta) U \, d\rho \\ &= 0\end{aligned}$$

□

*Remark 1* (a) Endow $\mathcal{C}^\infty(M)$ with the Fréchet structure of uniform convergence of functions and their derivatives. Corresponding semi-norms $\mathcal{N}_L$, for $L \in \mathbb{Z}_+$, are constructed as follow. Let $(x^{(n)})_{n \in [\![N]\!]}$ be a finite family of points from $M$ and $\epsilon > 0$ be such that $M$ is covered by the open balls $B(x^{(n)}, \epsilon)$, $n \in [\![N]\!]$, and each of the balls $B(x^{(n)}, 2\epsilon)$ is the basis of a system of coordinates $(x_k^{(n)})_{k \in [\![m]\!]}$, where $m$ is the dimension of $M$. For $L \in \mathbb{Z}_+$, we define for any $f \in \mathcal{C}^\infty(M)$,

$$\mathcal{N}_L(f) := \sum_{n \in [\![N]\!]} \sum_{l \in [\![0,L]\!]} \max \left\{ \sup_{x \in B(x^{(n)}, \epsilon)} |\partial_{k_1, k_2, \ldots, k_l} f(x)| \, : \, k_1, k_2, ..., k_l \in [\![m]\!] \right\}$$

where $\partial_{k_1, k_2, \ldots, k_l}$ stands for the derivation with respect to $x_{k_1}^{(n)}, x_{k_2}^{(n)}, ..., x_{k_l}^{(n)}$ in $B(x^{(n)}, \epsilon)$ (and when $l = 0$, it is just the identity operator).
This structure and Gateaux differentiability enable us to define the notion of a smooth curve $[0,1] \ni t \mapsto f_t \in \mathcal{C}^\infty(M)$, when the curve and all its $t$-derivatives are continuous. A mapping $\mathcal{U} : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is said to be **regular** if it transforms smooth curves into smooth curves, i.e. if $[0,1] \ni t \mapsto \mathcal{U}(f_t) \in \mathcal{C}^\infty(M)$ is a smooth curve when $[0,1] \ni t \mapsto f_t \in \mathcal{C}^\infty(M)$ is a smooth curve.

These notions can be extended to $\mathcal{P}_+(M)$ (since it can be seen as a convex subset of $\mathcal{C}^\infty(M)$) and to $\Sigma(TM)$ (e.g. by seeing it as a product of copies of $\mathcal{C}^\infty(M)$, via Whitney's embedding theorem).

(b) Let us come back to the solution $U$ of the Poisson equation (1), for given $\rho \in \mathcal{P}_+(M)$ and $b \in \Sigma(TM)$. For $x \in M$, consider $(X_x^\rho(s))_{s \geqslant 0}$ a diffusion of generator $L_\rho$ starting from $x$. We have the probabilistic representation

$$U(x) = -\int_0^{+\infty} \mathbb{E}[\operatorname{div}_\rho(b)(X_x^\rho)(s)] \, ds \tag{2}$$

where the integrant converges exponentially fast to 0 with respect to $s$, as the spectral gap of $\triangle_\rho$ can be bounded below uniformly in terms of the supremum norm of $\ln(\rho)$ (see also Appendix 3, where this property will be developed in the finite state space setting). In the sequel, a curve taking values in $\mathcal{P}_+(M)$ or $\Sigma(TM)$ will always implicitly assumed to be smooth.

Girsanov formula relates explicitly the law of $X_x^\rho[0,s] \coloneqq (X_x^\rho(u))_{u\in[0,s]}$ to that of the Riemannian Brownian motion $X_x^{\mathbb{1}}[0,s]$, for any $s \geqslant 0$, by stating that for any measurable bounded function $F$ on the $M$-valued trajectories over the time interval $[0,s]$,

$$\begin{aligned}
&\mathbb{E}[F(X_x^\rho[0,s])]\\
&= \mathbb{E}\Big[F(X_x^{\mathbb{1}}[0,s])\exp\Big(\int_0^s \big\langle \nabla \ln(\rho)(X_x^{\mathbb{1}}(v)), dX_x^{\mathbb{1}}(v)\big\rangle\\
&\quad -\frac{1}{2}\int_0^s |\nabla \ln(\rho)(X_x^{\mathbb{1}}(v))|^2\, dv\Big)\Big]\\
&= \mathbb{E}\Big[F(X_x^{\mathbb{1}}[0,s])\exp\Big(\ln(\rho)(X_x^{\mathbb{1}}(s)) - \ln(\rho)(x)\\
&\quad -\frac{1}{2}\int_0^s \triangle \ln(\rho)(X_X^{\mathbb{1}}(v)) + |\nabla \ln(\rho)(X_x^{\mathbb{1}}(v))|^2\, dv\Big)\Big]
\end{aligned}$$

(where the scalar product and the norms are taken above $X_X^{\mathbb{1}}(v)$).

In conjunction with (2), it can be deduced that the mapping $\mathcal{P}_+(M) \times \Sigma(TM) \ni (\rho, b) \mapsto U \in \mathcal{C}^\infty(M)$ is regular in the sense described above in (a).

Let us give a classical consequence of the Helmholtz decomposition of Theorem 1 concerning the introduction of a (infinite-dimensional) Riemannian-like structure on $\mathcal{P}_+(M)$ as in Ambrosio, Gigli and Savaré [1].

A curve $[0,1] \ni t \mapsto \rho_t \in \mathcal{P}_+(M)$ (recall it is assumed to be smooth according to Remark 1a) is said to be **generated by a curve** $[0,1] \ni t \mapsto b_t \in \Sigma(TM)$ **of vector fields** when

$$\forall\ t \in [0,1], \qquad \rho_t = \varphi_t(\rho_0) \tag{3}$$

where the time-inhomogenous flow $(\varphi_t)_{t\in[0,1]}$ is defined through

$$\forall\ t \in [0,1],\ \forall\ x \in M, \qquad \frac{d}{dt}\varphi_t(x) = b_t(\varphi_t(x))$$

(starting with $\varphi_0$ being the identity mapping).

When there is a curve $[0,1] \ni t \mapsto U_t \in \mathcal{C}^\infty(M)$ of functions such that furthermore

$$\forall\ t \in [0,1], \qquad b_t = \nabla U_t$$

we say $(\rho_t)_{t\in[0,1]}$ is **generated by a curve of gradient fields**.

Generation by vector fields is in fact equivalent to generation by gradient fields:

**Proposition 1** *Assume that* $(\rho_t)_{t\in[0,1]}$ *is generated by the vector field curve* $(b_t)_{t\in[0,1]}$. *Then it is also generated by the gradient field curve* $(\nabla U_t)_{t\in[0,1]}$, *where for any* $t \in [0,1]$, $\nabla U_t$ *comes from the Helmholtz decomposition of* $b_t$ *above* $\rho_t$*:*

$$b_t = \nabla U_t + \beta_t$$

*with* $\beta_t \in \mathcal{I}(\rho_t)$. *Furthermore* $(\nabla U_t)_{t\in[0,1]}$ *is the only gradient field curve generating* $(\rho_t)_{t\in[0,1]}$.

***Proof*** Taking into account Remark 1, we get $[0,1] \ni t \mapsto \nabla U_t$ is as smooth as $[0,1] \ni t \mapsto b_t$.

Denote by $(\widetilde{\rho}_t)_{t\in[0,1]}$ the flow generated by the gradient fields $(\nabla U_t)_{t\in[0,1]}$. Consider a test function $f \in \mathcal{C}^\infty(M)$, we have for any $t \in [0,1]$,

$$\begin{aligned}
\partial_t \rho_t[f] &= \partial_t \int f(\varphi_t(x))\,\rho(dx) \\
&= \int \langle b_t, \nabla f\rangle_{\varphi_t(x)}\;\rho(dx) \\
&= \int \langle b_t, \nabla f\rangle_{x}\;\rho_t(dx) \\
&= -\int f \mathrm{div}_{\rho_t}(b_t)\,d\rho_t \\
&= -\int f \mathrm{div}_{\rho_t}(\nabla U_t)\,d\rho_t \qquad (4)\\
&= \partial_t \widetilde{\rho}_t[f]
\end{aligned}$$

By integration, it follows that $(\rho_t)_{t\in[0,1]}$ coincides with $(\widetilde{\rho}_t)_{t\in[0,1]}$.

Assume there is another gradient field curve $(\nabla \widetilde{U}_t)_{t\in[0,1]}$ generating $(\rho_t)_{t\in[0,1]}$. According to (4), we have for any $t \in [0,1]$ and any test function $f \in \mathcal{C}^\infty(M)$,

$$\int f \mathrm{div}_{\rho_t}(\nabla U_t)\,d\rho_t = \int f \mathrm{div}_{\rho_t}(\nabla \widetilde{U}_t)\,d\rho_t$$

It follows that for any given $t \in [0,1]$,

$$\int \left\langle \nabla f, \nabla(U_t - \widetilde{U}_t)\right\rangle d\rho_t = 0$$

so that taking $f = U_t - \widetilde{U}_t$, we get $\nabla(U_t - \widetilde{U}_t) = 0$. Since this is true for any $t \in [0,1]$, it means the two gradient field curves $(\nabla U_t)_{t\in[0,1]}$ and $(\widetilde{\nabla} U_t)_{t\in[0,1]}$ coincide. □

*Remark 2* To make evolve the boundary $\partial A$ of a smooth domain $A$ in $M$, one usually resorts to a $T(M)$-valued vector field on $\partial A$ and "pushes the boundary points" according to this vector field. It is well-known, see e.g. Mantegazza [10], that it is sufficient to consider vector fields that are normal to $\partial A$, because their tangential components leave $\partial A$ invariant (and just act as an infinitesimal reparametrization). Proposition 1 is a result analogous to this orthogonal decomposition of vector fields on $\partial A$ into normal and tangential components. In fact we believe that both the Helmholtz decomposition and the normal and tangential decomposition are two instances of a more general result for measures on $M$ (where $\partial A$ would be identified with the Hausdorff trace of $\ell$ on $\partial A$) that is outside the scope of the present note.

To go in the direction of a Riemannian-type distance on $\mathcal{P}_+(M)$, we have to define the length of a curve. We start by associating to $\rho_0 \in \mathcal{P}_+(M)$ and to a curve of vector fields $(b_t)_{t\in[0,1]}$ the quantity

$$\mathcal{L}(\rho_0, (b_t)_{t\in[0,1]}) := \sqrt{\int_0^1 \|b_t\|_{\rho_t}^2 \, dt} \tag{5}$$

where $(\rho_t)_{t\in[0,1]}$ is deduced from $\rho_0$ and $(b_t)_{t\in[0,1]}$ via (3).

Next, if $(\rho_t)_{t\in[0,1]}$ is a curve in $\mathcal{P}_+(M)$, we define

$$\mathcal{L}((\rho_t)_{t\in[0,1]}) := \inf\left\{\mathcal{L}(\rho_0, (b_t)_{t\in[0,1]}) \;:\; (b_t)_{t\in[0,1]} \in \mathcal{B}((\rho_t)_{t\in[0,1]})\right\} \tag{6}$$

where $\mathcal{B}((\rho_t)_{t\in[0,1]})$ stands for the set of vector field curves generating $(\rho_t)_{t\in[0,1]}$.

**Lemma 2** *Let $(\rho_t)_{t\in[0,1]}$ be a curve in $\mathcal{P}_+(M)$ generated by a vector field curve. Consider $(\nabla U_t)_{t\in[0,1]}$ the unique gradient field curve generating $(\rho_t)_{t\in[0,1]}$ according to Proposition 1. We have*

$$\mathcal{L}((\rho_t)_{t\in[0,1]}) = \mathcal{L}(\rho_0, (\nabla U_t)_{t\in[0,1]})$$

***Proof*** Consider any vector field curve $(b_t)_{t\in[0,1]}$ generating $(\rho_t)_{t\in[0,1]}$ and decompose it as in Proposition 1. The orthogonality property of Theorem 1 then shows that

$$\forall\; t \in [0,1], \qquad \|b_t\|_{\rho_t}^2 = \|\nabla U_t\|_{\rho_t}^2 + \|\beta_t\|_{\rho_t}^2$$

implying that

$$\mathcal{L}(\rho_0, (b_t)_{t\in[0,1]}) \geqslant \mathcal{L}(\rho_0, (\nabla U_t)_{t\in[0,1]})$$

This shows that the infimum of (6) is attained at the unique gradient field generating $(\rho_t)_{t\in[0,1]}$. □

The last step in constructing a Riemannian-like distance $D$ on $\mathcal{P}_+(M)$ is to define, for any $\rho_0, \rho_1 \in \mathcal{P}_+(M)$,

$$D(\rho_0, \rho_1) \coloneqq \inf\{\mathcal{L}((\rho_t)_{t\in[0,1]}) \,:\, (\rho_t)_{t\in[0,1]} \in \mathcal{R}(\rho_0, \rho_1)\}$$

where $\mathcal{R}(\rho_0, \rho_1)$ is the set of curves in $\mathcal{P}_+(M)$, starting at $\rho_0$ and ending at $\rho_1$, and generated by vector field curves. Lemma 2 shows that in this definition, it is sufficient to consider gradient field curves, thus enabling us to recover the variational formulation of Benamou and Brenier [4] in terms of gradient field curves. It turns out that $D$ is a Wasserstein distance, but we will not go further in this direction.

## 3 The finite Markov process setting

We propose here a version of Theorem 1 valid for finite Markov processes (as a first step toward a full Markov process extension?).

The Riemannian structure of $M$ can be entirely encapsulated into the Laplace-Beltrami operator $\triangle$. By analogy, a finite framework consists in an irreducible Markov generator $L \coloneqq (L(x, y))_{x,y\in V}$ whose invariant probability $\pi$ is supposed to be reversible. Denote $V$ the underlying finite state space. It is endowed with a graph structure, by defining its edge set as

$$E \coloneqq \{(x, y) \in V^2 \,:\, L(x, y) > 0\} \tag{7}$$

Note that the edges are directed and that by reversibility we have

$$\forall\ x, y \in V, \qquad (x, y) \in E \Leftrightarrow (y, x) \in E$$

Thus it could be tempting to consider the corresponding set of undirected edges. It is more convenient to choose an orientation for any undirected edge. More precisely we consider a set $\mathbf{E} \subset E$, such that for any $(x, y) \in E$, either $(x, y) \in \mathbf{E}$ or $(y, x) \in \mathbf{E}$, but not both $(x, y)$ and $(y, x)$ belong to $\mathbf{E}$.

A function $F : E \to \mathbb{R}$ is said to be a **vector field** if and only if

$$\forall\ (x, y) \in E, \qquad F(x, y) = -F(y, x)$$

We denote by $\Sigma(E)$ the set of vector fields on the graph $(V, E)$. Note that it is in natural bijection with $\mathbb{R}^{\mathbf{E}}$ the set of mapping from $\mathbf{E}$ to $\mathbb{R}$. This definition is inspired by Kenyon [8] (see also Remark 5 below). The latter paper also writes $L$ in a form analogous to $\triangle = \mathrm{div} \circ \nabla$ in the Riemannian setting. More precisely, given a function $f : V \to \mathbb{R}$ (whose space is denoted $\mathbb{R}^V$), define the **gradient** of $f$ as the vector field given by

$$\forall\ (x, y) \in E, \qquad \nabla f(x, y) \coloneqq f(y) - f(x) \tag{8}$$

Endow $V$ with the probability measure $\pi$ and $\mathbf{E}$ with the measure $\mu$ defined by

$$\forall\ (x,y) \in \mathbf{E}, \qquad \mu(x,y) \coloneqq \pi(x)L(x,y) \tag{9}$$

The operator $\nabla$ can be seen as going from $\mathbb{L}^2(\pi)$ to $\mathbb{L}^2(\mu)$. By definition, the **divergence** div is the opposite of the adjoint operator to $\nabla$. Then we can write again:

$$L = \mathrm{div} \circ \nabla \tag{10}$$

Denote $\mathcal{G}$ the set of vector fields which are gradients.

Pursuing our analogies with the Riemannian case, let $\mathcal{P}_+(V)$ stands for all probability measures giving a positive weight to all points of $V$. Consider $F \in \Sigma(E)$, to make it act on elements of $\mathcal{P}_+(V)$, consider the Markov generator $L_F$ associated to $F$ via

$$\forall\ x \neq y \in V, \qquad L_F(x,y) \coloneqq F_+(x,y)L(x,y) \tag{11}$$

where we adopt the convention

$$\forall\ (x,y) \notin E, \qquad F(x,y) = 0 \tag{12}$$

and where $F_+(x,y)$ stands for the non-negative part of $F(x,y)$ (the entries of the diagonal of $L_F$ are such that the row sums all vanish).

*Remark 3* The Markov processes $X_F$ generated by $L_F$ mimick, as much as possible, dynamical systems generated by vector fields. In particular edges can be crossed only in one direction. Note that if $F$ is a gradient $\nabla f$, then the jumps of $X_F$ (strictly) increase $f$ and it follows that $X_F$ converges in finite but random time toward a (random) local maxima of $f$. Recall that $x \in V$ is a local maxima for $f$ (with respect to the neighborhood structure given by $E$) when for any $(x,y) \in E$, we have $f(y) \leqslant f(x)$.

The Markov generator $L_F$ enables to associate to $F$ the flow $(\varphi_t)_{t\geqslant 0}$ acting on $\mathcal{P}_+(V)$ via

$$\forall\ \nu \in \mathcal{P}_+(V),\ \forall\ t \geqslant 0, \qquad \varphi_t(\nu) \coloneqq \nu \exp(tL_F) \tag{13}$$

Contrary to the Riemannian case, we cannot consider negative times if we want to stay in $\mathcal{P}_+(V)$. The other major difference is that there is no underlying dynamical system acting on the points of $V$. Instead, it must be replaced by the Markov process generated by $L_F$ (whose time-marginal distribution are the $(\varphi_t(\nu))_{t\geqslant 0}$ when the initial distribution is $\nu$).

We say that a vector field $F \in \Sigma(E)$ leaves our fixed reversible probability $\pi$ invariant if and only if

$$\forall\ t \geqslant 0, \qquad \varphi_t(\pi) = \pi$$

Denote again by $\mathcal{I}(\pi)$ the set of $F \in \Sigma(E)$ leaving $\pi$ invariant.

Let us define a scalar product on $\Sigma(E)$, seen as a tangent space to $\mathcal{P}_+(V)$ above $\pi \in \mathcal{P}_+(V)$, to get a notion of $\pi$-**orthogonality**. We define the scalar product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\pi$ on $\Sigma(E)$ through

$$\forall\ F, F' \in \Sigma(E), \qquad \left\langle\!\left\langle F, F' \right\rangle\!\right\rangle_\pi = \sum_{x \in V} \left\langle F, F' \right\rangle_x \pi(x)$$

where

$$\forall\ x \in V,\ \forall\ F, F' \in \Sigma(E), \qquad \left\langle F, F' \right\rangle_x \coloneqq \sum_{y \in V \setminus \{x\}} F(x,y) F'(x,y) L(x,y)$$

Namely, we have

$$\forall\ F, F' \in \Sigma(E), \qquad \left\langle\!\left\langle F, F' \right\rangle\!\right\rangle_\pi = \sum_{x \neq y \in V} F(x,y) F'(x,y) \pi(x) L(x,y)$$

(this quantity can also be identified with $2\left\langle F, F' \right\rangle_{\mathbb{L}^2(\mu)}$, where $\mu$ is defined in (9)).

We have the analogue of Theorem 1, but restricted to the probability $\pi$:

**Theorem 2** *We have*

$$\Sigma(E) = \mathcal{G} \oplus \mathcal{I}(\pi)$$

*where the terms of the r.h.s. are $\pi$-orthogonal.*

Let us compute the divergence, as in Kenyon [8], who used the following explicit form to deduce (10).

**Lemma 3** *We have for any $F \in \Sigma(E)$:*

$$\forall\ x \in V, \qquad \mathrm{div}(F)(x) = \sum_{y \in V} F(x,y) L(x,y) \tag{14}$$

***Proof*** Consider a test function $f \in \mathbb{R}^V$. We have

$$\begin{aligned}
\mu[F \nabla f] &= \sum_{(x,y) \in \mathbf{E}} F(x,y)(f(y) - f(x)) \pi(x) L(x,y) \\
&= \frac{1}{2} \sum_{(x,y) \in E} F(x,y)(f(y) - f(x)) \pi(x) L(x,y) \\
&= \frac{1}{2} \sum_{x,y \in V} F(x,y)(f(y) - f(x)) \pi(x) L(x,y) \\
&= \frac{1}{2} \sum_{x,y \in V} F(x,y) f(y) \pi(x) L(x,y) - \frac{1}{2} \sum_{x,y \in V} F(x,y) f(x) \pi(x) L(x,y)
\end{aligned}$$

Note that by reversibility and by definition of a vector field,

$$\begin{aligned}\sum_{x,y\in V} F(x,y)f(y)\pi(x)L(x,y) &= \sum_{x,y\in V} F(x,y)f(y)\pi(y)L(y,x)\\ &= -\sum_{x,y\in V} F(y,x)f(y)\pi(y)L(y,x)\\ &= -\sum_{x,y\in V} F(x,y)f(x)\pi(x)L(x,y)\end{aligned}$$

thus we get

$$\begin{aligned}\mu[F\nabla f] &= -\sum_{x,y\in V} F(x,y)f(x)\pi(x)L(x,y)\\ &= -\sum_{x\in V}\pi(x)f(x)\sum_{y\in V} F(x,y)L(x,y)\end{aligned}$$ □

The desired formula follows.

We deduce the analogue of Lemma 1, at least for $\rho = \pi$:

**Lemma 4** *We have for any $F \in \Sigma(E)$,*

$$F \in \mathcal{I}(\pi) \Leftrightarrow \operatorname{div}(F) = 0$$

***Proof*** Consider a vector field $F$ satisfying $\operatorname{div}(F) = 0$. It is equivalent to the fact that for any $f \in \mathbb{R}^V$, we have $\pi[f\operatorname{div}(F)] = 0$. We compute, taking into account the convention (12), the reversibility of $\pi$ for $L$ and the anti-symmetry of $F$,

$$\begin{aligned}&\pi[f\operatorname{div}(F)]\\ &= \sum_{x,y\in V}\pi(x)L(x,y)F(x,y)f(x)\\ &= -\sum_{x,y\in V}\pi(y)L(y,x)F(y,x)f(x)\\ &= -\sum_{x,y\in V}\pi(x)L(x,y)F(x,y)f(y)\\ &= -\sum_{x\neq y\in V}\pi(x)L(x,y)F(x,y)f(y)\\ &= -\sum_{x\neq y\in V}\pi(x)L(x,y)F_+(x,y)f(y) + \sum_{x\neq y\in V}\pi(x)L(x,y)F_-(x,y)f(y)\end{aligned}$$

where $F_-(x,y) \coloneqq -\min(F(x,y),0)$ is the non-positive part of $F(x,y)$. By definition of $L_F$, the first sum is equal to

$$\sum_{x\neq y\in V}\pi(x)L_F(x,y)f(y) = \pi[L_F[f]] - \sum_{x\in V}\pi(x)L_F(x,x)f(x)$$

We compute

$$\begin{aligned}\sum_{x\in V}\pi(x)L_F(x,x)f(x) &= -\sum_{x\in V}\pi(x)f(x)\sum_{y\neq x}L_F(x,y)\\ &= -\sum_{x\in V}\pi(x)f(x)\sum_{y\neq x}F_+(x,y)L(x,y)\\ &= -\sum_{x\neq y\in V}\pi(x)L(x,y)F_+(x,y)f(x)\\ &= -\sum_{x\neq y\in V}\pi(y)L(y,x)F_+(y,x)f(y)\\ &= -\sum_{x\neq y\in V}\pi(x)L(x,y)F_+(y,x)f(y)\end{aligned}$$

Since $F$ is a vector field, we have

$$\forall\ x,y\in V,\qquad F_+(y,x) = F_-(x,y)$$

Putting together the above computations, we end up with

$$\pi[f\mathrm{div}(F)] = -\pi[L_F[f]] \tag{15}$$

□

and the desired equivalence follows.

Now we have at our disposal all the ingredients necessary to the

***Proof (of Theorem 2)*** The arguments are exactly the same as those of the proof of Theorem 1. The vector field $F\in\Sigma(E)$ being fixed, we consider the Poisson equation in $U$:

$$\begin{cases} L(U) = \mathrm{div}(F)\\ \pi[U] = 0\end{cases} \tag{16}$$

This equation is well-known to admit a unique solution if and only if $\pi[\mathrm{div}(F)] = 0$. From (15) and writing $\mathbb{1}$ for the constant function taking the value 1, we get

$$\begin{aligned}\pi[\mathrm{div}(F)] &= -\pi[L_F[\mathbb{1}]]\\ &= 0\end{aligned}$$

Consider $U$ the unique solution of (16) and define $G := b - \nabla U$. We have

$$\begin{aligned}\mathrm{div}(G) &= \mathrm{div}(F-\nabla U)\\ &= \mathrm{div}(F) - L(U)\\ &= 0\end{aligned}$$

by construction of $U$.

It remains to check that for any $\nabla U\in\mathcal{G}$ and any $G\in\mathcal{I}(\pi)$ are $\pi$-orthogonal. By definition we have

$$\langle\!\langle \nabla U, G\rangle\!\rangle_\pi = \sum_{x,y\in V}(U(y)-U(x))G(x,y)\,\pi(x)L(x,y)$$

By symmetry, we have

$$\sum_{x,y\in V}U(y)G(x,y)\,\pi(x)L(x,y) = -\sum_{x,y\in V}U(x)G(x,y)\,\pi(x)L(x,y)$$

so that

$$\begin{aligned}\langle\!\langle \nabla U, G\rangle\!\rangle_\pi &= -2\sum_{x,y\in V}U(x)G(x,y)\,\pi(x)L(x,y)\\ &= -2\pi[U\mathrm{div}(G)]\\ &= 0\end{aligned}$$ □

Of course, Theorem 2 can be extended to any $\rho \in \mathcal{P}_+(V)$, it is sufficient to be given a corresponding irreducible generator $L_\rho$ admitting $\rho$ as reversible probability measure and to consider all the above related notions. Let us assume that the corresponding graph $(V, E)$ remains the same, which amounts to assuming there is a function $\rho : E \to (0, +\infty)$ such that

$$\forall\ (x,y)\in V^2,\ x\neq y,\qquad L_\rho(x,y) = \rho(x,y)L(x,y)$$

(when $(x,y) \notin E$, $x \neq y$, the values of $\rho(x,y)$ are irrelevant since $L(x,y) = 0$, we can take for instance $\rho(x,y) = 0$). The reversibility of $\rho$ is equivalent to

$$\forall\ (x,y)\in E,\qquad \delta_\rho(x)\rho(x,y) = \delta_\rho(y)\rho(y,x) \tag{17}$$

where $\delta_\rho$ is the density of $\rho$ with respect to $\pi$:

$$\forall\ x\in V,\qquad \delta_\rho(x) \coloneqq \frac{\rho(x)}{\pi(x)}$$

*Remark 4* Maas in [9] also "extends" the probability $\rho$ into a function $\rho$ of two variables of the state space (rather corresponding to the sides of (17)). He makes further assumptions, in particular that there exists a function $\theta : (0, +\infty)^2 \to (0, +\infty)$ such that $\rho(x,y) = \theta(\rho(x), \rho(y))$, which are not necessary for our purposes.

The definition of the gradient remains the same, but in (9) we have to consider

$$\forall\ (x,y)\in \mathbf{E},\qquad \mu_\rho(x,y) \coloneqq \rho(x)L_\rho(x,y)$$

with the corresponding notion of divergence $\mathrm{div}_\rho$, so that $L_\rho = \mathrm{div}_\rho \circ \nabla$.

Definition (11) has to be replaced by

$$\forall\ x\neq y\in V,\qquad L_{\rho,F}(x,y) \coloneqq (F(x,y))_+L_\rho(x,y) \tag{18}$$

$$= (F(x,y))_+\rho(x,y)L(x,y)$$

The set $\mathcal{I}(\rho)$ can be seen as the set of vector fields $F$ leaving $\rho$ invariant, with respect to the linear flow $(\varphi_{\rho,t})_{t\geqslant 0}$ defined, as in (13), by

$$\forall\ \nu\in\mathcal{P}_+(V),\ \forall\ t\geqslant 0,\qquad \varphi_t(\nu) \coloneqq \nu\exp(tL_{\rho,F})$$

but it seems more natural (cf. (24) below) to see $\mathcal{I}(\rho)$ as the set of vector fields $F$ leaving $\rho$ invariant, with respect to the non-linear flow $(\psi_t)_{t\geqslant 0}$ described by its evolution:

$$\forall\ \nu\in\mathcal{P}_+(V),\ \forall\ t\geqslant 0,\qquad \partial_t\psi_t(\nu) = \psi_t(\nu)L_{\psi_t(\nu),F} \tag{19}$$

(and starting with $\psi_0(\nu)=\nu$). This non-linearity is a true discrepancy with respect to the Riemannian situation.

*Remark 5* The fact that (8) does not depend on $L$ nor $\rho$, suggests, as in Section 3.3 of Kenyon [8] (see the definition of $df(e)$ there, with $\phi_{ye}=\phi_{xe}=1$), that this definition corresponds more to an 1-form than to a (gradient) vector field, in analogy with differential geometry, where the differential of a function does not depend on the Riemannian structure, contrary to the gradient. Thus maybe we should have changed our terminology, replacing vector fields by 1-forms (and replace *Helmholtz decompositions* by *Hodge decompositions*, which is more traditional term for differential form fields, see for instance Section 3.44 of Aris [2]). In the Riemannian setting, vector fields are important to describe corresponding dynamical systems, reason why we preferred to work with them. Nevertheless, in the finite Markov setting this link is distorted by the mapping $F\mapsto L_{\rho,F}$, so it could be harmless to adopt the 1-form terminology.

Finally the scalar product $\langle\!\langle\cdot,\cdot\rangle\!\rangle_\rho$ on $\Sigma(E)$ should be defined through

$$\forall\ F,F'\in\Sigma(E),\qquad \langle\!\langle F,F'\rangle\!\rangle_\rho = \sum_{x\neq y\in V} F(x,y)F'(x,y)\rho(x)L_\rho(x,y) \tag{20}$$

leading to the notion of $\rho$-orthogonality on $\Sigma(E)$.

With these definitions, Theorem 2 extends immediately into

**Theorem 3** *For any $\rho\in\mathcal{P}_+(V)$, we have*

$$\Sigma(E) = \mathcal{G}\oplus\mathcal{I}(\rho)$$

*where the terms of the r.h.s. are $\rho$-orthogonal.*

Another difference with the Riemannian case is the choice of the $L_\rho$ for $\rho\in\mathcal{P}_+(V)$, role which previously were “naturally” played the $\triangle_\rho$ for $\rho\in\mathcal{P}_+(M)$. A classical choice is to consider the Metropolis generators, which corresponds to

$$\forall\ \rho \in \mathcal{P}_+(V),\ \forall\ (x,y) \in E, \qquad \rho(x,y) = \min\left(1, \frac{\rho(y)\pi(x)}{\pi(y)\rho(x)}\right) \tag{21}$$
$$= \min\left(1, \frac{\delta_\rho(y)}{\delta_\rho(x)}\right)$$

which satisfies (17), since

$$\forall\ (x,y) \in E, \qquad \delta_\rho(x)\rho(x,y) = \min(\delta_\rho(x), \delta_\rho(y)) \tag{22}$$

(there is a relation between this construction of $L_\rho$ from $L$ and $\rho$ and that of $\triangle_\rho$ from $\triangle$ and $\rho$, since they both correspond to the minimization of some trajectorial entropy, see [5]).

The favorite choice of Erbar and Maas [6], to extend the notions of Ricci lower bounds and their consequences for functional inequalities of finite Markov processes, is

$$\forall\ \rho \in \mathcal{P}_+(V),\ \forall\ (x,y) \in E, \quad \delta_\rho(x)\rho(x,y) = \frac{\delta_\rho(y) - \delta_\rho(x)}{\ln(\delta_\rho(y)) - \ln(\delta_\rho(x))} \tag{23}$$

Whatever the choice, some regularity of the mapping $\mathcal{P}_+(V) \ni \rho \mapsto L_\rho \in \mathbb{R}^{V\times V}$ must be assumed for the considerations of Remark 1 to hold (for more details in this respect see the following appendix). For instance let us assume this mapping is locally Lipschitzian (seeing $\mathcal{P}_+(V)$ as a subset of $(0,1)^V$), condition which also insures the needed local existence and uniqueness of the solution to (19). Note that both (22) and (23) satisfy this condition. Proposition 1 and Lemma 2 are then also valid in the finite setting, via the same proofs. Indeed, it is sufficient to extend the notion of generation by vector or gradient field curves via: a continuous curve $[0,1] \ni t \mapsto \rho_t \in \mathcal{P}_+(V)$ is said to be **generated by a continuous curve** $[0,1] \ni t \mapsto F_t \in \Sigma(E)$ **of vector fields** when

$$\forall\ t \in [0,1], \qquad \dot{\rho_t} = \rho_t L_{\rho_t, F_t} \tag{24}$$

We say furthermore that $[0,1] \ni t \mapsto \rho_t \in \mathcal{P}_+(V)$ is **generated by a continuous gradient field curve**, when $F_t$ belongs to $\mathcal{G}$ for all $t \in [0,1]$. Then Proposition 1 and Lemma 2 extend literally to the finite Markov setting.

More precisely, we are led to introduce as in (5), for any $\rho_0 \in \mathcal{P}_+(V)$ and any continuous curve $[0,1] \ni t \mapsto F_t \in \Sigma(E)$ of vector fields,

$$\mathcal{L}(\rho_0, (F_t)_{t\in[0,1]}) \coloneqq \sqrt{\int_0^1 \|F_t\|^2_{\rho_t}\ dt}$$

where $(\rho_t)_{t\in[0,1]}$ is the solution of (24). Next, as in (6), when $(\rho_t)_{t\in[0,1]}$ is a continuous curve in $\mathcal{P}_+(M)$, we define

$$\mathcal{L}((\rho_t)_{t\in[0,1]}) \coloneqq \inf\left\{\mathcal{L}(\rho_0, (F_t)_{t\in[0,1]})\ :\ (F_t)_{t\in[0,1]} \in \mathcal{B}((\rho_t)_{t\in[0,1]})\right\}$$

where $\mathcal{B}((\rho_t)_{t\in[0,1]})$ stands for the set of continuous vector field curves generating $(\rho_t)_{t\in[0,1]}$.

We get then that if $\mathcal{B}((\rho_t)_{t\in[0,1]}) \neq \emptyset$, then

$$\mathcal{L}((\rho_t)_{t\in[0,1]}) = \mathcal{L}(\rho_0, (\nabla U_t)_{t\in[0,1]})$$

where $(\nabla U_t)_{t\in[0,1]}$ is the unique gradient field curve generating $(\rho_t)_{t\in[0,1]}$. By the usual convention, when $\mathcal{B}((\rho_t)_{t\in[0,1]}) = \emptyset$, we take $\mathcal{L}((\rho_t)_{t\in[0,1]}) = +\infty$.

Following a traditional path, we construct a Riemannian-like distance $D$ on $\mathcal{P}_+(M)$ by defining, for any $\rho_0, \rho_1 \in \mathcal{P}_+(M)$,

$$D(\rho_0, \rho_1) \coloneqq \inf\{\mathcal{L}((\rho_t)_{t\in[0,1]}) \,:\, (\rho_t)_{t\in[0,1]} \in \mathcal{R}(\rho_0, \rho_1)\} \tag{25}$$

where $\mathcal{R}(\rho_0, \rho_1)$ is the set of continuous curves in $\mathcal{P}_+(M)$, starting at $\rho_0$ and ending at $\rho_1$.

To check this distance takes finite values, we have to show that $\mathcal{R}(\rho_0, \rho_1)$ contains at least one curve generated by a continuous vector field. The following result is the main step in this direction.

**Lemma 5** *Let be given $\rho \in \mathcal{P}_+(V)$ and a signed measure $\eta \in \mathcal{H}(V)$ with*

$$\mathcal{H}(V) \coloneqq \left\{ \eta \coloneqq (\eta(x))_{x\in V} \in \mathbb{R}^V \,:\, \sum_{x\in V} \eta(x) = 0 \right\}$$

*Then there exists a unique function $U \in \mathbb{R}^V$ with $\sum_{x\in V} U(x) = 0$ and such that*

$$\eta = \rho L_{\rho,\nabla U} \tag{26}$$

*Furthermore the mapping $(\rho, \eta) \mapsto U$ is continuous.*

***Proof*** Equation (26) amounts to

$$\forall\ x \in V, \qquad \eta(x) = \sum_{y\neq x} \rho(y)\nabla_+ U(y,x) L_\rho(y,x) + \rho(x) L_{\rho,\nabla U}(x,x) \tag{27}$$

where $\nabla_+ U(y,x)$ stands for $(\nabla U(y,x))_+$. By definition, we have

$$\begin{aligned} \rho(x) L_{\rho,\nabla U}(x,x) &= -\rho(x) \sum_{z\neq x} L_{\rho,\nabla U}(x,z) \\ &= -\sum_{z\neq x} \nabla_+ U(x,z)\rho(x) L_\rho(x,z) \\ &= -\sum_{z\neq x} \nabla_- U(z,x)\rho(z) L_\rho(z,x) \end{aligned}$$

so (27) rewrites

$$\forall\ x \in V, \qquad \eta(x) = \sum_{y \neq x} \rho(y) \nabla U(y,x) L_\rho(y,x)$$
$$= -\widetilde{L}_\rho[U](x) \tag{28}$$

where the Markov generator $\widetilde{L}_\rho$ is defined via

$$\forall\ x \neq y \in V, \qquad \widetilde{L}_\rho(x,y) \coloneqq \rho(y) L_\rho(y,x) \tag{29}$$

(and the diagonal of $\widetilde{L}_\rho$ is such that all the row sums vanish).

Note that $\widetilde{L}_\rho$ is a symmetric matrix so its reversible measure is the uniform distribution $\upsilon$ over $V$. Since $\widetilde{L}_\rho$ is irreducible, $\upsilon$ is also its unique invariant measure. These observations show that $U$ is the unique solution of the Poisson equation

$$\begin{cases} \widetilde{L}_\rho(U) = \eta \\ \upsilon[U] = 0 \end{cases} \tag{30}$$

□

The continuity of $U$ in $(\rho, \eta)$ is obtained as in the proof of Proposition 2 in Appendix 3.

The previous lemma proves that any $\mathcal{C}^1$ curve $(\rho_t)_{t\in[0,1]}$ in $\mathcal{P}_+(V)$ is generated by the continuous gradient field $(\nabla U_t)_{t\in[0,1]}$, where for any $t \in [0,1]$, $\nabla U_t$ is obtained as in (26), where $\eta$ is replaced by $\dot{\rho}_t$ and $\rho$ by $\rho_t$. In particular $\mathcal{R}(\rho_0, \rho_1) \neq \varnothing$, for any $\rho_0, \rho_1 \in \mathcal{P}_+(V)$.

The fact that in (26), $\eta = 0$ is equivalent to $\nabla U = 0$ implies that $D$ defined in (25) is a genuine Riemannian distance on $\mathcal{P}_+(V)$. Indeed, from the previous definitions, the scalar product above $\rho \in \mathcal{P}_+(V)$ of two tangent vectors $\eta$, $\eta' \in \mathcal{H}(V)$ is given by

$$\langle\langle\langle \eta, \eta' \rangle\rangle\rangle_\rho \coloneqq \langle\langle \nabla U, \nabla U' \rangle\rangle_\rho \tag{31}$$

where $\nabla U$ and $\nabla U'$ are the respective unique solutions to (26) and to $\eta' = \rho L_{\rho, \nabla U'}$.

The particular case where $(L_\rho)_{\rho \in \mathcal{P}_+(V)}$ correspond to the Metropolis construction (starting from a given irreducible and fixed couple $(\pi, L)$) could be called the **Metropolis-Riemann** structure of $\mathcal{P}_+(V)$. In Appendix 3 we compute the associated distance $D$ when $V$ has two points and the jump rates of $L$ are both equals to 1.

In general, let us compute more explicitly the scalar product given in (31):

**Lemma 6** *For any $\rho \in \mathcal{P}_+(V)$ and any $\eta$, $\eta' \in \mathcal{H}(V)$, we have*

$$\langle\langle\langle \eta, \eta' \rangle\rangle\rangle_\rho = \sum_{x \neq y \in V} \widetilde{A}_\rho(x,y)(\eta(y) - \eta(x))(\eta'(y) - \eta'(x))$$

*with $\widetilde{A}_\rho \coloneqq (\widetilde{A}_\rho(x,y))_{x,y\in V}$ the matrix given by*

$$\forall\ x, y \in V, \qquad \widetilde{A}_\rho(x,y) \coloneqq 2\int_{[0,+\infty)} \widetilde{P}_{\rho,t}(x,y) - \frac{1}{|V|}\, dt \tag{32}$$

*where*

$$\forall\ t \geqslant 0, \qquad \widetilde{P}_{\rho,t} \coloneqq \exp(t\widetilde{L}_\rho)$$

*(recall that the Markov generator $\widetilde{L}_\rho$ was defined in (29), that it is reversible with respect to the uniform distribution $\upsilon \equiv \frac{1}{|V|}$ and that the r.h.s. is absolutely convergent since the integrant is converging exponentially fast to zero).*

***Proof*** By definition, we have for any $\rho \in \mathcal{P}_+(V)$,

$$\begin{aligned}
\langle\!\langle \nabla U, \nabla U' \rangle\!\rangle_\rho &= \sum_{x\neq y\in V} \rho(x) L_\rho(x,y)(U(y)-U(x))(U'(y)-U'(x)) \\
&= \sum_{x\neq y\in V} \widetilde{L}_\rho(x,y)(U(y)-U(x))(U'(y)-U'(x)) \\
&= \sum_{x\in V} \widetilde{L}_\rho[(U-U(x))(U'-U'(x))](x)
\end{aligned}$$

Note that

$$\widetilde{L}_\rho[(U-U(x))(U'-U'(x))] = \widetilde{L}_\rho[UU'] - U(x)\widetilde{L}_\rho[U'] - U'(x)\widetilde{L}_\rho[U]$$

so we deduce, taking into account the reversibility of $\upsilon$ for $\widetilde{L}_\rho$

$$\begin{aligned}
\frac{\langle\!\langle \nabla U, \nabla U' \rangle\!\rangle_\rho}{|V|} &= \upsilon[\widetilde{L}_\rho[U^2] - U\widetilde{L}_\rho[U'] - U'\widetilde{L}_\rho[U]] \\
&= -\upsilon[U\widetilde{L}_\rho[U']] - \upsilon[U'\widetilde{L}_\rho[U]] \\
&= -2\upsilon[U\widetilde{L}_\rho[U']] \\
&= 2\upsilon[U\eta']
\end{aligned}$$

according to (26), where $\eta$ and $U$ are replaced by $\eta'$ and $U'$.

Recall that the solution of (30) is given by

$$\forall\ x \in V, \qquad U(x) = \int_0^{+\infty} \mathbb{E}[\eta(\widetilde{X}_{\rho,x}(t))]\, dt$$

where $\widetilde{X}_{\rho,x} \coloneqq (\widetilde{X}_{\rho,x}(t))_{t\geqslant 0}$ is a Markov process of generator $\widetilde{L}_\rho$ starting from $x \in V$. Taking into account that $\eta \in \mathcal{H}(V)$, the r.h.s. can written

$$\int_0^{+\infty} \mathbb{E}[\eta(\widetilde{X}_{\rho,x}(t))] - \upsilon[\eta]\, dt = \int_0^{+\infty} \widetilde{P}_{\rho,t}[\eta](x) - \frac{1}{|V|}\sum_{y\in V} \eta(y)\, dt$$

$$= \sum_{y\in V} \eta(y) \int_0^{+\infty} \left(\widetilde{P}_{\rho,t}(x,y) - \frac{1}{|V|}\right) dt$$

It follows that

$$\begin{aligned}\left\langle\!\!\left\langle \nabla U, \nabla U' \right\rangle\!\!\right\rangle_\rho &= 2 \sum_{x,y\in V} \eta'(x)\eta(y) \int_0^{+\infty} \left(\widetilde{P}_{\rho,t}(x,y) - \frac{1}{|V|}\right) dt \\ &= 2 \sum_{x,y\in V} \widetilde{A}_\rho(x,y)\eta'(x)\eta(y)\end{aligned}$$

Note that $\widetilde{A}_\rho \coloneqq (\widetilde{A}_\rho(x,y))_{x,y\in V}$ is a symmetric matrix, since the same is true for $\widetilde{L}_\rho$ and thus for $\widetilde{P}_{\rho,t}$ for any $t \geqslant 0$. Furthermore we have

$$\forall\ x \in V, \qquad \sum_{y\in V} \left(\widetilde{P}_{\rho,t}(x,y) - \frac{1}{|V|}\right) = 0$$

and by symmetry

$$\forall\ y \in V, \qquad \sum_{x\in V} \left(\widetilde{P}_{\rho,t}(x,y) - \frac{1}{|V|}\right) = 0$$

It follows that

$$\forall\ x \in V, \qquad \sum_{y\in V} A_\rho(x,y) \ = \ 0 \ = \ \sum_{y\in V} A_\rho(y,x) \qquad \square$$

and the announced result follows.

Consider any scalar product, $\langle\!\langle\!\langle \cdot, \cdot \rangle\!\rangle\!\rangle$ on $\mathcal{H}(V)$. It can be extended into a semi-definite scalar product on $\mathbb{R}^V$, still written $\langle\!\langle\!\langle \cdot, \cdot \rangle\!\rangle\!\rangle$, by imposing that $\mathbb{1}$, the vector whose entries are all equal to 1, is orthogonal to $\mathcal{H}(V)$ and that $\langle\!\langle\!\langle \mathbb{1}, \mathbb{1} \rangle\!\rangle\!\rangle = 0$. Consider $A \coloneqq (A(x,y))_{x,y\in V}$ the associated symmetric matrix. It is a non-negative matrix of rank $|V| - 1$ and all the row and column sums vanish and we can write

$$\forall\ \eta, \eta' \in \mathcal{H}(V), \qquad \langle\!\langle\!\langle \eta, \eta' \rangle\!\rangle\!\rangle = \sum_{x\neq y\in V} A(x,y)(\eta(y) - \eta(x))(\eta'(y) - \eta'(x))$$

Any Riemannian structure on $\mathcal{P}_+(V)$ is thus equivalent to the datum of a smooth mapping

$$\mathcal{P}_+(V) \ni \rho \mapsto A_\rho \in \mathcal{A} \tag{33}$$

where $\mathcal{A}$ is the space of all symmetric non-negative matrices of rank $|V| - 1$ whose row and column sums all vanish.

It is natural to wonder if all Riemannian structures on $\mathcal{P}_+(V)$ come from a smooth family $(L_\rho)_{\rho\in\mathcal{P}}$ of irreducible Markov generators, respectively reversible with respect to the $\rho$. Or equivalently, if the mapping (33) is constructed, as in Lemma 6, from a family $(\widetilde{L}_\rho)_{\rho\in\mathcal{P}}$ of irreducible Markov generators reversible with respect to the uniform distribution $\upsilon$. Let us call these metric structures **Markov-Riemannian**.

To investigate the existence of non-Markov-Riemannian structures, we consider a generic $A \in \mathcal{A}$ and we wonder if we can find on $V$ an irreducible Markov generator $\widetilde{L}$ reversible with respect to $\upsilon$ such that $A = \widetilde{A} \coloneqq (\widetilde{A}(x, y))_{x,y\in V}$ with

$$\forall\ x, y \in V, \qquad \widetilde{A}(x, y) \coloneqq 2\int_{[0,+\infty)} \exp(t\widetilde{L})(x, y) - \frac{1}{|V|}\, dt$$

Consider a spectral decomposition of $-\widetilde{L}$: $(\widetilde{\lambda}_n, \widetilde{\varphi}_n)_{n\in[\![0,|V|-1]\!]}$, where $(\widetilde{\varphi}_n)_{n\in[\![0,|V|-1]\!]}$ is an orthogonal basis of $\mathbb{R}^V$ for its usual scalar product, where

$$\forall\ n \in [\![0, |V| - 1]\!], \qquad \widetilde{L}[\widetilde{\varphi}_n] = -\widetilde{\lambda}_n\widetilde{\varphi}_n$$

and where $\widetilde{\lambda}_0 = 0$ and $\widetilde{\varphi}_0 = \mathbb{1}$.

It appears that a spectral decomposition of $\widetilde{A}$ is $(\lambda_n, \widetilde{\varphi}_n)_{n\in[\![0,|V|-1]\!]}$, where

$$\forall\ n \in [\![0, |V| - 1]\!], \qquad \lambda_n \coloneqq \begin{cases} 0 & \text{, if } n = 0 \\ \frac{2}{\widetilde{\lambda}_n} & \text{, otherwise} \end{cases}$$

We are thus led to the following question. Consider an orthogonal basis $(\varphi_n)_{n\in[\![0,|V|-1]\!]}$ of $\mathbb{R}^V$, with $\varphi_0 = \mathbb{1}$. Let be given positive numbers $(\lambda_n)_{n\in[\![|V|-1]\!]}$ and take $\lambda_0 = 0$. The matrix of the operator $A : \mathbb{R}^V \to \mathbb{R}^V$ described by

$$\forall\ n \in [\![0, |V| - 1]\!], \qquad A[\varphi_n] = \lambda_n\varphi_n$$

corresponds to a generic element of $\mathcal{A}$. When is the operator $\widetilde{L}$ defined by $\widetilde{L}[\varphi_0] = 0$ and

$$\forall\ n \in [\![|V| - 1]\!], \qquad \widetilde{L}[\varphi_n] = -\frac{2}{\lambda_n}\varphi_n \tag{34}$$

a Markov generator? It amounts to check that the off-diagonal entries of $\widetilde{L}$ are non-negative.

This problem is related to the determination of Markov sequences, see Definition 2.3 of Bakry and Huet [3] and to the hypergroup property of the basis $(\varphi_n)_{n\in[\![0,|V|-1]\!]}$ .

When $|V|=2$, the operator $\widetilde{L}$ defined by (34) is always a Markov generator, since it is given by

$$\widetilde{L} = \begin{pmatrix} -1/\lambda_1 & 1/\lambda_1 \\ 1/\lambda_1 & -1/\lambda_1 \end{pmatrix}$$

But as soon as $|V|\geqslant 3$, it is well-known that $\widetilde{L}$ defined by (34) may not be a Markov generator.

Here is an example with $V=\{1,2,3\}$ (which can be extended to any $V$ with $|V|\geqslant 3$). We take

$$\varphi_0 \;=\; \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \varphi_1 \;=\; \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad \varphi_2 \;=\; \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Consider $\widetilde{L}$ described by (34), with $\lambda_1, \lambda_2 > 0$ to be chosen later. Introduce the function

$$f \coloneqq 2\varphi_0 + \varphi_1 + 3\varphi_2 \;=\; \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$$

If $\widetilde{L}$ was to be Markovian we would have

$$\forall\ t\geqslant 0, \qquad f_t \;\coloneqq\; 2\varphi_0 + \exp(-2t/\lambda_1)\varphi_1 + 3\exp(-2t/\lambda_2)\varphi_2 \geqslant 0$$

In particular we should have $\partial_t f_t(1)|_{t=0}\geqslant 0$. But we compute that

$$\partial_t|_{t=0} f_t(1) = -\frac{2}{\lambda_1} + \frac{6}{\lambda_2}$$

quantity which is negative with $\lambda_1 = 2$ and $\lambda_2 = 7$.

**Acknowledgements** I would particularly like to thank Jérôme Bolte and Stéphane Villeneuve for the discussions we had about this paper, and Thăng Lê for finding mistakes in a first version. Fundings from the grants ANR-17-EURE-0010 and AFOSR-22IOE016 are acknowledged.

## Appendix 1: Continuity of the solution to the Poisson equation

The purpose of this appendix is to show that the solution $U$ of (16) is continuous in terms of $F, L$ and $\pi$.

On the finite set $V$, let $\mathcal{R}(V)$ be the set of couples $(\pi, L)$ consisting of an irreducible Markov generator $L$ reversible with respect to the probability measure $\pi$. Endow $V$ with a total order $\preccurlyeq$ and consider

$$\overrightarrow{V} := \{(x,y) \in V^2 \,:\, x \prec y\}$$

The mapping

$$\mathcal{R}(V) \ni (\pi, L) \mapsto ((\pi(x))_{x\in V}, (L(x,y))_{(x,y)\in\overrightarrow{V}}) \in (0,1)^V \times [0,+\infty)^{\overrightarrow{V}}$$

is a bijection on its image and enables us to endow $\mathcal{R}(V)$ with a natural topology.

Let $\Sigma(V^2)$ be the set of **general vector fields**, which are the functions $F : V^2 \to \mathbb{R}$ satisfying

$$\forall\ (x,y) \in V^2, \qquad F(x,y) = -F(y,x) \tag{35}$$

(in particular $F$ vanishes on the diagonal). Again, $\Sigma(V^2)$ is endowed with the topology inherited from $\mathbb{R}^{\overrightarrow{V}}$.

The **divergence associated to** $L$ is the mapping transforming any vector field $F \in \Sigma(V^2)$ into the function $\mathrm{div}_L(F)$ defined on $V$ via

$$\forall\ x \in V, \qquad \mathrm{div}_L(F)(x) = \sum_{y\in V} F(x,y)L(x,y)$$

For any vector field $F \in \Sigma(V^2)$ and any $(\pi, L) \in \mathcal{R}(V)$, consider the Poisson equation in the unknown function $U_{L,F}$ described by

$$\begin{cases} L(U_{L,F}) = \mathrm{div}_L(F) \\ \pi[U_{L,F}] = 0 \end{cases} \tag{36}$$

By reversibility of $\pi$ for $L$ and by the anti-symmetry property (35), we have $\pi[\mathrm{div}_L(F)] = 0$ so there is a unique solution $U_{L,F}$ to (36). Our main result here is:

**Proposition 2** *The mapping*

$$\Sigma(V^2) \times \mathcal{R}(V) \ni (F, \pi, L) \mapsto U_{L,F} \in \mathbb{R}^V$$

*is continuous.*

***Proof*** Recall the probabilistic representation of $U_{L,F}$:

$$\forall\ x \in V, \qquad U_{L,F}(x) = -\int_0^{+\infty} \mathbb{E}[\mathrm{div}_L(F)(X_x(t))]\,dt \tag{37}$$

where $X_x := (X_x(t))_{t\geqslant 0})$ stands for a left-limit and right-continuous Markov process starting from $x$ and admitting $L$ (resp. $L$) as generator.

To see the r.h.s. of (37) is well-defined, introduce the spectral gap of $L$:

$$\lambda(L) := \min\left\{-\frac{\pi[fL[f]]}{\pi[f^2]} \ :\ f \in \mathbb{R}^V\backslash\{0\},\, \pi[f]=0\right\}$$

which is the smallest non-zero eigenvalue of $-L$ (which is diagonalizable in $\mathbb{R}_+$ by reversibility). We have $\Lambda(L) > 0$ by irreducibility of $L$.

For $x \in V$ and $t \geqslant 0$, denote $m_{x,t}$ the law of $X_x(t)$, seen as a row vector. Using matrix product, we have

$$\forall\ t \geqslant 0, \qquad m_{x,t} = m_{x,0}\exp(tL)$$

and by duality, il follows that

$$\forall\ t \geqslant 0, \qquad \frac{m_{x,t}}{\pi} = \exp(tL)\left[\frac{m_{x,0}}{\pi}\right]$$

where the densities $\frac{m_{x,t}}{\pi}$ and $\frac{m_{x,0}}{\pi}$ are seen as a function, thus represented by column vectors.

We deduce that for any test function $f \in \mathbb{R}^V$,

$$\begin{aligned}
\forall\ t \geqslant 0, \qquad |m_{x,t}[f]-\pi[f]| &= \left|\pi\left[\left(\frac{m_{x,t}}{\pi}-1\right)f\right]\right| \\
&\leqslant \sqrt{\pi\left[\left(\frac{m_{x,t}}{\pi}-1\right)^2\right]}\sqrt{\pi[f^2]} \\
&= \sqrt{\pi\left[\left(\exp(tL)\left[\frac{m_{x,0}}{\pi}-1\right]\right)^2\right]}\sqrt{\pi[f^2]} \\
&\leqslant \exp(-\lambda(L)t)\sqrt{\pi\left[\left(\frac{m_{x,0}}{\pi}-1\right)^2\right]}\sqrt{\pi[f^2]} \\
&= \exp(-\lambda(L)t)\sqrt{\pi\left[\left(\frac{\delta_x}{\pi}-1\right)^2\right]}\sqrt{\pi[f^2]} \\
&\leqslant \sqrt{\frac{\pi[f^2]}{\pi(x)}}\exp(-\lambda(L)t)
\end{aligned}$$

Applying this bound with $f = \mathrm{div}_L(F)$, we get

$$\begin{aligned}
\forall\ x \in V,\, \forall\ t \geqslant 0, \qquad |\mathbb{E}[\mathrm{div}_L(F)(X_x(t))]| &= |m_{x,t}[\mathrm{div}_L(F)] - \pi[\mathrm{div}_L(F)]| \\
&\leqslant \sqrt{\frac{\pi[\mathrm{div}_L(F)^2]}{\pi(x)}}\exp(-\lambda(L)t)
\end{aligned}$$

which shows that the integral in (37) is absolutely converging.

It is well-known, see for instance Kato [7], that the mapping

$$\mathcal{R}(V) \ni (\pi, L) \mapsto \lambda(L) \in (0, +\infty)$$ □

is continuous. More precisely, to come back to the symmetric setting of Chapter 2 Section 5 of Kato [7], note that $\lambda(L)$ is also the spectral gap of the symmetric matrix $(L(x,y)\sqrt{\pi(x)/\pi(y)})_{x,y\in V}$.

It follows we can apply the convergence under integral theorem to get the desired continuity from (37).

The previous result enables us to extend Proposition 1 to the finite setting, by showing that continuous curve $[0,1] \ni t \mapsto \rho_t \in \mathcal{P}_+(V)$ generated (in the sense of (24)) by a continuous vector field curve is also generated by a continuous gradient vector field curve. Proposition 1 is even more than what we need. Indeed in Section 3, we fixed an element $(\pi, L)$ of $\mathcal{R}(V)$ and only considered the other elements $(\widetilde{\pi}, \widetilde{L}) \in \mathcal{R}(V)$ sharing with $(\pi, L)$ the same oriented edge set $E$ defined in (7). We end up with the previous set $\mathcal{R}(V)$ only when $E$ coincides with $V^2$ minus its diagonal. Otherwise, the set of such elements $(\widetilde{\pi}, \widetilde{L})$ is a proper subset $\mathcal{R}(E)$ of $\mathcal{R}(V)$. Of course by restriction, Proposition 2 is also valid if $\mathcal{R}(V)$ is replaced by $\mathcal{R}(E)$.

The interest of $\mathcal{R}(E)$ is that a Girsanov formula holds between its elements, a finite setting analogue of Remark 2. More precisely, let $(\widetilde{\pi}, \widetilde{L}) \in \mathcal{R}(E)$ and denote $\widetilde{X}_x \coloneqq (\widetilde{X}_x(t))_{t\geqslant 0}$ be a left-limit and right-continuous Markov process starting from $x$ and admitting $\widetilde{L}$ as generator. Then for any $T \geqslant 0$, the laws $\mathcal{L}(\widetilde{X}[0,T])$ and $\mathcal{L}(X[0,T])$ of $\widetilde{X}[0,T]$ and $X[0,T]$ are equivalent and the Radon-Nikodym derivative of $\mathcal{L}(\widetilde{X}[0,T])$ with respect to $\mathcal{L}(X[0,T])$ is given by

$$\frac{d\mathcal{L}(\widetilde{X}[0,T])}{d\mathcal{L}(X[0,T])} = \exp\left(\sum_{(x,y)\in E} \ln\left(\frac{\widetilde{L}(x,y)}{L(x,y)}\right) N_T(x,y) - \int_0^T H(X_t)\, dt\right)$$

where for any $(x,y) \in E$, $N_T(x,y)$ is the number of jumps of $X[0,T]$ from $x$ to $y$ and

$$\forall\ x \in V, \qquad H(x) \coloneqq \widetilde{L}(x,x) - L(x,x)$$

For a proof of this result, see e.g. the lecture notes [11].

## Appendix 2: On the two point state space

The goal of this appendix is to compute a Metropolis distance on the set of positive probability measures on $V \coloneqq \{0,1\}$, as an illustration of the constructions of Section 3.

As reference framework we choose

$$L \coloneqq \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

so $\pi = (1/2, 1/2)$ and the corresponding edge set is $E = \{(0,1),(1,0)\}$.

Any $\rho \in \mathcal{P}_+(V)$ writes $(1-\rho(1), \rho(1))$, where $\rho(1) \in (0,1)$. From now on, to simplify notations, $\rho$ is identified with $\rho(1)$ which is denoted $\rho \in (0,1)$. We consider the Metropolis choice of generators described in (21):

$$\forall\ \rho \in (0,1), \qquad L_\rho = \begin{pmatrix} -\rho(0,1) & \rho(0,1) \\ \rho(1,0) & -\rho(1,0) \end{pmatrix}$$

with

$$\begin{aligned} \rho(0,1) &= 1 \wedge \frac{\rho}{1-\rho} \\ \rho(1,0) &= 1 \wedge \frac{1-\rho}{\rho} \end{aligned}$$

Any vector field $F \in \Sigma(E)$ is determined by the value $F(0,1) \in \mathbb{R}$ which will simply be denoted $F \in \mathbb{R}$ in the sequel. Note that in the present situation, any vector field is a gradient field. With this convention, the Markov generator (18) is given by

$$\forall\ \rho \in (0,1),\ \forall\ F \in \mathbb{R}, \qquad L_{\rho,F} = \begin{cases} \begin{pmatrix} -F\rho(0,1) & F\rho(0,1) \\ 0 & 0 \end{pmatrix} & \text{, if } F \geqslant 0 \\ \\ \begin{pmatrix} 0 & 0 \\ -F\rho(1,0) & F\rho(1,0) \end{pmatrix} & \text{, if } F < 0 \end{cases}$$

We also compute that, as in (20) and with the above notations,

$$\begin{aligned} \forall\ \rho \in (0,1),\ \forall\ F \in \mathbb{R}, \qquad \|F\|_\rho^2 &= \sum_{x \neq y \in V} F^2(x,y)\rho(x)L_\rho(x,y) \\ &= 2[\rho \wedge (1-\rho)]F^2 \end{aligned}$$

Given a continuous curve $[0,1] \ni t \mapsto F_t \in \mathbb{R}$, Equation (24) writes

$$\begin{aligned} \forall\ t \in [0,1], \qquad \dot{\rho}_t &= \begin{cases} (1-\rho_t)F_t\rho_t(0,1) & \text{, if } F_t \geqslant 0 \\ \rho_t F_t \rho_t(1,0) & \text{, if } F_t < 0 \end{cases} \\ &= [\rho \wedge (1-\rho)]F_t \end{aligned} \tag{38}$$

Let be given $\rho_0, \rho_1 \in (0,1)$, the distance $D(\rho_0, \rho_1)$ defined in (25) is described by

$$D^2(\rho_0,\rho_1) = 2\min\left\{\int_0^1 [\rho_t \wedge (1-\rho_t)]F_t^2\,dt \ :\ (F_t)_{t\in[0,1]} \in \mathcal{D}(\rho_0,\rho_1)\right\} \tag{39}$$

where $(\rho_t)_{t\in[0,1]}$ is the solution of (38) and where $\mathcal{D}(\rho_0,\rho_1)$ is the set of stepwise continuous mappings $[0,1]\ni t\mapsto F_t\in\mathbb{R}$ which are such that the corresponding $(\rho_t)_{t\in[0,1]}$ does start at $\rho_0$ and end at $\rho_1$. Indeed, our previous continuity assumption can be relaxed into stepwise continuity by classical arguments of regularization by convolution.

To compute $D(\rho_0,\rho_1)$, there is no loss of generality in assuming that $\rho_0\leqslant\rho_1$ and even $\rho_0<\rho_1$, since $D(\rho_0,\rho_0)=0$. Furthermore, we can restrict our attention to non-negative $[0,1]\ni t\mapsto F_t\in\mathbb{R}_+$. Indeed, if we have $F_{t_0}<0$ for some $t_0\in[0,1]$, then, since $\dot\rho_t$ has the same sign as $F_t$ for all $t\in[0,1]$, we can find $t_1<t_2\in[0,1]$ with $t_0\in(t_1,t_2)$ such that $\rho_{t_1}=\rho_{t_2}$. It follows that in the minimization (39), it is advantageous to replace $(F_t)_{t\in[0,1]}$ by $(G_t)_{t\in[0,1]}$ defined by

$$\forall\ t\in[0,1],\qquad G_t\coloneqq\begin{cases}0 & \text{, if } t\in[t_1,t_2]\\ F_t & \text{, otherwise}\end{cases}$$

since $(G_t)_{t\in[0,1]}\in\mathcal{D}(\rho_0,\rho_1)$ and

$$\int_0^1[\rho_t\wedge(1-\rho_t)]F_t^2\,dt>\int_0^1[\rho_t\wedge(1-\rho_t)]G_t^2\,dt$$

Let us consider the case where $\rho_0\geqslant 1/2$. The situation where $\rho_1\leqslant 1/2$ can be treated similarly and the case where $\rho_0<1/2$ and $\rho_1>1/2$ is deduced by writting $D(\rho_0,\rho_1)=D(\rho_0,1/2)+D(1/2,\rho_1)$.

**Lemma 7** *For $1/2\leqslant\rho_0<\rho_1$, we have*

$$D(\rho_0,\rho_1)=4(\sqrt{1-\rho_0}-\sqrt{1-\rho_1})$$

***Proof*** Assuming, as we are allowed to, $F_t\geqslant 0$ for any $t\in[0,1]$, (38) reduces to

$$\forall\ t\in[0,1],\qquad \dot\rho_t=(1-\rho_t)F_t$$

since $\rho_t\geqslant 1/2$ for all $t\in[0,1]$.

We deduce that

$$\forall\ t\in[0,1],\qquad \rho_t=1-(1-\rho_0)\exp(-\phi_t)$$

where

$$\forall\ t\in[0,1],\qquad \phi_t=\int_0^t F_s\,ds$$

It appears that $(F_t)_{t\in[0,1]}$ belongs to $\mathcal{D}(\rho_0,\rho_1)$ if and only if

$$\phi_1=\ln\left(\frac{1-\rho_0}{1-\rho_1}\right)\tag{40}$$

Introduce the mapping $h$ defined on $\mathbb{R}_+$ via

$$\forall\ u \in \mathbb{R}_+, \qquad h(u) := \int_0^u \sqrt{(1-\rho_0)e^{-q}}\, dq$$

so that

$$\begin{aligned}\int_0^1 [\rho_t \wedge (1-\rho_t)]F_t^2\, dt &= \int_0^1 (1-\rho_t)F_t^2\, dt\\ &= \int_0^1 (h'(\phi_t)\dot{\phi}_t)^2\, dt\end{aligned}$$

The optimization problem (39) amounts to minimize twice the above r.h.s. under the condition

$$h(\phi_1) = h\left(\ln\left(\frac{1-\rho_0}{1-\rho_1}\right)\right) \quad =: \ A$$

which is equivalent to (40). Writing for any $t \in [0,1]$, $\varphi_t := h'(\phi_t)\dot{\phi}_t$, we are led to the simple problem of minimizing $\int_0^1 \varphi_t^2\, dt$ under the contraint $\int_0^1 \varphi_t\, dt = A$. It is well-known that the minimizer $[0,1] \ni t \mapsto \varphi_t$ is constant and so we end up with $A^2$ for the minimal value.

Thus we have shown that

$$\begin{aligned}\frac{D^2(\rho_0,\rho_1)}{2} &= h^2\left(\ln\left(\frac{1-\rho_0}{1-\rho_1}\right)\right)\\ &= \left(\int_0^{\ln\left(\frac{1-\rho_0}{1-\rho_1}\right)} \sqrt{(1-\rho_0)e^{-q}}\, dq\right)^2\\ &= (1-\rho_0)\left(\int_0^{\ln\left(\frac{1-\rho_0}{1-\rho_1}\right)} e^{-q/2}\, dq\right)^2\\ &= 8(1-\rho_0)\left(1-\sqrt{\frac{1-\rho_1}{1-\rho_0}}\right)^2\\ &= 8(\sqrt{1-\rho_0}-\sqrt{1-\rho_1})^2 \qquad \square\end{aligned}$$

which is the desired result.

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