On finite interweaving relations

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Plan of the talk

1. Interweaving relations
2. Characterizations
3. Sketchs of proofs
4. Matthews' result
5. Markov kernels with non-negative eigenvalues
6. The continuous-time situation

Plan

Discrete setting

Contrary to the historical examples of Rogers-Pitman (Brownian motion and Bessel-3 process) and Aldous-Diaconis (top-to-random shuffle), here we only consider relations between Markov kernels P and $\widetilde{P}$ defined on the same finite state space V.

Given initial distributions μ_0 and $\widetilde{\mu}_0$, $X := (X_n)_{n\in\mathbb{Z}_+}$ and $\widetilde{X} := (\widetilde{X}_n)_{n\in\mathbb{Z}_+}$ will stand for corresponding Markov chains.

Intertwining relations

Intertwining from P to $\widetilde{P}$:

$$P\Lambda = \Lambda\widetilde{P}$$

where the **link** Λ is another Markov kernel on V. When Λ is invertible, the relation is said to be **faithful**.
Bi-intertwining relation between P and $\widetilde{P}$, when in addition:

$$\widetilde{P}\widetilde{\Lambda} = \widetilde{\Lambda}P$$

The probabilistic interest of an intertwining relation from an absorbed P to an ergodic $\widetilde{P}$ is that it enables one to construct strong stationary times for $\widetilde{P}$ for *certain* initial distributions: those of the form $\mu_0\Lambda$.

Interweaving relations (1)

Strengthening of bi-intertwining relations: **interweaving relations**, when furthermore there exists a probability distribution $q = (q_n)_{n \in \mathbb{Z}_+}$ on $\mathbb{Z}_+$ such that

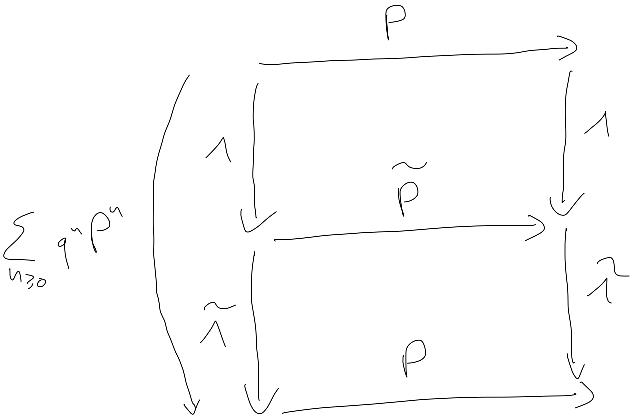
$$\Lambda\widetilde{\Lambda} = \sum_{n \in \mathbb{Z}_+} q_n P^n$$

It is a **bi-interweaving relation**, when for a probability distribution $\widetilde{q} = (\widetilde{q}_n)_{n \in \mathbb{Z}_+}$ on $\mathbb{Z}_+$,

$$\widetilde{\Lambda}\Lambda = \sum_{n \in \mathbb{Z}_+} \widetilde{q}_n \widetilde{P}^n$$

These relations are said to be **faithful** when Λ and $\widetilde{\Lambda}$ are invertible.

Interweaving relations (2)



Interweaving relations (3)

When there are both a faithful bi-intertwining relation between P and $\widetilde{P}$ and an interweaving relation then there is a faithful bi-interweaving relation with $\widetilde{q} = q$.
In the sequel all faithful bi-interweaving relations are with $\widetilde{q} = q$.

The probabilistic interest of an interweaving relation from P to $\widetilde{P}$ is to transfer information from $\widetilde{X}$ to X, for *any* initial distribution for X, but after a warming time distributed according to q.

With Pierre Patie, we introduced interweaving relations, first between square Bessel processes and their birth-and-death analogues, with a deterministic warming time equal to 1, in both directions. Our goal here is to investigate if such relations are common or not, in the finite context to begin with.

Plan

The irreducible case

Theorem 1

Assume that P and $\widetilde{P}$ are irreducible and similar. Then there exists a faithful bi-interweaving relation between them, with a probability q whose support contains at most $m+1$ points, where m is the common period of P and $\widetilde{P}$. Thus when P is aperiodic, there exists a faithful bi-interweaving relation between P and $\widetilde{P}$ with a probability q having a support with at most two points. When in addition of aperiodicity, we assume that none of the eigenvalues of P vanishes, then there exists a faithful bi-interweaving relation between P and $\widetilde{P}$ with q a Dirac mass.

The non-transient case (1)

Assume P and $\widetilde{P}$ are similar and non-transient kernels. Denote by C_1, C_2, ..., C_ℓ (respectively $\widetilde{C}_1$, $\widetilde{C}_2$, ..., $\widetilde{C}_\ell$) the irreducible classes of P (resp. $\widetilde{P}$). They are in the same number $\ell \in \mathbb{N}$, because this is the multiplicity of the eigenvalue 1. For all $l \in [\![\ell]\!] := \{1, 2, ..., \ell\}$, denote P_{C_l} (resp. $\widetilde{P}_{\widetilde{C}_l}$) the restriction of P (resp. $\widetilde{P}$) to C_l (resp. $\widetilde{C}_l$).

Theorem 2

There exists a faithful bi-interweaving relation between P and $\widetilde{P}$ if and only if there exists a permutation $\sigma \in \mathcal{S}_\ell$ and a probability q on $\mathbb{Z}_+$ such that for any $l \in [\![\ell]\!]$, $|C_l| = |\widetilde{C}_{\sigma(l)}|$ and there is a faithful bi-interweaving relation between P_{C_l} and $\widetilde{P}_{\widetilde{C}_{\sigma(l)}}$ with the same probability q. It can furthermore be imposed that q has a finite support.

The non-transient case (2)

By contrast, two non-transient Markov matrices P and $\widetilde{P}$ are similar if and only if there exists a faithful bi-intertwining relation between them. Thus there is a faithful bi-intertwining relation but no faithful bi-interweaving relation between

$$P := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 1/3 \end{pmatrix} \qquad \widetilde{P} := \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

Plan

The reversible case (1)

Proposition 3

Assume that P and $\widetilde{P}$ are similar and that P and $\widetilde{P}$ are irreducible and reversible. Then there exists a faithful bi-interweaving relation between them, with a probability q whose support contains at most three points. When P is aperiodic (and by consequence $\widetilde{P}$ too), we can find such a relation with a probability q whose support contains at most two points. When in addition to aperiodicity, none of the common eigenvalues of P and $\widetilde{P}$ vanishes, we can furthermore impose that q is a Dirac mass.

The reversible case (2)

Since P is irreducible and reversible, denote π, $1 = \theta_1 > \theta_2 \geqslant \theta_3 \geqslant \cdots \geqslant \theta_N \geqslant -1$ and $(\varphi_k)_{k\in[\![N]\!]}$, the invariant probability, the ordered eigenvalues and a corresponding orthonormal basis of $\mathbb{L}^2(\pi)$ of eigenvectors, with $N := |V|$.

Similarly for $\widetilde{P}$ with the same eigenvalues and a corresponding orthonormal basis of eigenvectors $(\widetilde{\varphi}_k)_{k\in[\![N]\!]}$ in $\mathbb{L}^2(\widetilde{\pi})$. We assume that $\varphi_1 = \widetilde{\varphi}_1 = \mathbb{1}$.

To any sequences $b := (b_k)_{k\in[\![2,|V|]\!]}$ and $\widetilde{b} := (\widetilde{b}_k)_{k\in[\![2,|V|]\!]}$ of real numbers, associate the operators A_b and $\widetilde{A}_{\widetilde{b}}$ defined by

$$\forall\ k \in [\![N]\!], \qquad A_b[\widetilde{\varphi}_k] \ := \ \begin{cases} b_k\varphi_k & \text{, if } k \geqslant 2 \\ 0 & \text{, if } k = 1 \end{cases}$$

$$\forall\ k \in [\![N]\!], \qquad \widetilde{A}_{\widetilde{b}}[\varphi_k] \ := \ \begin{cases} \widetilde{b}_k\widetilde{\varphi}_k & \text{, if } k \geqslant 2 \\ 0 & \text{, if } k = 1 \end{cases}$$

The reversible case (3)

We have for the corresponding matrices, for any $x, y \in V$,

$$
\begin{aligned}
|A_b(x,y)| &\leqslant \frac{1}{\sqrt{\pi_\wedge \widetilde{\pi}_\wedge}} \max_{k \in [\![2,N]\!]} |b_k| \widetilde{\pi}(y) \\
|\widetilde{A}_{\widetilde{b}}(x,y)| &\leqslant \frac{1}{\sqrt{\pi_\wedge \widetilde{\pi}_\wedge}} \max_{k \in [\![2,N]\!]} |\widetilde{b}_k| \pi(y)
\end{aligned}
$$

where $\pi_\wedge := \min_{x \in V} \pi(x)$ and $\widetilde{\pi}_\wedge := \min_{x \in V} \widetilde{\pi}(x)$.
The operator

$$
\Lambda_b := \widetilde{\pi} + A_b
$$

is Markovian as soon as

$$
\forall\ x, y \in V, \qquad \widetilde{\pi}(y) - |A_b(x,y)| \geqslant 0
$$

and in particular when

$$
\max_{k \in [\![2,N]\!]} |b_k| \leqslant \sqrt{\pi_\wedge \widetilde{\pi}_\wedge} \tag{1}
$$

The reversible case (4)

Similar arguments hold for $\widetilde{\Lambda}_{\widetilde{b}} := \pi + \widetilde{A}_{\widetilde{b}}$, so we get a faithful bi-intertwining relation between P and $\widetilde{P}$, with Λ_b and $\widetilde{\Lambda}_{\widetilde{b}}$ as links, by choosing any b and $\widetilde{b}$ with coordinates belonging to $[-\sqrt{\pi_\wedge \widetilde{\pi}_\wedge}, \sqrt{\pi_\wedge \widetilde{\pi}_\wedge}] \setminus \{0\}$.
Note that

$$\forall\ k \in [\![N]\!], \qquad \Lambda_b \widetilde{\Lambda}_{\widetilde{b}}[\varphi_k] \;=\; \begin{cases} \varphi_1 & \text{, if } k = 1 \\ \widetilde{b}_k b_k \varphi_k & \text{, if } k \geqslant 2 \end{cases}$$

and

$$\forall\ k \in [\![N]\!], \qquad \sum_{n \in \mathbb{Z}_+} q_n P^n[\varphi_k] \;=\; \sum_{n \in \mathbb{Z}_+} q_n \theta_k^n \varphi_k$$

So a bi-interweaving relation is equivalent to

$$\forall\ k \in [\![2, N]\!], \qquad \sum_{n \in \mathbb{Z}_+} q_n \theta_k^n \;=\; \widetilde{b}_k b_k$$

The reversible case (5)

In particular if we look for a Dirac mass $q = \delta_{n_0}$ while respecting (1), we can take

$$\forall\ k \in [\![2, N]\!], \qquad \left\{ \begin{array}{rcl} b_k & := & \sqrt{|\theta_k^{n_0}|} \\ \widetilde{b}_k & := & \sqrt{|\theta_k^{n_0}|}\mathrm{sign}(\theta_k^{n_0}) \end{array} \right.$$

with

$$\begin{array}{rcl} n_0 & := & 1 + \left\lfloor \dfrac{\ln(\pi_\wedge \widetilde{\pi}_\wedge)}{\ln(\zeta)} \right\rfloor \\ \zeta & := & \max\{|\theta_k| \ : \ k \in [\![2, N]\!]\} \end{array}$$

This is possible if P is aperiodic ($\zeta < 1$). Furthermore the quantities b_k and $\widetilde{b}_k$ do not vanish if none of the eigenvalues of P vanish.

The reversible case (6)

If some of the eigenvalues of P vanish, take

$$q \quad := \quad \frac{\pi_\wedge \widetilde{\pi}_\wedge}{2}\delta_0 + \left(1 - \frac{\pi_\wedge \widetilde{\pi}_\wedge}{2}\right)\delta_{n_1}$$

with

$$n_1 \quad := \quad 1 + \left\lfloor \frac{\ln(\pi_\wedge \widetilde{\pi}_\wedge / 4)}{\ln(\zeta)} \right\rfloor$$

If P is periodic (necessarily of period 2 by reversibility), rather take

$$q \quad := \quad \frac{\pi_\wedge \widetilde{\pi}_\wedge}{2}\delta_0 + \left(1 - \frac{\pi_\wedge \widetilde{\pi}_\wedge}{2}\right)\frac{\delta_{n_1} + \delta_{n_1+1}}{2}$$

Proof of Theorems 1 and 2 (1)

• Theorem 1 requires more care in the handling of the bases associated to the Jordan blocks, especially when some of the eigenvalues are non-real.

• Concerning Theorem 2, the reverse implication is simple: up to renaming the points it is sufficient to work with block diagonal matrices. For the direct implication, we first note that Λ must preserve the eigenspaces associated to the eigenvalue 1, which are respectively generated by the indicator functions $\mathbb{1}_{C_l}$ and $\mathbb{1}_{\widetilde{C}_l}$, for $l \in [\![\ell]\!]$.

Proof of Theorems 1 and 2 (2)

Thus there exist Markov matrices $M := (M_{k,l})_{k,l\in[\![\ell]\!]}$ and $\widetilde{M} := (\widetilde{M}_{k,l})_{k,l\in[\![\ell]\!]}$ so that for any $l \in [\![\ell]\!]$,

$$\begin{aligned} \Lambda[\mathbb{1}_{\widetilde{C}_l}] &= \sum_{k\in[\![\ell]\!]} M_{k,l}\mathbb{1}_{C_k} \\ \widetilde{\Lambda}[\mathbb{1}_{C_l}] &= \sum_{k\in[\![\ell]\!]} \widetilde{M}_{k,l}\mathbb{1}_{\widetilde{C}_k} \end{aligned}$$

From the interweaving relation, we deduce

$$\Lambda\widetilde{\Lambda}[\mathbb{1}_{C_l}] = \sum_{n\in\mathbb{Z}_+} q_n P^n[\mathbb{1}_{C_l}] = \sum_{n\in\mathbb{Z}_+} q_n \mathbb{1}_{C_l} = \mathbb{1}_{C_l}$$

Thus $\Lambda\widetilde{\Lambda}$ restricted to eigenspace associated to the eigenvalue 1 of P is the identity, namely $M\widetilde{M}$ is the identity matrix. The Markovian feature of M and $\widetilde{M}$ implies the latter are permutation matrices.

Plan

Framework and notations (1)

Let P be an irreducible Markov kernel on V with a reversible probability π. Assume its eigenvalues are non-negative and denote them $1 = \theta_1 > \theta_2 \geqslant \theta_3 \geqslant \cdots \geqslant \theta_N \geqslant 0$. Let $(\varphi_k)_{k \in [\![N]\!]}$ be a corresponding orthonormal eigenvector basis of $\mathbb{L}^2(\pi)$.
Let $X := (X(n))_{n \in \mathbb{Z}_+}$ be a Markov chain admitting P for transition kernel. Let μ_0 be the law of $X(0)$. For any $n \in \mathbb{Z}_+$, consider the probability distribution $\widetilde{\mu}_0^{(n)}$ on $[\![N]\!]$ given by

$$\forall\ k \in [\![N]\!], \qquad \widetilde{\mu}_0^{(n)}(k) \ := \ \begin{cases} \frac{\|\varphi_k\|_\infty |\mu_0[\varphi_k]|}{Z(\mu_0, n)} \theta_k^n & \text{, if } k \geqslant 2 \\ 0 & \text{, if } k = 1 \end{cases} \tag{2}$$

with

$$Z(\mu_0, n) \ := \ \sum_{l \in [\![N]\!] \setminus \{1\}} \|\varphi_l\|_\infty |\mu_0[\varphi_l]| \theta_l^n$$

Framework and notations (2)

Introduce the times

$$n_0 := \min\{n \in \mathbb{Z}_+ \ : \ Z(\mu_0, n) \leqslant 1\}$$
$$\bar{n}_0 := \min\left\{n \in \mathbb{Z}_+ \ : \ \frac{1}{\pi_\wedge} \sum_{k \in [\![2,N]\!]} \theta_k^n \leqslant 1\right\}$$

Consider $(G_k)_{k \in [\![2,N]\!]}$ a family of independent geometric random variables of respective parameters $(\theta_k)_{k \in [\![2,N]\!]}$, namely

$$\forall\ k \in [\![2,N]\!], \forall\ j \in \mathbb{N}, \qquad \mathbb{P}[G_k = j] = \theta_k^{j-1}(1 - \theta_k) \quad (3)$$

Construct a random variable $\mathcal{G}$ taking values in $\mathbb{Z}_+$ in the following way. First we sample an element K from $[\![2,N]\!]$ according to $\widetilde{\mu}_0^{(n_0)}$. We take $\mathcal{G} := G_K$.

Matthews' result

Recall that a strong stationary time for X is a finite stopping time τ such that τ and X_τ are independent and X_τ is distributed according to π.

Theorem 4 (Matthews)

Assume that P is irreducible, reversible and that its eigenvalues are all non-negative. Then there exists a strong stationary time for X which is stochastically dominated by

$$n_0 + \mathcal{G} \tag{4}$$

This random variable is itself stochastically dominated by $\bar{n}_0 + G_2 \leqslant \left\lceil \frac{\ln(N/\pi_\wedge)}{\ln(1/\theta_2)} \right\rceil + G_2$, *where G_2 is a geometric random variable of parameter θ_2.*

Idea of the proof

A proof of this result adapts that of Proposition 3 to the degenerate setting where $\widetilde{P}$ is an "absorbed model" for P:

$$\forall\ k, l \in [\![N]\!], \qquad \widetilde{P}(k,l) \ := \ \begin{cases} 1 & \text{, if } k = l = 1 \\ \theta_k & \text{, if } k = l \geqslant 2 \\ 1 - \theta_k & \text{, if } k \geqslant 2 \text{ and } l = 1 \\ 0 & \text{, otherwise} \end{cases}$$

and Λ is not Markovian:

$$\forall\ x \in V,\ \forall\ k \in [\![N]\!], \qquad \Lambda(x,k) \ := \ \begin{cases} \frac{\|\varphi_k\|_\infty \varphi_k(x)}{Z(\mu_0, n_0)} \theta_k^{n_0} & \text{, if } k \geqslant 2 \\ 0 & \text{, if } k = 1 \end{cases}$$

(once the φ_k, $k \in [\![N]\!]$, have been chosen so that $\mu_0(\varphi_k) \geqslant 0$).
The main job is to construct a "true" link $\widetilde{\Lambda}$ from $[\![N]\!]$ to V ensuring a generalised interweaving relation with warming time n_0.

Example of the discrete hypercube (1)

For $d \in \mathbb{N}$, consider the state space $V := \{-1, 1\}^d$, endowed with the transition kernel P of the lazy random walk whose entries are given by

$$P(x, x') := \begin{cases} \frac{1}{2} & \text{, if } x = x' \\ \frac{1}{2d} & \text{, if } x \text{ and } x' \text{ only differ at one coordinate} \\ 0 & \text{, otherwise} \end{cases}$$

The uniform distribution π on V is reversible for P.
Products over subsets of coordinates give eigenvectors which are thus well-controlled in the $\|\cdot\|_\infty$-norm.

Example of the discrete hypercube (2)

We compute that, on one hand, for any $\chi' < \ln(2) < \chi''$, for d large enough, we have

$$d \ln(d/\chi'') \leqslant n_0(d) \leqslant d \ln(d/\chi')$$

and on the other hand that

$$\lim_{d\to\infty} \frac{\mathbb{E}[\mathcal{G}(d)]}{d} = \sum_{k\in\mathbb{N}} \frac{1}{k} \frac{\ln(2)^k}{k!}$$

It follows that the estimate of Theorem 4 provides the right order for the upper bound in the separation cut-off on the hypercube $\{-1,1\}^d$ for large d.

Plan

Jordan blocs

Recall that a Jordan block of type (θ, n) is a $n \times n$ matrix whose diagonal entries are equal to θ, whose first above diagonal entries are equal to 1 and whose other entries vanish. Any $N \times N$-matrix P is similar to a block matrix, whose blocks are of Jordan types (θ_1, γ_1), (θ_2, γ_2), ..., (θ_r, γ_r), where $\theta_1, \theta_2, ..., \theta_r$ are the eigenvalues of P. The Jordan blocks are characterised by the existence of a basis $(\varphi_{(k,l)})_{(k,l)\in S}$ with $S := \{(k, l) \; : \; k \in [\![r]\!] \text{ and } l \in [\![\gamma_k]\!]\}$, such that

$$\forall \; (k, l) \in S, \qquad P[\varphi_{(k,l)}] \;\; = \;\; \theta_k \varphi_{(k,l)} + \varphi_{(k,l-1)} \qquad (5)$$

where by convention, $\varphi_{(k,0)} = 0$ for all $k \in [\![r]\!]$.

Assume now that P is an irreducible transition matrix on V whose eigenvalues are non-negative. Then all the above objects are real, we order the eigenvalues by

$$1 = \theta_1 > \theta_2 \geqslant \theta_3 \geqslant \cdots \geqslant \theta_r \geqslant 0$$

and take $\varphi_{(1,1)} = \mathbb{1}$.

Preliminary definitions (1)

Consider $S_0 := S \backslash \{(1,1)\}$ and the Gramian matrix R_0

$$\forall\ (k',l'),(k'',l'') \in S_0 \qquad R_0((k',l'),(k'',l'')) \quad := \quad \pi[\varphi_{(k',l')}\varphi_{(k'',l'')}]$$

where π is the invariant probability associated to P.

Let $\upsilon_\vee \geqslant \upsilon_\wedge > 0$ be the largest and the smallest eigenvalues of R_0.

Introduce for any $x \in V$, the vector

$$\varphi_0(x) \quad := \quad (\varphi_{(k,l)}(x))_{(k,l)\in S_0} \in \mathbb{R}^{S_0}$$

and define another vector $\alpha_0(x) := (\alpha_{(k,l)}(x))_{(k,l)\in S_0} \in \mathbb{R}^{S_0}$ by

$$\alpha_0(x) \quad := \quad R_0^{-1}\varphi_0(x)$$

Preliminary definitions (2)

Define

$$\forall\ k \in [\![2, r]\!], \qquad B_k \;:=\; \max\left\{ \sum_{l \in [\![\gamma_k]\!]} |\alpha_{(k,l)}(x)| \;:\; x \in V \right\}$$

and

$$n_0 \;:=\; \min\left\{ n \geqslant 2\Gamma \;:\; \sum_{(k,l) \in S_0} B_k \theta_k^n |\mu_0[\varphi_{(k,l)}]| \leqslant 1 \right\}$$

with $\Gamma := \max\{\gamma_k \;:\; k \in [\![r]\!]\}$.
Furthermore introduce

$$\bar{n}_0 \;:=\; (2\Gamma) \vee \left\lceil \frac{1}{2\ln(1/\theta_2)} \ln\left(\frac{\Gamma |V| \upsilon_\vee}{\pi_\wedge^2 \upsilon_\wedge} \right) \right\rceil$$

Preliminary definitions (3)

Introduce a probability on S and supported by S_0 via

$$\forall\ (k,l) \in S_0, \qquad \widetilde{\mu}_0((k,l)) \ := \ \frac{B_k \theta_k^{n_0} |\mu_0[\varphi_{(k,l)}]|}{Z_0}$$

where μ_0 is the initial distribution of X, Z_0 is the normalising constant, and the Markov kernel $\widetilde{P}$ on S whose entries are given by

$$\widetilde{P}((k,l),(k',l')) \ := \ \begin{cases} 1 & \text{, if } (k,l) = (k',l') = (1,1) \\ \theta_k & \text{, if } k = k' \geqslant 2 \text{ and } l = l' \\ 1-\theta_k & \text{, if } k = k' \geqslant 2 \text{ and } l' = l-1 \geqslant 1 \\ 1-\theta_k & \text{, if } k \geqslant 2,\ l = 1 \text{ and } (k',l') = (1,1) \\ 0 & \text{, otherwise} \end{cases}$$

Consider $\widetilde{X} := (\widetilde{X}(n))_{n\in\mathbb{Z}_+}$ a Markov chain associated with $(\widetilde{\mu}_0, \widetilde{P})$ and define

$$\mathcal{G} \ := \ \inf\{n \in \mathbb{Z}_+ \ :\ \widetilde{X}(n) = (1,1)\}$$

Extension of Matthews' result

Theorem 5

Assume that P is irreducible and that its eigenvalues are all non-negative. Then there exists a strong stationary time for X which is stochastically dominated by

$$n_0 + \mathcal{G} \tag{6}$$

This random variable is itself stochastically dominated by $\bar{n}_0 + \mathcal{H}_2$, where $\mathcal{H}_2$ is the convolution of Γ independent geometric random variables of parameter θ_2.

The proof uses $\widetilde{X}$ as a simple "spectral model" for X and provides a generalised interweaving between them.

Plan

Interweaving in continuous time

Replace the transition matrices by Markov generators on V and the Markov chains by Markov processes.
Interweaving relations now require the existence of a probability q on $\mathbb{R}_+$ such that

$$\Lambda\widetilde{\Lambda} \;=\; \int_{\mathbb{R}_+} \exp(tL)\, q(dt)$$

The previous considerations can be extended to this framework.
E.g. the analogue of Proposition 1 is:

Proposition 6

Assume that the Markov generators L and $\widetilde{L}$ are irreducible and similar. Then there exists a faithful bi-interweaving relation between them, with a probability q which can be taken to be a Dirac mass.

Non-transient generators

Consider L and $\widetilde{L}$ two non-transient Markov generators and introduce their irreducible classes as in the discrete-time setting. Here is the equivalent of Theorem 2:

Theorem 7

There exists a faithful bi-interweaving relation between L and $\widetilde{L}$ if and only if there exists a permutation $\sigma \in \mathcal{S}_\ell$ and a probability q on $\mathbb{R}_+$ such that for any $l \in [\![\ell]\!]$, $|C_l| = |\widetilde{C}_{\sigma(l)}|$ and there is a faithful bi-interweaving relation between L_{C_l} and $\widetilde{L}_{\widetilde{C}_{\sigma(l)}}$ with probability q. It can furthermore be imposed that q is a Dirac mass.

Real eigenvalues (1)

Let L be an irreducible Markov generator whose eigenvalues are real. The eigenvalues of $-L$ are denoted

$$0 = \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_r$$

We consider again the decomposition of L into Jordan blocks and in particular $(\varphi_{(k,l)})_{(k,l)\in S}$ is an adapted basis satisfying

$$\forall\ (k,l) \in S, \qquad L[\varphi_{(k,l)}] \quad = \quad -\lambda_k \varphi_{(k,l)} + \varphi_{(k,l-1)}$$

Given an initial distribution for X and with similar definitions as before, define

$$\begin{aligned} t_0 &:= \min\left\{ t \geqslant \Gamma \,:\, \sum_{(k,l)\in S_0} B_k \exp(-\lambda_k t) |\mu_0[\varphi_{(k,l)}]| \leqslant 1 \right\} \\ \bar{t}_0 &:= \Gamma \vee \frac{1}{\lambda_2} \ln\left(\frac{\Gamma |V| \upsilon_\vee}{\pi_\wedge^2 \upsilon_\wedge} \right) \end{aligned}$$

Real eigenvalues (2)

Introduce a probability on S and supported by S_0 via

$$\forall\ (k,l) \in S_0, \qquad \widetilde{\mu}_0((k,l)) \;:=\; \frac{B_k \exp(-\lambda_k t_0)|\mu_0[\varphi_{(k,l)}]|}{Z_0}$$

where Z_0 is the normalising constant, and the Markov generator $\widetilde{L}$ on S whose off-diagonal entries are given by

$$\widetilde{L}((k,l),(k',l')) \;:=\; \begin{cases} \lambda_k & \text{, if } k = k' \geqslant 2 \text{ and } l' = l-1 \geqslant 1 \\ \lambda_k & \text{, if } k \geqslant 2,\ l = 1 \text{ and } (k',l') = (1,1) \\ 0 & \text{, otherwise} \end{cases}$$

Consider $\widetilde{X} := (\widetilde{X}(t))_{n \in \mathbb{R}_+}$ a Markov process associated with $(\widetilde{\mu}_0, \widetilde{L})$ and define

$$\mathcal{G} \;:=\; \inf\{t \in \mathbb{R}_+ \;:\; \widetilde{X}(t) = (1,1)\}$$

Real eigenvalues (3)

Here is the analogue of Theorem 5 for continuous time:

Theorem 8

Assume that L is irreducible and that its eigenvalues are all real. Then there exists a strong stationary time for X which is stochastically dominated by

$$t_0 + \mathcal{G}$$

This random variable is itself stochastically dominated by $\bar{t}_0 + \mathcal{H}_2$, where $\mathcal{H}_2$ is a gamma distribution of shape Γ and scale $1/\lambda_2$.

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